Global stability of a susceptible-infected-susceptible epidemic model on networks with individual awareness*

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Recent research results indicate that individual awareness can play an important influence on epidemic spreading in networks. By local stability analysis, a significant conclusion is that the embedded awareness in an epidemic network can increase its epidemic threshold. In this paper, by using limit theory and dynamical system theory, we further give global stability analysis of a susceptible-infected-susceptible (SIS) epidemic model on networks with awareness. Results show that the obtained epidemic threshold is also a global stability condition for its endemic equilibrium, which implies the embedded awareness can enhance the epidemic threshold globally. Some numerical examples are presented to verify the theoretical results.

Keywords: epidemic spreading, global stability, awareness

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1. Introduction

Complex networks exist almost everywhere in our daily life. Examples of networks include the Internet,[1] biological networks,[2,3] electricity networks,[4] social relation networks,[5] epidemic networks,[6,7] traffic networks,[8,9] and so on. A typical network comprises nodes and links, where the nodes represent different individuals of a real system and each link denotes a certain relation between two nodes. In the past few decades, the epidemic spreading on complex networks has been studied widely and deeply.[10–16] It has been verified that the spreading behavior in heterogeneous networks behaves very differently from those traditional homogeneous ones. Among this research, a striking result[12] is the absence of an epidemic threshold in a scale-free network with a power-law degree distribution in the infinite size limit.

To predict the behavior of general epidemics on complex networks, global analysis of their equilibrium points must be made to guide the real applications. Lajmainovitch and Yorke[17] have presented an effective method to investigate the global stability properties of a deterministic model for gonorrhea in a nonhomogeneous population. In recent years, this method has been generalized to solve more general epidemic spreading models on complex networks.[18,19] By using limit theory and dynamical system theory, a strict mathematical proof was given in Ref. [15] to show the global stability of the well-known susceptible-infected-susceptible (SIS) epidemic model established by Pastor–Satorras and Vespignani. Recently, the method of global Lyapunov functions based on graph-theoretic approach was proposed in Refs. [20]–[22] to study the global stability of multi-group epidemic models.

The interplay between awareness and epidemic dynamics in networks has recently received much attention.[23–28] The human responses from self-protection awareness (e.g., avoiding assemblage and washing hands) to disease outbreak can result in reduction of susceptibility to infection, which in turn, affect epidemic dynamics. By local stability analysis, the authors in Ref. [29] have studied an SIS model and found that the contact awareness and the local awareness can enhance the epidemic threshold, while the global awareness can only decrease the epidemic prevalence. It is natural to raise the following problem: under certain conditions for this SIS epidemic model, can the global stability of its equilibrium point be guaranteed? Therefore, the aim of this paper is to investigate the global stability of this model to see if the awareness can enhance its epidemic threshold globally.

In this paper, we will give a positive answer to the above question. By using limit theory and dynamical system theory to address the global stability of its endemic equilibrium, we find that the embedded awareness can enhance the epidemic threshold globally. Some numerical examples are presented to illustrate the theoretical result.

This paper is organized as follows. In Section 2, some preliminaries about the considered model are presented. In Section 3, we study the global stability of the endemic equilibrium of the model. In Section 4, some numerical examples are given. In Section 5, we conclude this paper.

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2. Preliminaries

In the SIS model, each node can be in one of two distinct states at each time: susceptible (S) or infected (I). Each infected node can recover to be susceptible with probability δ. Each susceptible node has a probability ν of contagion with each infected neighbor. So we can define an effective spreading rate $\lambda = \nu/\delta$.

In Ref. [29], the authors have explored the impact of awareness on epidemic spreading in networks. Considering the complexity of individual awareness, they have defined three forms of awareness: local awareness, global awareness, and contact awareness. The authors have established the following mean-field rate equation:

$$\begin{aligned}\frac{\mathrm{d}\rho_k(t)}{\mathrm{d}t} &= -\nu\rho_k(t) + \lambda k\psi(k)(1-\rho_k(t))\Theta(t)(1-\alpha\Theta(t)) \\ &\quad \times (1-\beta\rho(t)) - \lambda\alpha\psi(k)(1-\rho_k(t)) \\ &\quad \times (1-\beta\rho(t))\Theta(t)(1-\Theta(t)), \end{aligned} \tag{1}$$

where $k = 1,2,\ldots,n$, parameters $\alpha,\beta \in [0,1]$, $\psi(k)$ is a decreasing function of k. In model (1), ν denotes the recovery rate of an infected node, and λ is the spreading rate. $\psi(k),\alpha$, and β represent the strength of contact awareness, local awareness, and global awareness, respectively. $\rho_k(t)$ denotes the relative density of infected nodes with degree k, n denotes the maximum degree of all nodes. $\rho(t) = \sum_{k=1}^{n}\rho_k(t)P(k)$ is the global infection density, where $P(k)$ denotes the degree distribution of the network. $\Theta(t) = \langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)\rho_i(t)$ represents the probability of a randomly chosen link pointing to an infected node, where $\langle k\rangle = \sum_{i=1}^{n} iP(i)$ is the average degree, $\langle k^2\rangle = \sum_{i=1}^{n} i^2P(i)$ denotes the second-order moment of the degree distribution.

Obviously, the equilibrium points of model (1) satisfy the following algebraic equation:

$$\begin{aligned}&-\nu\rho_k + \lambda k\psi(k)(1-\rho_k)\langle k\rangle^{-1}\sum_{k=1}^{n} kP(k)\rho_k \\ &\times\left(1-\alpha\langle k\rangle^{-1}\sum_{k=1}^{n} kP(k)\rho_k\right)\left(1-\beta\sum_{k=1}^{n}\rho_k P(k)\right) \\ &-\lambda\alpha\psi(k)(1-\rho_k)\left(1-\beta\sum_{k=1}^{n}\rho_k P(k)\right) \\ &\times\langle k\rangle^{-1}\sum_{k=1}^{n} kP(k)\rho_k\left(1-\langle k\rangle^{-1}\sum_{k=1}^{n} kP(k)\rho_k\right) \\ &= 0. \end{aligned} \tag{2}$$

By linear stability analysis,[29] if $\lambda/\nu < \lambda_c = \langle k\rangle/[\langle k^2\psi(k)\rangle - \alpha\langle k\psi(k)\rangle]$, then the disease-free equilibrium of model (1) is locally stable. Here, λ_c denotes the epidemic threshold: if λ/ν is below λ_c, the infection will gradually die out, while if λ/ν is above λ_c, the infection spreads and becomes endemic. In the next Section, we will prove that $\lambda/\nu > \lambda_c$ is a global stability condition for the endemic equilibrium of model (1).

3. Global stability of epidemic spreading

First, we introduce four lemmas about model (1) which will be utilized in the proof of the main theorem in this paper.

Lemma 1[30] If $0 \le \rho_k(0) \le 1$ satisfies $\sum_{k=1}^{n} kP(k)\rho_k(0) > 0$, then as $t > 0$, the solution $\rho_k(t)$ of model (1) satisfies $0 < \Theta(t) < 1$, and $0 < \rho_k(t) < 1$.

Lemma 2[30] Suppose $0 \le \rho_k(0) \le 1$ satisfies $\sum_{k=1}^{n} kP(k)\rho_k(0) > 0$ and

$$\lambda/\nu > \lambda_c = \frac{\langle k\rangle}{\langle k^2\psi(k)\rangle - \alpha\langle k\psi(k)\rangle},$$

then the solution $\rho_k(t)$ of model (1) satisfies $\inf_{t\ge 0}\Theta(t) > 0$; for any $\tau > 0$, $\inf_{t\ge\tau}\rho_k(t) > 0$.

Let two vectors $\theta = (P(1),P(2),\ldots,P(n))^{\mathrm{T}} \in R^n$, $\omega = (P(1),2P(2),\ldots,nP(n))^{\mathrm{T}} \in R^n$, matrix $A = \mathrm{diag}(P(1),2P(2),\ldots,nP(n))$. Define

$$\begin{aligned} g(x_1,x_2,\ldots,x_n) &= (1-\beta\sum_{i=1}^{n} x_iP(i))\times\sum_{i=1}^{n} iP(i)x_i \\ &\quad\times[k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)x_i]. \end{aligned}$$

Lemma 3 Suppose $0 < x_i < y_i < 1$, for each $i \in \{1,2,\cdots,n\}$. If

$$0 < \alpha \le \frac{1-\beta(1+\langle k\rangle)}{2},\ 0 < \beta \le \frac{1}{1+\langle k\rangle}, \tag{3}$$

or

$$\begin{aligned}&(k-\alpha)\min_{1\le i\le n}\{ip(i)\} + [(\alpha-\alpha k)\langle k\rangle^{-1}n + \beta(\alpha-k)] \\ &\times\sum_{i=1}^{n} nP^2(n) \ge 0, \end{aligned} \tag{4}$$

for every $k \in \{1,2,\ldots,n\}$, then we have

$$g(x_1,x_2,\ldots,x_n) \le g(y_1,y_2,\ldots,y_n). \tag{5}$$

From these two inequalities, we can see that both the local and global awareness strengths should not be too large, and they should maintain certain relation with network structure to meet inequality (3) or (4).

Lemma 4 Let $\mu_k^{(1)} = 1$, $\xi^{(m)}(k,\alpha) = k - \alpha + (\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)\mu_i^{(m)}$, $\xi(k,\alpha) = k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)\mu_i$ and $\eta^{(m)}(k,\alpha) = k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(m)}$, $\eta(k,\alpha) = k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i$, $1 \le k \le n$, $m = 1,2,\ldots$ Define the sequences

$$\mu_k^{(m+1)} = \frac{\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}\mu_i^{(m)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)\mu_i^{(m)}\xi^{(m)}(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}\mu_i^{(m)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)\mu_i^{(m)}\xi^{(m)}(k,\alpha)},$$

$$l_k^{(m+1)} = \frac{\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}l_i^{(m)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)l_i^{(m)}\eta^{(m)}(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}l_i^{(m)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)l_i^{(m)}\eta^{(m)}(k,\alpha)},$$

and $\mu_k=\lim_{m\to\infty}\mu_k^{(m)}$, $l_k=\lim_{m\to\infty}l_k^{(m)}$. Suppose the solution $\rho_k(t)$ satisfies $\limsup_{t\to+\infty}\rho_k\le\mu_k$, and $\liminf_{t\to+\infty}\rho_k(t)\ge l_k$, where $\mu_k\ge 0,\ l_k\ge 0$, then

$$\limsup_{t\to+\infty}\rho_k(t) \le \frac{\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}\mu_iP(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)\mu_i\xi(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}\mu_iP(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)\mu_i\xi(k,\alpha)},$$

$$\liminf_{t\to+\infty}\rho_k(t) \ge \frac{\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}l_iP(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)l_i\eta(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}l_iP(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)l_i\eta(k,\alpha)}.$$

For the sake of clarity, the proofs of Lemma 3 and 4 are presented in Appendix A. Now we present the main theorem of the paper.

Theorem 1 Suppose that parameters α and β satisfy Eq. (3) or Eq. (4) to guarantee the global stability for the endemic equilibrium of model (1). If $\sum_{k=1}^{n}kP(k)\rho_k(0)>0$ with $0\le\rho_k(0)\le 1$ and $\lambda/\nu>\lambda_c$, then the fraction of infected nodes of model (1) satisfies for every $k\in\{1,2,\ldots,n\}$,

$$\lim_{t\to\infty}\rho_k(t)=\rho_k>0.$$

Proof In order to prove this theorem, we first prove that $\lim_{t\to\infty}\rho_k(t)$ exists. For this purpose, we turn to prove that

$$\liminf_{t\to+\infty}\rho_k(t)=\limsup_{t\to+\infty}\rho_k(t).$$

Letting $\mu_k^{(1)}=1$, $\xi(k,\alpha)=k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)\mu_i$, and $\xi^{(m)}(k,\alpha)=k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)\mu_i^{(m)}$, define the sequence

$$\mu_k^{(m+1)} = \frac{\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}\mu_i^{(m)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)\mu_i^{(m)}\xi^{(m)}(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}\mu_i^{(m)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)\mu_i^{(m)}\xi^{(m)}(k,\alpha)},\quad 1\le k\le n,\ m=1,2,\ldots \tag{6}$$

According to Lemma 1, for $1\le k\le n$, $\limsup_{t\to+\infty}\rho_k(t)\le 1=\mu_k^{(1)}$. Applying Lemma 4 repeatedly, we obtain

$$\limsup_{t\to+\infty}\rho_k(t)\le\mu_k^{(m)},\ 1\le k\le n,\ m=1,2,\ldots \tag{7}$$

Now, we consider the convergence of sequence (6). By Eq. (6), $\mu_k^{(2)}\le 1=\mu_k^{(1)}$ for all k. If for all k, $\mu_k^{(m+1)}\le\mu_k^{(m)}$, then by Lemma 3, it follows that $\forall k,\ \mu_k^{(m+2)}\le\mu_k^{(m+1)}$.

By induction, the sequence $\mu_k^{(m)}$ is decreasing for each k, so its limit exists and is denoted by $\mu_k=\lim_{m\to\infty}\mu_k^{(m)}$. Letting $m\to\infty$ on both sides of Eqs. (6) and (7), we deduce that $\mu_k,\ k=1,2,\ldots,n$, satisfies the following relations:

$$-\nu\mu_k+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}\mu_iP(i)\right)(1-\mu_k)\langle k\rangle^{-1}\times\sum_{i=1}^{n}iP(i)\mu_i\xi(k,\alpha)=0,$$

$$\limsup_{t\to+\infty}\rho_k(t)\le\mu_k,\ 1\le k\le n. \tag{8}$$

On the other hand, we consider a function

$$\begin{aligned} f(x) &= -x+\frac{\psi(k)\left[1-\beta\sum_{k=1}^{n}\frac{\lambda x}{\nu+\lambda x}P(k)\right]\langle k\rangle^{-1}\sum_{k=1}^{n}\lambda k^2P(k)}{\nu+\lambda x}\\ &\quad\times\left[\left(1-\frac{\alpha}{k}\right)x+\left(\frac{\alpha}{k}-\alpha\right)\langle k\rangle^{-1}\sum_{k=1}^{n}\lambda k\frac{x^2}{\nu+\lambda x}P(k)\right]. \end{aligned}\tag{9}$$

Here, $f(0)=0$. Moreover, under the condition

$$\lambda/\nu>\lambda_c=\frac{\langle k\rangle}{\langle k^2\psi(k)\rangle-\alpha\langle k\psi(k)\rangle},$$

we have

$$f'(0) = \langle k\rangle^{-1} \sum_{k=1}^{n} \frac{\lambda k^2 P(k)\psi(k)\left(1-\frac{\alpha}{k}\right)\nu}{\nu^2} - 1 > 0. \quad (10)$$

By the definition of derivatives, if $x > 0$ is sufficiently small, then $f(x) > f(0) = 0$.

According to Lemma 2 and Eq. (10), we can take $l_k^{(1)}$ such that $\forall k,\ 0 < l_k^{(1)} < \liminf_{t\to\infty} \rho_k(t)$ and

$$f\left(\psi(k)\left(1-\beta\sum_{i=1}^{n} l_i^{(1)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(1)} \times \left[k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(1)}\right]\right) > 0. \quad (11)$$

Then, we define the following sequence:

$$l_k^{(m+1)} = \frac{\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n} l_i^{(m)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(m)}\eta^{(m)}(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n} l_i^{(m)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(m)}\eta^{(m)}(k,\alpha)}, \quad (12)$$

where $\eta^{(m)}(k,\alpha) = k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(m)}$, $1 \le k \le n$, $m = 1,2,\ldots$ According to Eq. (11) and applying Lemma 4 repeatedly, we have

$$\liminf_{t\to\infty} \rho_k(t) \ge l_k^{(m)},\ 1 \le k \le n,\ m = 1,2,\ldots \quad (13)$$

Now we consider the convergence of the sequence (12). According to Eqs. (9), (11), and (12), we have

$$\psi(k)\left(1-\beta\sum_{k=1}^{n} l_i^{(2)}P(k)\right)\langle k\rangle^{-1}\sum_{k=1}^{n} kP(k)l_i^{(2)} \times \left[k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{k=1}^{n} kP(k)l_i^{(2)}\right] > \psi(k)\left(1-\beta\sum_{i=1}^{n} l_i^{(1)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(1)} \times \left[k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(1)}\right]. \quad (14)$$

According to Eqs. (12) and (14), we obtain $\forall k$,

$$\begin{aligned} l_k^{(3)} &= \frac{\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n} l_i^{(2)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(2)}\eta^{(2)}(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n} l_i^{(2)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(2)}\eta^{(2)}(k,\alpha)} \\ &> \frac{\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n} l_i^{(1)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(1)}\eta^{(1)}(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n} l_i^{(1)}P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i^{(1)}\eta^{(1)}(k,\alpha)} \\ &= l_k^{(2)}. \end{aligned} \quad (15)$$

If for all k, $l_k^{(m+1)} > l_k^{(m)}$, by Lemma 3, it follows from Eq. (11) that $\forall k,\ l_k^{(m+2)} > l_k^{(m+1)}$.

By induction, for each k, the sequence $l_k^{(m)}$, $m \ge 2$, is increasing, so its limit exists and is denoted by $l_k = \lim_{m\to\infty} l_k^{(m)}$. Letting $m \to \infty$ on both sides of Eqs. (12) and (13), we deduce that the limit l_k, $k = 1,2,\ldots,n$, satisfies the following relations:

$$\begin{aligned} &-\nu l_k + \lambda\psi(k)\left(1-\beta\sum_{i=1}^{n} l_i P(i)\right)(1-l_k)\langle k\rangle^{-1} \\ &\times \sum_{i=1}^{n} iP(i)l_i\eta(k,\alpha) = 0, \\ &l_k \le \liminf_{t\to+\infty} \rho_k(t),\ \ 1 \le k \le n, \end{aligned} \quad (16)$$

where $\eta(k,\alpha) = k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)l_i$.

According to Eqs. (8) and (16), both $\mu_k = \lim_{m\to\infty} \mu_k^{(m)}$ and $l_k = \lim_{m\to\infty} l_k^{(m)}$ are positive stationary points of the rate equation; thus by the uniqueness of the positive stationary point of the rate equation, we have $\mu_k = l_k = \rho_k$ and

$$\rho_k \le \liminf_{t\to+\infty} \rho_k(t) \le \limsup_{t\to+\infty} \rho_k(t) \le \rho_k,\ 1 \le k \le n.$$

That is, $\lim_{t\to+\infty} \rho_k(t) = \rho_k$.

Theorem 1 indicates that if $\lambda/\nu > \lambda_c$ and the network has infected nodes initially, then the infected process becomes stable and $\rho_k(t)$ approach globally the unique positive stationary point of rate equation (1). Therefore, from

$$\lambda_c = \frac{\langle k\rangle}{\langle k^2\psi(k)\rangle - \alpha\langle k\psi(k)\rangle}$$

in this Theorem we can see that both the contact awareness and the local awareness can enhance the epidemic threshold, which is indeed a global result.

4. Simulations

In this section, we will perform some simulations to verify the global result in Theorem 1. Without loss of generality, we use WS small-world network[25] and BA network[13] as the topological structures of model (1). The network size N is set as 200. The WS network is generated with probability $p = 0.1$ for rewiring links, where each node is symmetrically connected with its $2m$ nearest neighbors in its initial nearest-neighbor network. The BA network is produced with initial nodes m_0 and adding a new node with m edges in each time step.

Based on model (1), figure 1 gives some realizations on a WS small-world network where $m = 3$ and the minimum degree, maximum degree and average degree are 3, 8, and 6, respectively. The two groups of parameter setting ($\alpha = 0.15$, $\beta = 0.1$, $\psi(k) = k^{-0.2}$ and $\alpha = 0.08$, $\beta = 0.12$, $\psi(k) = k^{-0.4}$) in Fig. 1 all satisfy condition (3). From Theorem 1, we get the theoretical epidemic thresholds $\lambda_c = 0.35$, 0.23 with respect to two groups of parameter settings. In order to verify the global stability, two kinds of initial conditions are chosen as $\rho_3(0) = 0.01$, $\rho_k(0) = 0$, $k = 4,5,\ldots,8$ and $\rho_3(0) = 0.1$, $\rho_k(0) = 0$, $k = 4,5,\ldots,8$, respectively. In Fig. 1(a), without loss of generality, we set $\lambda = 0.01$ and $\nu = 0.15$ to satisfy $\lambda/\nu < \lambda_c$. As the growth of time t, corresponding to the two kinds of initial conditions and awareness strengths, all the total infected densities $\rho(t)$ tend to zero, i.e., the disease will die out. In Fig. 1(b), we set $\lambda = 0.1$ and $\nu = 0.15$ to meet $\lambda/\nu > \lambda_c$. In this case, the infection spreads and becomes endemic. By performing plentiful simulations with other different parameter settings and initial conditions, we can get the same convergence property.

Next we test this model on the BA network. Based on model (1), figure 2 gives some realizations where the minimum degree, maximum degree, and average degree of the BA network are 3, 40, and 5.94, respectively. The two groups of parameter settings in Fig. 2 all satisfy condition (3). From Theorem 1, we get the theoretical epidemic thresholds $\lambda_c = 0.26$, 0.15 with respect to the two groups of parameter settings. When $\lambda/\nu = 0.01/0.15 < \lambda_c$, from the Fig. 2(a), we can see that the total infected densities $\rho(t)$ tend to zero. When $\lambda/\nu = 0.1/0.15 > \lambda_c$, in Fig. 2(b), the total infected densities $\rho(t)$ converge to the endemic equilibrium. In conclusion, these simulations verify the correctness of Theorem 1 very well.

In order to further verify the theoretical epidemic threshold in Theorem 1, figures 3 and 4 show the relationship between λ/ν and stable density ρ on some different WS networks and BA networks, respectively. One can see that the numerical epidemic threshold is consistent with theoretical results very well.

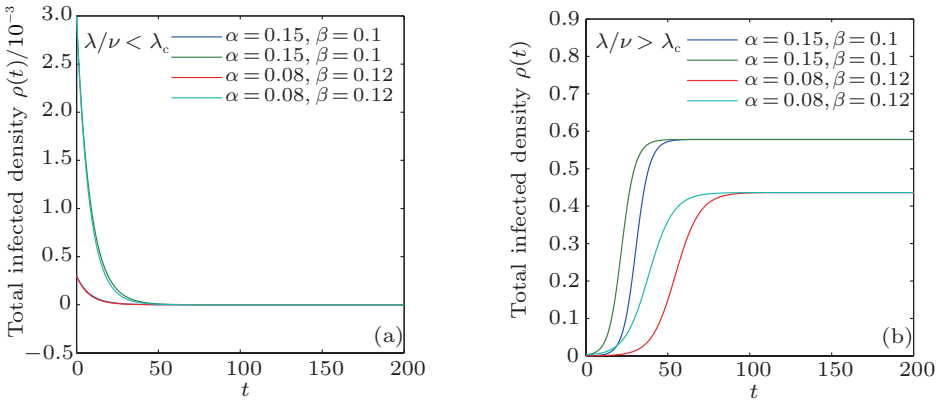


Fig. 1. (color online) The total infected density $\rho(t)$ on a WS small-world network. In each panel, the blue and red curves correspond to the initial condition $\rho_3(0) = 0.01$, $\rho_k(0) = 0$, $k = 4,5,\ldots,8$, the other two curves correspond to $\rho_3(0) = 0.1$, $\rho_k(0) = 0$, $k = 4,5,\ldots,8$, respectively. (a) $\lambda/\nu < \lambda_c$, (b) $\lambda/\nu > \lambda_c$.

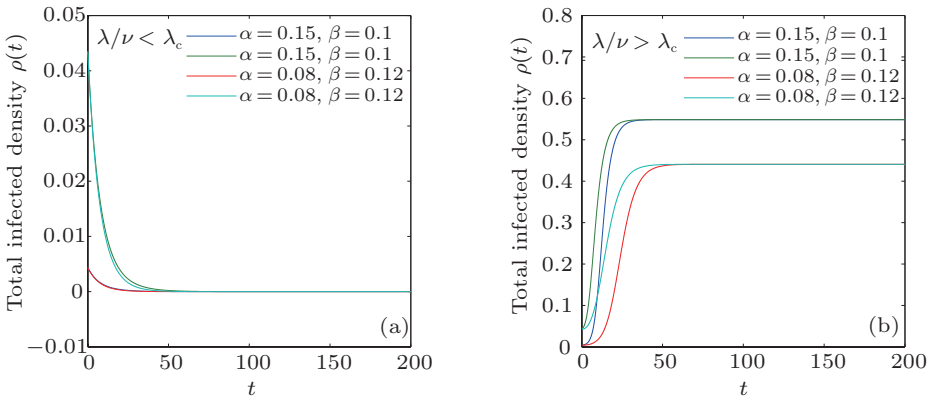


Fig. 2. (color online) The total infected density $\rho(t)$ on a BA network. In each panel, the blue and red curves correspond to the initial condition $\rho_3(0) = 0.01$, $\rho_k(0) = 0$, $k = 3,4,\ldots,40$ and the other two curves correspond to $\rho_3(0) = 0.1$, $\rho_k(0) = 0$, $k = 3,4,\ldots,40$, respectively. (a) $\lambda/\nu < \lambda_c$, (b) $\lambda/\nu > \lambda_c$.

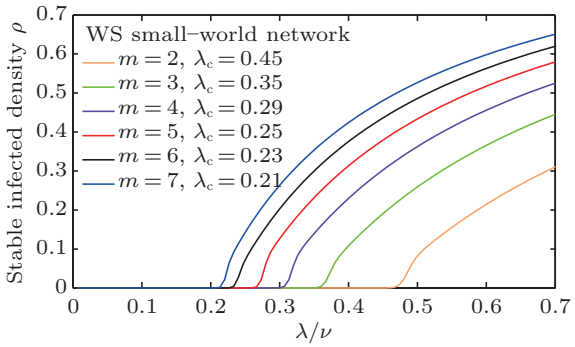


Fig. 3. (color online) The stable infected density ρ with respect to effective spreading rate λ/ν. The topological structure of model (1) is WS small-world network with different structure parameters m, λ_c denotes the theoretical epidemic threshold. This figure is obtained by 20 independent realizations.

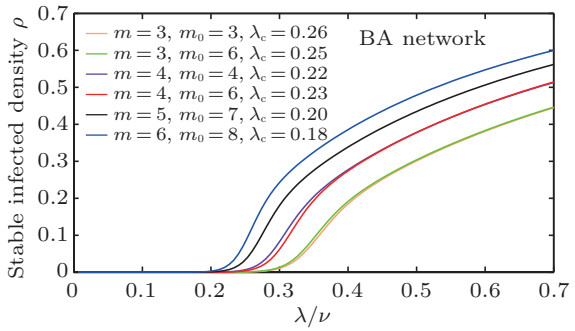


Fig. 4. (color online) The stable infected density ρ with respect to effective spreading rate λ/ν. The topological structure of model (1) is the BA network with different structure parameters m and m_0, λ_c denotes the theoretical epidemic threshold. This figure is obtained by 20 independent realizations.

5. Conclusion

The interplay between individual awareness and epidemic dynamics in networks has recently received much attention. The individual awareness can have an important influence on epidemic spreading on networks. By the local stability analysis method, a significant conclusion is that the embedded awareness in an epidemic network can increase its epidemic threshold. In particular, both the contact awareness and the local awareness can raise the epidemic threshold, while the global awareness can only decrease the epidemic prevalence.

In this paper, we give global stability analysis of an SIS epidemic model on networks with individual awareness. Because of the powerful nonlinear terms, the considered model is more complicated than the general SV rate equation (i.e., the well-known SIS epidemic model established by Pastor-Satorras and Vespignani). Based on limit theory and dynamic system theory, we obtain a sufficient condition of global stability of the endemic equilibrium. As we know, to seize the behavior of epidemics on complex networks, global analysis of their equilibrium points must be made. Hence, this work will provide certain theoretical support for real applications in epidemic control on complex networks.

From the simulation part, we can see that the strength of awareness and network structure play an important role in stable infected density of the spreading process. Hence, it is possible and valuable to further discuss theoretically the relationship between them in the future.

Appendix A

Now we give the proofs of Lemma 3 and Lemma 4 in Section 3.

Proof of Lemma 3 On one hand, expanding function $g(x_1,x_2,\ldots,x_n)$, we obtain

$$\begin{aligned} & g(x_1,x_2,\ldots,x_n) \\ =\ & (k-\alpha)\sum_{i=1}^{n} iP(i)x_i + (\alpha-\alpha k)\langle k\rangle^{-1}\left(\sum_{i=1}^{n} iP(i)x_i\right)^2 \\ & -\beta(k-\alpha)\sum_{i=1}^{n} x_iP(i)\sum_{i=1}^{n} iP(i)x_i - \beta(\alpha-\alpha k) \\ & \times\langle k\rangle^{-1}\sum_{i=1}^{n} x_iP(i)\left(\sum_{i=1}^{n} iP(i)x_i\right)^2. \end{aligned} \tag{A1}$$

For each $j\in\{1,2,\ldots,n\}$, computing the partial derivative of g with respect to x_j gives

$$\begin{aligned} \frac{\partial g}{\partial x_j} =\ & (k-\alpha)jP(j) + 2(\alpha-\alpha k)\langle k\rangle^{-1}jP(j)\sum_{i=1}^{n} iP(i)x_i \\ & -\beta(k-\alpha)P(j)\sum_{i=1}^{n} iP(i)x_i - \beta(k-\alpha)jP(j) \\ & \times\sum_{i=1}^{n} x_iP(i) - \beta(\alpha-\alpha k)\langle k\rangle^{-1}P(j)\left(\sum_{i=1}^{n} iP(i)x_i\right)^2 \\ & -2\beta(\alpha-\alpha k)\langle k\rangle^{-1}jP(j) \\ & \times\sum_{i=1}^{n} x_iP(i)\sum_{i=1}^{n} iP(i)x_i \\ \geq\ & (k-\alpha)jP(j) + 2(\alpha-\alpha k)jP(j) - \beta(k-\alpha)jP(j)\langle k\rangle \\ & -\beta(k-\alpha)jP(j) \\ =\ & (2(\alpha-\alpha k) + (k-\alpha)[1-\beta\langle k\rangle-\beta])jP(j). \end{aligned}$$

If $2(\alpha-\alpha k)+(k-\alpha)[1-\beta\langle k\rangle-\beta]\geq 0$, then we obtain $\partial g/\partial x_j\geq 0$, which means function $g(x_1,x_2,\ldots,x_n)$ is monotonically increasing with respect to all variables. Furthermore, let $h(k)=2(\alpha-\alpha k)+(k-\alpha)[1-\beta\langle k\rangle-\beta]$, $k\in\{1,2,\ldots,n\}$. If

$$0<\alpha\leq\frac{1-\beta(1+\langle k\rangle)}{2},\quad 0<\beta\leq\frac{1}{1+\langle k\rangle},$$

we obtain $h(1)=(1-\alpha)[1-\beta\langle k\rangle-\beta]\geq 0$, $h'(k)=1-2\alpha-\beta-\beta\langle k\rangle\geq 0$, which leads to $h(k)\geq 0$ for all k. So the inequality (5) is valid.

On the other hand, calculating directly the following difference:

$$\begin{aligned} & g(y_1,y_2,\ldots,y_n)-g(x_1,x_2,\ldots,x_n) \\ =\ & (k-\alpha)\sum_{i=1}^{n} iP(i)(y_i-x_i) + (\alpha-\alpha k)\langle k\rangle^{-1}\left[\left(\sum_{i=1}^{n} iP(i)y_i\right)^2\right. \end{aligned}$$

$$- \left(\sum_{i=1}^{n} iP(i)x_i\right)^2\Bigg] - \beta(k-\alpha)\Bigg[\sum_{i=1}^{n} P(i)y_i \sum_{i=1}^{n} iP(i)y_i - \sum_{i=1}^{n} P(i)x_i \sum_{i=1}^{n} iP(i)x_i\Bigg] + \beta(\alpha k-\alpha)\langle k\rangle^{-1} \times \Bigg[\sum_{i=1}^{n} P(i)y_i\left(\sum_{i=1}^{n} iP(i)y_i\right)^2 - \sum_{i=1}^{n} P(i)x_i\left(\sum_{i=1}^{n} iP(i)x_i\right)^2\Bigg]. \tag{A2}$$

Define three vectors as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad e = \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \\ \vdots \\ y_n - x_n \end{bmatrix}.$$

Then, the first term of the right-hand side of Eq. (A2) can be reduced to

$$\begin{aligned}(k-\alpha)\sum_{i=1}^{n} iP(i)(y_i - x_i) &\geq (k-\alpha)\sum_{i=1}^{n} iP(i)(y_i - x_i)^2 \\ &= e^{\mathrm{T}}(k-\alpha)Ae.\end{aligned}$$

The second term of the right-hand side of Eq. (A2) can be transformed as

$$\begin{aligned}&(\alpha-\alpha k)\langle k\rangle^{-1}\left[\left(\sum_{i=1}^{n} iP(i)y_i\right)^2 - \left(\sum_{i=1}^{n} iP(i)x_i\right)^2\right] \\ &= (\alpha-\alpha k)\langle k\rangle^{-1}\left[y^{\mathrm{T}}\omega\omega^{\mathrm{T}}y - x^{\mathrm{T}}\omega\omega^{\mathrm{T}}x\right] \\ &= e^{\mathrm{T}}(\alpha-\alpha k)\langle k\rangle^{-1}\omega\omega^{\mathrm{T}}e.\end{aligned}$$

The third term of the right-hand side of Eq. (A2) can be rewritten as

$$\begin{aligned}&-\beta(k-\alpha)\left[\sum_{i=1}^{n} P(i)y_i \sum_{i=1}^{n} iP(i)y_i - \sum_{i=1}^{n} P(i)x_i \sum_{i=1}^{n} iP(i)x_i\right] \\ &= -\beta(k-\alpha)\left[y^{\mathrm{T}}\theta\omega^{\mathrm{T}}y - x^{\mathrm{T}}\theta\omega^{\mathrm{T}}x\right] \\ &= e^{\mathrm{T}}\beta(\alpha-k)\theta\omega^{\mathrm{T}}e.\end{aligned}$$

At the same time, we note that the fourth term of the right-hand side of Eq. (A2) is non-negative. Integrating the above analysis concludes that

$$\begin{aligned}&g(y_1, y_2, \ldots, y_n) - g(x_1, x_2, \ldots, x_n) \\ &\geq e^{\mathrm{T}}\left[(k-\alpha)A + (\alpha-\alpha k)\langle k\rangle^{-1}\omega\omega^{\mathrm{T}} + \beta(\alpha-k)\theta\omega^{\mathrm{T}}\right]e \\ &= e^{\mathrm{T}}\left[(k-\alpha)A + (\alpha-\alpha k)\langle k\rangle^{-1}\omega\omega^{\mathrm{T}} + \frac{\beta(\alpha-k)(\theta\omega^{\mathrm{T}} + \omega\theta^{\mathrm{T}})}{2}\right]e.\end{aligned} \tag{A3}$$

Therefore, if the matrix

$$(k-\alpha)A + (\alpha-\alpha k)\langle k\rangle^{-1}\omega\omega^{\mathrm{T}} + \frac{\beta(\alpha-k)(\theta\omega^{\mathrm{T}} + \omega\theta^{\mathrm{T}})}{2} \tag{A4}$$

is semi-positive, then we can obtain inequality (5).

Let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ be the minimum and maximum eigenvalues of a matrix, respectively. Defining the matrix (A4) as B, it is semi-positive, if and only if $\lambda_{\min}(B) \geq 0$.

From Eq. (A4), we have

$$\begin{aligned}\lambda_{\min}(B) &\geq (k-\alpha)\lambda_{\min}(A) + (\alpha-\alpha k)\langle k\rangle^{-1}\lambda_{\max}(\omega\omega^{\mathrm{T}}) \\ &\quad + \frac{\beta(\alpha-k)}{2}(\lambda_{\max}(\theta\omega^{\mathrm{T}}) + \lambda_{\max}(\omega\theta^{\mathrm{T}})) \\ &= (k-\alpha)\min_{1\leq i\leq n}\{ip(i)\} + [(\alpha-\alpha k)\langle k\rangle^{-1}n \\ &\quad + \beta(\alpha-k)]\sum_{i=1}^{n} nP^2(n),\end{aligned} \tag{A5}$$

which means that if $(k-\alpha)\min_{1\leq i\leq n}\{ip(i)\} + [(\alpha-\alpha k)\langle k\rangle^{-1}n + \beta(\alpha-k)]\sum_{i=1}^{n} nP^2(n) \geq 0$, we get $\lambda_{\min}(B) \geq 0$, then we can obtain inequality (5).

Proof of Lemma 4 Since $\limsup_{t\to+\infty}\rho_k \leq \mu_k$, for all $\varepsilon > 0$, there exists $\tau > 0$ as $t \geq \tau$, $\rho_k(t) \leq \mu_k + \varepsilon$. Lemma 1 shows that $1 - \rho_k(t) > 0$. Hence, according to Eq. (1) we know that, as $t \geq \tau$, the following inequality holds:

$$\begin{aligned}\frac{\mathrm{d}\rho_k(t)}{\mathrm{d}t} &= -\nu\rho_k + \lambda\psi(k)\big(1-\beta\rho(t)\big)\Theta(t) \\ &\quad \times [k-\alpha+(\alpha-\alpha k)\Theta(t)](1-\rho_k) \\ &\leq -\nu\rho_k + \lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}(\mu_i+\varepsilon)P(i)\right)\langle k\rangle^{-1} \\ &\quad \times \sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon)\left(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\right. \\ &\quad \times \left.\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon)\right)(1-\rho_k) \\ &= -\left[\nu + \lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}(\mu_i+\varepsilon)P(i)\right)\right. \\ &\quad \times \langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon) \\ &\quad \times \left.\left(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon)\right)\right]\rho_k \\ &\quad + \lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}(\mu_i+\varepsilon)P(i)\right) \\ &\quad \times \langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon) \\ &\quad \times \left(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon)\right).\end{aligned} \tag{A6}$$

Let $\Delta = \lambda\psi(k)(1-\beta\sum_{i=1}^{n}(\mu_i+\varepsilon)P(i))\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon)(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon))$. Multiplying both sides of the above inequality by $\exp\{[\nu+\lambda\psi(k)(1-\beta\sum_{i=1}^{n}(\mu_i+\varepsilon)P(i))\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon)(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n} iP(i)(\mu_i+\varepsilon))]t\}$, and integrating from $\tau \to t$, we obtain

$$\rho_k(t) \leq \frac{\rho_k(\tau)}{\exp(\nu+\Delta)(t-\tau)} + \frac{\Delta}{\nu+\Delta}. \tag{A7}$$

Taking the limit as $t \to +\infty$, we obtain

$$\limsup_{t\to+\infty}\rho_k(t) \leq \frac{\Delta}{\nu+\Delta}. \tag{A8}$$

Taking the limit as $\varepsilon \to +\infty$, we obtain

$$\limsup_{t\to+\infty}\rho_k(t)\le\frac{\lambda\psi(k)\left(1-\beta\sum\limits_{i=1}^{n}\mu_iP(i)\right)\langle k\rangle^{-1}\sum\limits_{i=1}^{n}iP(i)\mu_i\xi(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum\limits_{i=1}^{n}\mu_iP(i)\right)\langle k\rangle^{-1}\sum\limits_{i=1}^{n}iP(i)\mu_i\xi(k,\alpha)}. \tag{A9}$$

Since $\liminf_{t\to+\infty}\rho_k\ge l_k$, for all $\varepsilon>0$, there exists $\tau>0$ such that $\rho_k(t)\ge l_k-\varepsilon$ as $t\ge\tau$. From Eq. (1), we know that, as $t\ge\tau$, the following inequality holds:

$$\begin{aligned}\frac{\mathrm{d}\rho_k(t)}{\mathrm{d}t}&=-\nu\rho_k+\lambda\psi(k)\big(1-\beta\rho(t)\big)\Theta(t)\big[k-\alpha+(\alpha-\alpha k)\Theta(t)\big](1-\rho_k)\\&\ge-\nu\rho_k+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}(l_i-\varepsilon)P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon)\left(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon)\right)(1-\rho_k)\\&=-\left[\nu+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}(l_i-\varepsilon)P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon)\left(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon)\right)\right]\rho_k\\&\quad+\lambda\psi(k)\left(1-\beta\sum_{i=1}^{n}(l_i-\varepsilon)P(i)\right)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon)\left(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon)\right).\end{aligned} \tag{A10}$$

Let $\nabla=\lambda\psi(k)(1-\beta\sum_{i=1}^{n}(l_i-\varepsilon)P(i))\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon)(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon))$. Multiplying both sides of the above inequality by $\exp\{[\nu+\lambda\psi(k)(1-\beta\sum_{i=1}^{n}(l_i-\varepsilon)P(i))\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon)(k-\alpha+(\alpha-\alpha k)\langle k\rangle^{-1}\sum_{i=1}^{n}iP(i)(l_i-\varepsilon))]t\}$, integrating from $\tau\to t$, and taking the limit as $t\to+\infty$, we obtain

$$\liminf_{t\to+\infty}\rho_k(t)\ge\frac{\nabla}{\nu+\nabla}. \tag{A11}$$

Taking the limit as $\varepsilon\to+\infty$, we obtain

$$\liminf_{t\to+\infty}\rho_k(t)\ge\frac{\lambda\psi(k)\left(1-\beta\sum\limits_{i=1}^{n}l_iP(i)\right)\langle k\rangle^{-1}\sum\limits_{i=1}^{n}iP(i)l_i\eta(k,\alpha)}{\nu+\lambda\psi(k)\left(1-\beta\sum\limits_{i=1}^{n}l_iP(i)\right)\langle k\rangle^{-1}\sum\limits_{i=1}^{n}iP(i)l_i\eta(k,\alpha)}.$$

The end of the proof.

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