Anti-plane problem analysis for icosahedral quasicrystals under shear loadings*

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(Received 11 January 2014; revised manuscript received 28 May 2014; published online 24 September 2014)

The present paper is concerned with the longitudinal shear elasticity of three-dimensional icosahedral quasicrystals. By virtue of the Dugdale hypothesis along with the method of complex potential theory, it involves two defect problems of the icosahedral quasicrystals. The first one is the calculation of stress intensity factors and the size of the cohesive force zone in a half-infinite crack. Meanwhile, the crack tip tearing displacements can be exactly derived. The other is the demonstration of the generalized stress intensity factors induced by a sharp V-notch as an extension of a crack. The generalized E-integral around the notch tip gives the energy release rate when the V-notch degenerates into a crack. Apart from their own usefulness in carrying out some simplified crack analyses, the results obtained in this work can particularly serve as a basis for fracture mechanics of anti-plane defect problems of icosahedral quasicrystals.

Keywords: quasicrystal, stress intensity factor, crack tip tearing displacement, energy release rate

PACS: 62.20.D–, 61.44.Br, 62.20.M– **DOI:** 10.1088/1674-1056/23/11/116201

1. Introduction

Quasicrystals are a form of solids different from both crystals and glassy solids, which possess quasi-periodic long-range translational symmetry and non-crystallographic rotational symmetry. They were discovered in 1984 by Shechtman, who was awarded the Nobel prize in chemistry in 2011.[1] In 2011, some scholars found 12-fold and 18-fold diffraction symmetric quasicrystals in colloids, which aroused great interest.[2] The discovery of this novel matter with long-range order and non-crystallographic symmetry changes the traditional concept of classifying solids into two classes, crystal and non-crystal. It has a strong impact on traditional crystallography, and brings new physical ideas into structure and symmetry of matter. The unusual structure of quasicrystals leads to a series of properties, which has attracted a great deal of attention from researchers in a range of fields, such as physics, mechanics, chemistry, etc. The phenomenological theory of Landau and Lifshitz on the elementary excitation of condensed matter was essentially taken as the physical basis of elasticity of quasicrystals, and two types of excitation, phonons and phasons, were considered for quasi-periodicity of materials.[3–7] The phonon field is similar to that in crystals, and its gradient describes changes of a cell in volume and shape. The gradient of the phason field characterizes the local rearrangement of atoms in a cell. From general phenomenological analysis, the classical linear elasticity theory of conventional periodic crystals has been extended to a generalized linear elasticity theory of quasicrystals.[8] It is noted that in quasicrystals the number of independent material constants involved in the linear constitutive equations can be derived by using the group representation theory, and is greatly increased as compared with that of periodic crystals.[9]

Several researchers in continuum physics attempted to establish a mathematical modeling of the elasticity of quasicrystals under mechanical effects soon after their discovery (e.g. Bak,[10] Levine *et al.*,[11] Lubensky *et al.*,[12] Socolar *et al.*,[13] Chatzopoulos and Trebin[14]). Also, Yang *et al.*[15] presented the linear elasticity theory of three-dimensional cubic quasicrystals. A generalized theory applicable to the elasticity of all quasicrystals was established by Ding *et al.*[8] on the basis of the laws of motion in continuum physics. In addition, Wang and his co-workers have studied quasicrystals by experimental observation and pointed out that long-period structure in magnesium alloys as well as precipitation of the quasicrystalline phase is a very important area of this research.[16–18] Li *et al.*,[19] Gao,[20] Wang *et al.*,[21] Gong *et al.*,[22] and Li[23] have used cracked quasicrystals with a variety of methods. Li *et al.*[24] have also studied the physical property tensors of one-dimensional quasicrystals. Various static and dynamic problems were solved for the elasticity of one-, two-, and three-dimensional quasicrystals by Fan *et al.*[25,26] In addition, a recent comprehensive book by Fan for the development of quasicrystals, including material properties, theories of elasticity, and some applications, has been published.[26] However, for the elastic problem of icosahedral and cubic quasicrystals, the analytical and numerical results are not fruitful.

In linear elasticity, the stresses at the tip of a crack in a body are singular and the stresses in the neighborhood of the

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11402158 and 11272053) and the Qualified Personnel Foundation of Taiyuan University of Technology of China (Grant No. tyut-rc201358a).

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crack tip may exceed the yield stress of the material constituting the body, whereas inelastic deformations are not permissible anywhere. To avoid this paradox, Dugdale[27] suggested a crack model which allows the occurrence of yielding and gives an estimate for the spread of plasticity in terms of external applied loading. Since then, Dugdale's model has been widely applied and developed to deal with a variety of crack or crack-like problems (see, e.g., Refs. [28]–[32]). By applying the Bilby–Cottrell–Swinden–Schmith (BCS) model, Fan *et al.* first probed the study of the plasticity of quasicrystals.[25] Using the complex function method and dislocation theory, they obtained the solutions for the plasticity of quasicrystals.[33,34] On the basis of the method proposed by Fan *et al.*, the work reported in this paper is on a half-infinite crack in a three-dimensional icosahedral or cubic quasicrystal under longitudinal shear, aiming to analytically determine the stress intensity factors and the extent of the plastic zone size on the basis of Dugdale's crack model and to explicitly calculate the corresponding crack tip tearing displacement in some cases. More precisely, two cases are investigated. First, a pair of longitudinal shear loadings is exerted on the upper and lower surfaces of a half-infinite crack in an infinite strip. Second, the quasicrystal strip is of finite height. In this section, the conformal mapping adopted originates from Ref. [26], and was first employed by Fan *et al.* for the study of quasicrystals. Last, we present an asymptotic solution of a sharp V-notch in longitudinal shear problem for three-dimensional icosahedral or cubic quasicrystals. Similar to Irwin's and Fan's theory, we make an effort to obtain the stress intensity factors and generalized E-integral. Under some loading conditions, an analytical approach has been used to establish the corresponding equations for the longitudinal shear problem, and the relationship of the stress field and the stress intensity factors near V-notch tip is also established. The generalized E-integral around the V-notch tip in the longitudinal shear problem gives the energy release rate in the case where the V-notch degenerates into a crack.

2. Basic equations and complex potential theory

In this section, some preliminary results given by Fan using the complex potential theory will be recalled for a crack in a quasicrystalline medium.[26]

A three-dimensional quasicrystal refers to a three-dimensional solid structure with quasi-periodic arrangement in all three directions, where phonon and phason fields exist simultaneously. According to the cut- and -projection method, quasicrystals are understood as a three-dimensional projection of a higher-dimensional periodic space, with a huge number of field variables and field equations involving elasticity. In addition to the ordinary phonon field, there is another so-called phason field describing the rearrangements of local atomic configurations. The displacement vectors are denoted as $\boldsymbol{u} = (u_1, u_2, u_3)$ for the phonon field and $\boldsymbol{w} = (w_1, w_2, w_3)$ for the phason field, which are both dependent on the spatial point $\boldsymbol{x} = (x_1, x_2, x_3)$ in the real space. According to the above description and Refs. [8] and [9], the equations of deformation geometry can be expressed as

$$\varepsilon_{ij} = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j), \quad w_{ij} = \partial_j w_i, \tag{1}$$

where an abbreviated notation $\partial_j f = \partial f/\partial x_j$ is used, with f being a function in x_j, and ε_{ij} and w_{ij} denote the phonon and phason strains, respectively.

In the case of small deformation of quasicrystals, the phonon and phason stresses, i.e. σ_{ij} and H_{ij}, obey a generalized Hooke's law. That is, they are linearly dependent upon both the phonon strains and phason strains; namely, they can be given through the following generalized Hooke's law:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl}, \quad H_{ij} = R_{klij}\varepsilon_{kl} + K_{ijkl}w_{kl}, \tag{2}$$

where C_{ijkl} and K_{ijkl} represent the elastic constant tensors for phonons and phasons, respectively, and R_{ijkl} the phonon–phason coupling elastic constant tensor. Due to the presence of K_{ijkl} and R_{ijkl}, the number of elastic constants appearing in the above constitutive equation (2) is much greater than that for conventional crystals. This leads to a mathematical complexity in solving associated boundary value problems or initial boundary value problems.

Due to the huge number of variables and field equations, the solution is highly complicated. One way to solve the elasticity problem is to reduce the number of field variables and equations mentioned above. For this purpose we can utilize the eliminating element method in classical mathematical physics. We can find that the equations can be simplified for some physically meaningful cases. One of them is the so-called anti-plane case where the non-zero displacements are only u_z and w_z in the Cartesian coordinate system, and the other components vanish. In particular, these two displacements and relevant strains and stresses are independent of the direction x_3 (or z). For example, there is a crack along the direction x_3 or a straight dislocation line along the direction, etc; in addition, the applied field is independent of x_3, so we have $\partial/\partial x_3 (= \partial/\partial z) = 0$. The corresponding strains are only

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\frac{\partial u_z}{\partial y}, \quad \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\frac{\partial u_z}{\partial x},$$
$$w_{zy} = \frac{\partial w_z}{\partial y}, \quad w_{zx} = \frac{\partial w_z}{\partial x}. \tag{3}$$

In this anti-plane case, first we consider icosahedral quasicrystals, which have the constitutive relations[26]

$$\begin{cases} \sigma_{zx} = \sigma_{xz} = 2\mu\varepsilon_{zx} + Rw_{zx}, \\ \sigma_{zy} = \sigma_{yz} = 2\mu\varepsilon_{zy} + Rw_{zy}, \\ H_{zx} = (K_1 - K_2)w_{zx} + 2R\varepsilon_{zx}, \\ H_{zy} = (K_1 - K_2)w_{zy} + 2R\varepsilon_{zy}, \end{cases} \tag{4}$$

where we only determine four stress components here, and others can also be determined by the same method below.

In the absence of any body forces, the equilibrium equations are

$$\frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}=0, \quad \frac{\partial H_{zx}}{\partial x}+\frac{\partial H_{zy}}{\partial y}=0. \tag{5}$$

Substituting Eq. (3) into Eq. (4), then into the equilibrium equation (5), yields the final governing equations[26]

$$\nabla^2 u_z=0, \quad \nabla^2 w_z=0, \tag{6}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. Equation (6) indicates that the terminal governing equations of the anti-plane elasticity problem for coupled phonon–phason fields are two harmonic equations. To find solutions of the anti-plane shear problem, the complex variable method can be used. In the Cartesian coordinate system, we define the complex variable as

$$t=x+\mathrm{i}y(=x_1+\mathrm{i}x_2). \tag{7}$$

According to the theory of complex variable functions, u_z and w_z can be represented as the real part or imaginary part of two analytic functions. We assume that

$$u_z=\mathrm{Im}\left(\varphi(t)\right), \quad w_z=\mathrm{Im}\left(\phi(t)\right), \tag{8}$$

where the symbol "Im" indicates the imaginary part of a complex number. It is well known that if a function $F(t)$ is analytic then

$$\frac{\partial F(t)}{\partial x}=\frac{\partial F(t)}{\partial t}, \quad \frac{\partial F(t)}{\partial y}=\mathrm{i}\frac{\partial F(t)}{\partial t}. \tag{9}$$

In addition, the Cauchy–Riemann relations of an analytic function are

$$\begin{aligned}&\frac{\partial\left(\mathrm{Re}\,F(t)\right)}{\partial x}=\frac{\partial\left(\mathrm{Im}F(t)\right)}{\partial y},\\ &\frac{\partial\left(\mathrm{Re}\,F(t)\right)}{\partial y}=-\frac{\partial\left(\mathrm{Im}F(t)\right)}{\partial x}.\end{aligned} \tag{10}$$

With the aid of relations (8) and (9), equations (5) and (6) lead to

$$\begin{cases}\sigma_{zx}=\sigma_{xz}=\mu\dfrac{\partial}{\partial x}[\mathrm{Im}\,\varphi(t)]+R\dfrac{\partial}{\partial x}[\mathrm{Im}\,\phi(t)],\\ \sigma_{zy}=\sigma_{yz}=\mu\dfrac{\partial}{\partial y}[\mathrm{Im}\,\varphi(t)]+R\dfrac{\partial}{\partial y}[\mathrm{Im}\,\phi(t)],\\ H_{zx}=(K_1-K_2)\dfrac{\partial}{\partial x}[\mathrm{Im}\,\phi(t)]+R\dfrac{\partial}{\partial x}[\mathrm{Im}\,\varphi(t)],\\ H_{zy}=(K_1-K_2)\dfrac{\partial}{\partial y}[\mathrm{Im}\,\phi(t)]+R\dfrac{\partial}{\partial y}[\mathrm{Im}\,\varphi(t)].\end{cases} \tag{11}$$

In terms of the Cauchy–Riemann relations (10), equation (11) can be written as

$$\begin{cases}\sigma_{zy}+\mathrm{i}\sigma_{zx}=\mu\phi'(t)+R\varphi'(t),\\ H_{zy}+\mathrm{i}H_{zx}=R\varphi'(t)+(K_1-K_2)\phi'(t),\end{cases} \tag{12}$$

where $\phi'(t)=\mathrm{d}\phi/\mathrm{d}t$, and $\varphi'(t)=\mathrm{d}\varphi/\mathrm{d}t$. According to Eq. (12), we have

$$\begin{cases}\sigma_{zy}=\dfrac{1}{2}\left[\mu(\phi'(t)+\overline{\phi'(t)})+R(\varphi'(t)+\overline{\varphi'(t)})\right],\\ H_{zy}=\dfrac{1}{2}\left[R(\varphi'(t)+\overline{\varphi'(t)})+(K_1-K_2)(\phi'(t)+\overline{\phi'(t)})\right].\end{cases} \tag{13}$$

3. Statement of the Dugdale problem for anti-plane shear and solutions

Figure 1 shows a schematic of a mode III crack containing a cohesive zone ahead of the crack tip. The cohesive zone is defined as a transition region in which the surface traction smoothly changes from zero at the crack tip to a certain magnitude at the cohesive zone tip, where the displacement discontinuity across the zone disappears. The cohesive zone is a mathematical extension of the crack and can be viewed physically as the fracture process zone. A three-dimensional quasicrystal strip has a half-infinite crack embedded at the mid-plane with the crack tip corresponding to the coordinate origin. Two coincident lines, namely $y=0^+$ and $y=0^-$, are used to represent the surfaces of the crack, respectively. The portions $y=0^{\pm}$, $-a<x<0$ of the crack surfaces are assumed to be under the action of uniform longitudinal shearing loadings $\sigma_{zy}=-\tau_0$, $H_{zy}=0$, where a is used to simulate a finite size crack length. Then the cohesive forces are incorporated into the material. Near the crack tip there is a so-called cohesive force zone with a length of R whose value is unknown temporarily. This zone is subjected to shear traction τ_Y (phonon field), in which τ_Y represents the shear yield strength of the material. In essence, the Dugdale model may be regarded as a superposition of two linear elastic crack solutions: the stress field resulting from the uniform longitudinal shear traction τ_0 which tends to tear the crack; and the surface tractions over the yield zones, which are equal in magnitude to the material's yield shear stress τ_Y and act to prevent this tearing action such that the stresses are bounded at the crack tip. Based on the concept of the Dugdale model, the unknown strip yield zone sizes are determined by the condition of vanishing shear stress singularity at the crack tip, namely,

$$K_{\tau_0}+K_{\tau_Y}=0, \tag{14}$$

where K_{τ_0} is the stress intensity factor of the fictitious crack tip under uniform shear stress τ_0, and K_{τ_Y} represents the stress intensity factor due to the shear yield strength σ_Y, which is determined by the experimental results.

Our problem based on the preceding description will be explained by the solutions of two component problems contributing towards the stress singularity at the tips of the cracks. These two problems, termed problems I and II, are appropriately derived as stated.

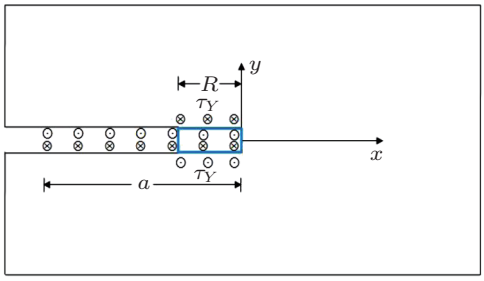


Fig. 1. (color online) A schematic diagram of a mode III crack containing a cohesive zone ahead of the crack tip in a quasicrystal.

3.1. Problem I

The configuration of problem I is an infinite height elastic–perfectly plastic body. We intend in this paragraph to reduce these problems to a few Cauchy integral equations following the works of Muskhelishvili and give its solution. Mathematically, the boundary conditions of the problem may be expressed as

(i) for $\left(x^2+y^2\right)^{1/2}\to\infty$,

$$\sigma_{zx}=\sigma_{zy}=0,\quad H_{zx}=H_{zy}=0; \tag{15}$$

(ii) for $y=0$, $-\infty<x<-a$,

$$\sigma_{zy}=-\tau_0,\quad H_{zy}=0; \tag{16}$$

(iii) for $y=0$, $-a<x<0$,

$$\sigma_{zy}=\tau_Y-\tau_0,\quad H_{zy}=0. \tag{17}$$

As mentioned above, problem I can be seen as a superposition of two sub-problems. We firstly solve the stress intensity factor resulting from the uniform shear traction τ_0 at the crack surfaces. Substituting Eq. (13) into boundary condition (16), we can obtain

$$\begin{cases}\mu(\phi'(t)+\overline{\phi'(t)})+R(\varphi'(t)+\overline{\varphi'(t)})=2\tau_0 f,\\ R(\varphi'(t)+\overline{\varphi'(t)})+(K_1-K_2)(\phi'(t)+\overline{\phi'(t)})=0,\end{cases} \tag{18}$$

where

$$f=\begin{cases}0, & x<-1,\\ 1, & -1<x<0.\end{cases}$$

In the crack problems involving complicated geometries, it is convenient to use conformal transformation where the physical domain is mapped to a region in which the crack is transformed to a unit circle. Here the constructed conformal mapping corresponding to the transformation between the region in the physical plane and the interior of the unit circle γ in the mapping plane reads[35]

$$t=\omega(\zeta)=m\left(1+\frac{2\zeta}{1-\zeta}\right)^2, \tag{19}$$

in which the crack tip $t=0$ is mapped to $\zeta=-1$.

Using the complex integral formula under this mapping (19), equation (18) becomes

$$\begin{cases}\mu\phi'(\zeta)+R\varphi'(\zeta)=2\tau_0\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_\gamma\frac{\omega'(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma=2\tau_0F(\zeta),\\ (K_1-K_2)\phi'(\zeta)+R\varphi'(\zeta)=0,\end{cases} \tag{20}$$

where $\phi(\zeta)$ and $\varphi(\zeta)$ are single-valued analytic functions in $|\zeta|<1$.

After some calculation in Eq. (20), we can obtain

$$\phi'(\zeta)=\frac{2\mathrm{i}[(K_1-K_2)\tau_0]}{\mu(K_1-K_2)-R^2}F(\zeta). \tag{21}$$

The stress intensity factor of longitudinal shear in the phonon field can be written as[36]

$$K_{\mathrm{III}}^{\mathrm{phon}}=\lim_{\zeta\to-1}\tau_0\sqrt{2\pi}\frac{F(\zeta)}{\sqrt{\omega''(\zeta)}}. \tag{22}$$

The function $F(-1)$ can be easily obtained. Substituting it and Eq. (19) into Eq. (22) yields

$$K_{\mathrm{III}}^{\mathrm{phon}}=\tau_0\frac{2\sqrt{2a\pi}}{\pi}. \tag{23}$$

Similarly, the stress intensity factor due to the shear yield strength τ_Y is given by

$$K_{\mathrm{III}}^{\mathrm{phon}}=-\tau_Y\frac{2\sqrt{2R\pi}}{\pi}. \tag{24}$$

The requirement given by Eq. (14) leads to

$$R=a\left(\frac{\tau_0}{\tau_Y}\right)^2. \tag{25}$$

Based on the method and principle of Muskhelishvili, the conformal mapping must be coincident in the calculation of the crack tip tearing displacement. Hence, the crack tip tearing displacement cannot be obtained. However, we can derive its exact solution by implementing the limit for the solution of the next section.

3.2. Problem II

The difference between this problem and problem I is that the body reduces to a finite height plastic strip. Almost all of the formulas are similar to those of the foregoing section. In the light of the treatment in Refs. [26] and [32], the conformal mapping should be modified as

$$t=\omega(\zeta)=\frac{H}{\pi}\ln\left[1+\left(\frac{1+\zeta}{1-\zeta}\right)^2\right], \tag{26}$$

in which the crack tip $t=0$ is mapped to $\zeta=-1$, and the two points $(-a,0^{\pm})$ on the upper and lower lips of the crack are mapped to

$$\begin{cases}\sigma_{-a}=\dfrac{-\mathrm{e}^{-\pi a/H}+2\mathrm{i}\sqrt{1-\mathrm{e}^{-\pi a/H}}}{2-\mathrm{e}^{-\pi a/H}},\\ \overline{\sigma}_{-a}=\dfrac{-\mathrm{e}^{-\pi a/H}-2\mathrm{i}\sqrt{1-\mathrm{e}^{-\pi a/H}}}{2-\mathrm{e}^{-\pi a/H}},\end{cases} \tag{27}$$

where $\sigma = e^{i\theta} = \zeta|_{|\zeta|=1}$ represents the values of the unit circle γ in the ζ plane.

In order to determinate the size of cohesive force zone, we need to seek the stress intensity factor. In terms of the definition, we simply give the results

$$K_{\mathrm{III}}^{\mathrm{phon}} = g\frac{\sqrt{2H}}{\pi}\ln\frac{1+\sqrt{1-e^{-\pi a/H}}}{1-\sqrt{1-e^{-\pi a/H}}}, \tag{28}$$

where $g = \tau_0$ for uniform shear stress and $g = -\tau_Y$ for yield strength.

The requirement given by Eq. (14) leads to

$$R = \frac{\pi}{8}\left(\frac{K_{\mathrm{III}}^{\mathrm{phon}}}{\tau_Y}\right)^2. \tag{29}$$

Ignoring the details of the calculation, and only for small scale yield, the crack tip tearing displacement is

$$\delta_{\mathrm{III}}^t = \frac{(K_1-K_2)(K_{\mathrm{III}}^{\mathrm{phon}})^2}{\tau_Y\left[\mu(K_1-K_2)-R^2\right]}. \tag{30}$$

In the last part of this section, we conclude some new results for the cases of cubic quasicrystals and one-dimensional hexagonal quasicrystals, and one can find that we only need to make a little alteration. The formula and the solution process still hold, and we can obtain the identical stress intensity factor and cohesive force zone, but for cubic quasicrystals the crack tip tearing displacement should be modified as

$$\delta_{\mathrm{III}}^t = \frac{K_{44}(K_{\mathrm{III}}^{\mathrm{phon}})^2}{\tau_Y\left[C_{44}K_{44}-R_{44}^2\right]}, \tag{31}$$

where C_{44} and K_{44} represent the elastic constants for phonons and phasons, respectively, and R_{44} represents the phonon–phason coupling elastic constant. Substituting K_{44} and R_{44} in Eq. (31) by K_2 and R_3, we have the crack tip tearing displacement for one-dimensional hexagonal quasicrystals. The corresponding phonon and phason elastic constants are C_{44} and K_2, and the phonon–phason coupling elastic constant is R_3.

The relationship between problem I and problem II can be implemented when we let $H \to \infty$ or $a/H \to 0$. We can find that equations (28) and (29) turn into Eqs. (23) and (25), in which

$$\ln\frac{1+\sqrt{1-e^{-\pi a/H}}}{1-\sqrt{1-e^{-\pi a/H}}} \to 2\sqrt{\pi}\sqrt{\frac{a}{H}}$$

is used. Meanwhile, we can deduce the crack tip tearing displacement for the case of problem I as follows:

$$\delta_{\mathrm{III}}^t = \frac{8a(K_1-K_2)\tau_0^2}{\pi\tau_{\mathrm{Y}}\left[\mu(K_1-K_2)-R^2\right]}. \tag{32}$$

Although the phason field and phonon–phason coupling have no effect on the stress intensity factor and the plastic zone size, the crack tip tearing displacements are related to the phason field and phonon–phason coupling. Thus, they cannot be ignored in the deformation of quasicrystals.

4. Application of complex potential approach to a sharp V-notch

Stress in the polar coordinate system can be transformed into a fixed rectangular coordinate system such that

$$\begin{cases} \sigma_{z\theta} + i\sigma_{zr} = [\mu\phi'(t) + R\varphi'(t)]e^{i\theta}, \\ H_{z\theta} + iH_{zr} = [R\varphi'(t) + (K_1-K_2)\phi'(t)]e^{i\theta}. \end{cases} \tag{33}$$

Consider a sharp V-notch in longitudinal shear problem for an icosahedral quasicrystal. Let us suppose now that both corner faces with vertex angle 2β for $\beta \in (0, 180^\circ)$ are free from traction and under anti-plane conditions imposed on the body, corresponding to longitudinal shearing of the V-notch. Boundary conditions in this case are

$$\sigma_{z\theta} = 0, \quad H_{z\theta} = 0, \quad \theta = \pm\beta. \tag{34}$$

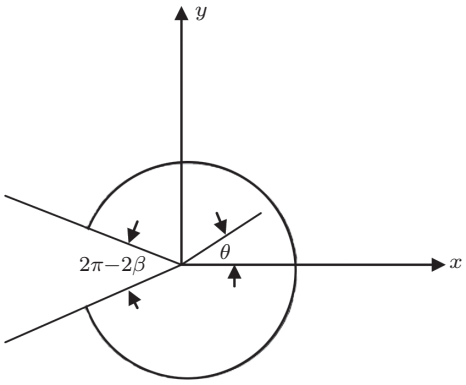


Fig. 2. A mode III sharp V-notch in an icosahedral quasicrystal.

We choose two functions as follows:

$$\begin{bmatrix} \phi \\ \varphi \end{bmatrix}(t) = \begin{bmatrix} M \\ N \end{bmatrix} t^{\lambda}. \tag{35}$$

Substituting Eq. (35) into Eq. (33), we can obtain

$$\begin{bmatrix} \sigma_{z\theta} \\ H_{z\theta} \end{bmatrix} + i\begin{bmatrix} \sigma_{zr} \\ H_{zr} \end{bmatrix} = \lambda z^{\lambda-1}e^{i\theta}\begin{bmatrix} \mu M + RN \\ RM + (K_1-K_2)N \end{bmatrix}. \tag{36}$$

For arbitrary M and N, we take

$$A = \mu M + RN, \quad B = RM + (K_1-K_2)N, \tag{37}$$

where A and B are arbitrary. If we let $z = re^{i\theta}$, the stresses can be taken as a unified representation such that

$$\begin{bmatrix} \sigma_{z\theta} \\ H_{z\theta} \end{bmatrix} + i\begin{bmatrix} \sigma_{zr} \\ H_{zr} \end{bmatrix} = \lambda r^{\lambda-1}e^{i\lambda\theta}\begin{bmatrix} A \\ B \end{bmatrix}. \tag{38}$$

If we take the separation of the real part and the imaginary part from A and B, i.e. suppose $A = A_1 + iA_2$ and $B = B_1 + iB_2$, we can obtain the displacement as

$$u_z = \frac{r^{\lambda}}{\mu(K_1-K_2)-R^2}[(A_1(K_1-K_2)-B_1R)\sin(\lambda\theta)$$

$$+(A_2(K_1-K_2)-B_2R)\cos(\lambda\theta)], \tag{39}$$

$$w_z = \frac{r^{\lambda}}{R^2-\mu(K_1-K_2)}[(B_1\mu-A_1R)\sin(\lambda\theta) + (B_2\mu-A_2R)\cos(\lambda\theta)]. \tag{40}$$

Substituting Eq. (38) into the boundary conditions (34), we can obtain the following linear matrix equations:

$$\begin{bmatrix} 0 & -2\sin(2\lambda\beta) \\ 2\sin(2\lambda\beta) & 0 \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0, \tag{41}$$

where $C_1 = A_1, C_2 = A_2$, or $C_1 = B_1, C_2 = B_2$. Because A and B are arbitrary, the coefficient determinant of the above matrix form equations must be zero, that is,

$$\sin(2\lambda\beta) = 0. \tag{42}$$

The fact that the displacement near the sharp V-notch tip is a finite value leads to

$$\lambda_n = \frac{n\pi}{2\beta} \quad (n = 1,2,\ldots). \tag{43}$$

Substituting Eq. (43) into Eq. (38), the stress relationship can be expressed as a linear combination such that

$$\begin{bmatrix} \sigma_{z\theta}^{(n)} \\ H_{z\theta}^{(n)} \end{bmatrix} + \mathrm{i} \begin{bmatrix} \sigma_{zr}^{(n)} \\ H_{zr}^{(n)} \end{bmatrix} = \frac{n\pi}{2\beta} r^{n\pi/2\beta-1} \mathrm{e}^{\mathrm{i}n\pi\theta/2\beta} \left\{ \begin{bmatrix} A_1^{(n)} \\ B_1^{(n)} \end{bmatrix} + \mathrm{i} \begin{bmatrix} A_2^{(n)} \\ B_2^{(n)} \end{bmatrix} \right\} \quad (n = 1,2,\ldots), \tag{44}$$

where

$$\begin{bmatrix} A_2^{(n)} \\ B_2^{(n)} \end{bmatrix} = \cot\left(\frac{n\pi}{2\beta}\theta\right) \begin{bmatrix} A_1^{(n)} \\ B_1^{(n)} \end{bmatrix} \quad (n = 1,2,\ldots).$$

Similar to Shi,[37] we introduce the stress intensity factors near the V-notch tip as follows:

$$\begin{bmatrix} K_{\mathrm{III}}^{\mathrm{phon}} \\ K_{\mathrm{III}}^{\mathrm{phas}} \end{bmatrix} = \sqrt{2\pi} \lim_{r\to 0} r^{1-\pi/2\beta} \begin{bmatrix} \sigma_{z\theta}^{(1)} \\ H_{z\theta}^{(1)} \end{bmatrix} \quad (\theta = 0), \tag{45}$$

where $K_{\mathrm{III}}^{\mathrm{phon}}$ is the stress intensity factor for the phonon field and $K_{\mathrm{III}}^{\mathrm{phas}}$ that for the phason field. Substituting Eq. (45) into Eq. (44), we can obtain the relationship of the stress field and the stress intensity factors near the V-notch tip

$$\begin{bmatrix} \sigma_{z\theta} \\ \sigma_{zr} \end{bmatrix} = \frac{K_{\mathrm{III}}^{\mathrm{phon}}}{\sqrt{2\pi}} r^{\pi/2\beta-1} \begin{bmatrix} \cos\left(\frac{\pi\theta}{2\beta}\right) \\ \sin\left(\frac{\pi\theta}{2\beta}\right) \end{bmatrix},$$

$$\begin{bmatrix} H_{z\theta} \\ H_{zr} \end{bmatrix} = \frac{K_{\mathrm{III}}^{\mathrm{phas}}}{\sqrt{2\pi}} r^{\pi/2\beta-1} \begin{bmatrix} \cos\left(\frac{\pi\theta}{2\beta}\right) \\ \sin\left(\frac{\pi\theta}{2\beta}\right) \end{bmatrix}, \tag{46}$$

where

$$A_1^{(1)} = \frac{2\beta}{\pi}\frac{K_{\mathrm{III}}^{\mathrm{phon}}}{\sqrt{2\pi}}, \quad A_2^{(1)} = 0, \quad B_1^{(1)} = \frac{2\beta}{\pi}\frac{K_{\mathrm{III}}^{\mathrm{phas}}}{\sqrt{2\pi}}, \quad B_2^{(1)} = 0.$$

Substituting Eqs. (43) and (45) into Eqs. (39) and (40), we can obtain the displacement

$$u_z = \frac{\sqrt{2}\beta}{\pi\sqrt{\pi}} \frac{r^{\pi/2\beta}}{\mu(K_1-K_2)-R^2} \times \left[((K_1-K_2)K_{\mathrm{III}}^{\mathrm{phon}} - RK_{\mathrm{III}}^{\mathrm{phas}})\sin\left(\frac{\pi\theta}{2\beta}\right)\right], \tag{47}$$

$$w_z = \frac{\sqrt{2}\beta}{\pi\sqrt{\pi}} \frac{r^{\pi/2\beta}}{R^2-\mu(K_1-K_2)} \times \left[(RK_{\mathrm{III}}^{\mathrm{phon}} - \mu K_{\mathrm{III}}^{\mathrm{phas}})\sin\left(\frac{\pi\theta}{2\beta}\right)\right], \tag{48}$$

when $\beta \to \pi$ and $R = 0$. It is found that equations (46) and (47) can reduce the solutions of the isotropic medium, which can be seen in Ref. [35].

Similar to Fan,[26] we apply the generalized E-integral (Eshelby integral) for the V-notch tip in quasicrystals as

$$E = \int_{-\beta}^{\beta} (W\delta_{j1} - \sigma_{3j}u_{3,1} - H_{3j}w_{3,1})n_j r\mathrm{d}\theta, \tag{49}$$

where the path of integration is a circle of the V-notch tip, W is the strain energy density, and δ_{j1} is conventionally named the Kronecker delta. The strain energy density can be written as

$$W = \frac{1}{2}\sigma_{31}u_{3,1} + \frac{1}{2}\sigma_{32}u_{3,2} + \frac{1}{2}H_{31}w_{3,1} + \frac{1}{2}H_{32}w_{3,2}. \tag{50}$$

Substituting Eqs. (46)–(50), after proper calculation, one can obtain

$$E = \frac{\beta\sin\beta}{2\pi(\pi-\beta)} \frac{r^{\pi/\beta-1}}{\mu(K_1-K_2)-R^2} [(K_1-K_2)(K_{\mathrm{III}}^{\mathrm{phon}})^2 - 2RK_{\mathrm{III}}^{\mathrm{phon}}K_{\mathrm{III}}^{\mathrm{phas}} + \mu(K_{\mathrm{III}}^{\mathrm{phas}})^2]. \tag{51}$$

The result shows that if $r \to 0$, then $E \to 0$. This corresponds to the result for an isotropic medium; if $\beta \to \pi$, then the generalized E-integral gives the energy release rate when the crack begins to propagate in an icosahedral quasicrystal

$$E = \frac{1}{2}\frac{1}{\mu(K_1-K_2)-R^2} [(K_1-K_2)(K_{\mathrm{III}}^{\mathrm{phon}})^2 - 2RK_{\mathrm{III}}^{\mathrm{phon}}K_{\mathrm{III}}^{\mathrm{phas}} + \mu(K_{\mathrm{III}}^{\mathrm{phas}})^2]. \tag{52}$$

It is obvious that, when the phonon–phason elastic constant $R = 0$, equation (52) reduces to the energy release rate form of the classical elastic materials.[35] As pointed out in Section 3, similarly, the results of cubic and one-dimensional hexagonal quasicrystals can be obtained.

5. Conclusion and discussion

This work employed the conformal mapping technique developed in Ref. [26] for elasticity theory of quasicrystals and provided some successful applications to the fracture theory of quasicrystals. We studied two new problems by using the old method. The stress intensity factors can be expressed in exact analytic forms. Although there exists a phason field in quasicrystals, the results for quasicrystalline materials are consistent. The Dugdale–Barenblatt model is a simple and efficient method to study elastic–plastic cracks. Problems I and II are studied in this paper and our results show that phason field and phonon–phason coupling have no effect on the stress intensity factor or the plastic zone size. The cohesive force zone size has first been obtained, which is useful for plastic deformation of the elastic body. The crack tip tearing displacements can be expressed in exact forms. They are related to the phason field and phonon–phason coupling. Although the phason field and phonon–phason coupling have no effect on the stress intensity factor or the plastic zone size, the crack tip tearing displacements are related to the phason field and phonon–phason coupling. Therefore, they cannot be ignored in the deformation of quasicrystals. In Section 4, a sharp V-notch in longitudinal shear problem for an icosahedral quasicrystal is considered, and the results are given by means of the complex potential method. All results of Sections 3 and 4 can be reduced to the corresponding results for the classical isotropic body when the phason elastic constant R is equal to zero.

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