Noether's theorem for non-conservative Hamilton system based on El-Nabulsi dynamical model extended by periodic laws*

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This paper focuses on the Noether symmetries and the conserved quantities for both holonomic and nonholonomic systems based on a new non-conservative dynamical model introduced by El-Nabulsi. First, the El-Nabulsi dynamical model which is based on a fractional integral extended by periodic laws is introduced, and El-Nabulsi–Hamilton's canonical equations for non-conservative Hamilton system with holonomic or nonholonomic constraints are established. Second, the definitions and criteria of El-Nabulsi–Noether symmetrical transformations and quasi-symmetrical transformations are presented in terms of the invariance of El-Nabulsi–Hamilton action under the infinitesimal transformations of the group. Finally, Noether's theorems for the non-conservative Hamilton system under the El-Nabulsi dynamical system are established, which reveal the relationship between the Noether symmetry and the conserved quantity of the system.

Keywords: Noether's theorem, non-conservative Hamilton system, El-Nabulsi dynamical model, fractional integral extended by periodic laws

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1. Introduction

The principle of symmetry is a rule of higher level physics, and it occupies an important position in mathematics, physics, and mechanics. Studying the symmetry properties of a dynamic system is also a very effective method to solve practical problems. There are many applications for the theory of symmetry, one of which is used for seeking conservation laws. By analyzing the forces acting on a system of mass particles, Newtonian mechanics can find the law of conservation of mechanical energy, the law of conservation of momentum, and the law of conservation of moment of momentum, which have a clear physical meaning. Analytical mechanics goes further than Newtonian mechanics, and it can find the conservation law of generalized momentum (cyclic integral) and the conservation law of generalized energy (generalized energy integral) by utilizing the expression of the Lagrangian and Hamiltonian. The theory of symmetry goes further than the theory of Newtonian mechanics and analytical mechanics, and it can find more conservation laws than Newtonian mechanics and analytical mechanics. From the Newtonian mechanics to analytical mechanics and to the theory of symmetry, it is a progress of the conserved quantities from less to more, the physical meaning from obvious to vague, and the mathematical tools from simple to complex.[1]

The Noether symmetry is an invariance of Hamilton action under the action of infinitesimal transformations.[2] The theory of Noether symmetry first reveals the relationship between a symmetry and a conserved quantity. Candotti[3] and Desloge[4] applied the Noether theory to classical mechanics, and established Noether's theorem of classical mechanics. Djukić *et al.*[5,6] established Noether's theorem in classical non-conservative mechanics. Bahar[7] presented the extension of Noether's theorem to constrained non-conservative dynamical systems based on the work of Kalotas *et al.*[8] Vujanovic[9] studied the conservation laws of dynamical systems by means of the differential variational principles of Jourdain and Gauss. Palmieri, Rosen, and Candotti[10–12] further extended Noether's theorem to classical field theory. After that, Sarlet,[13,14] Carinena,[15] Rosenhous,[16] Sniatychi,[17] and Crasmareanu,[18] *et al.* developed the Noether theory from different aspects. In 1981, Li[19] gave Noether's theorem for nonholonomic systems, which was six years earlier than Bahar's similar results.[7] Since then, Mei,[20–23] Luo,[24,25] Zhang,[26–28] Fu,[29,30] Jia,[31] Shang,[32] Fang,[33] and Xie,[34] *et al.* have all studied the theory of Noether symmetry and gained a lot of significant achievements.

In 2007, Frederico and Torres gave a formulation of Noether's theorem for fractional problems of the calculus of variations by defining a fractional conserved quantity;[35] they further extended the results to the fractional variational problems with Riesz–Caputo derivatives,[36] the fractional optimal control problems in the sense of Caputo[37] and the optimal control problems for fractional Hamilton system.[38] Malinowska further extended the results to conservative and non-conservative generalized physical systems.[39] Atanack-

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ović[40] established the invariance conditions and Noether's theorem for variational problems with fractional derivatives, which are based on the definition of conserved quantity, just like in the classical theory. Zhou and Fu[41] established the fractional Noether theory of a Hamiltonian system by using Frederico's definition of fractional conserved quantity. Cresson[42] defined a class of fractional operators and established Noether's theorem under this operator. In order to establish a non-conservative dynamical model, El-Nabulsi presented a new model known as the fractional action-like variational approach,[43–45] which may also be called El-Nabulsi's dynamics model. The novelty of the El-Nabulsi dynamics model is that the fractional integral over time only needs one parameter, and the derived Euler-Lagrange equations are similar to the classical ones, in absence of fractional derivatives, but in the presence of the fractional generalized external force acting on the system. Frederico and Torres presented the constants of motion for fractional action-like variational problems.[46,47] Recently, Noether's theorems based on the El-Nabulsi dynamics model were studied and some important results have been found.[48–50] In short, the research on the fractional conserved quantities and fractional variational symmetries has only just begun.

In this paper, we further study Noether's theorems for a non-conservative Hamilton system with holonomic or nonholonomic constraints under the El-Nabulsi dynamical model which is based on a fractional integral extended by periodic laws. The definitions and criteria of El-Nabulsi–Noether symmetrical transformations and quasi-symmetrical transformations are presented, Noether's theorems for non-conservative Hamilton system with holonomic or nonholonomic constraints are established, and the conserved quantities led by the Noether symmetries are given.

2. El-Nabulsi–Hamilton's canonical equations for non-conservative Hamilton system based on the El-Nabulsi dynamical model

Definition 1[45] Let $f(t)$ be a continuous function in the interval $[a,b]$. For $t\in[a,b]$, the left and right extended periodic integrals of order $\alpha>0$ are defined as

$$\begin{aligned}&\left\{\begin{matrix}S^{\alpha}_{(a+)}\\ C^{\alpha}_{(a+)}\end{matrix}\right\}f(t)\\ &=\frac{1}{\Gamma(\alpha)}\int_a^t f(\tau)\left\{\begin{matrix}\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\\ \cos\left((\alpha-1)(t-\tau)\right)\end{matrix}\right\}\mathrm{d}\tau,\quad(1)\\ &\left\{\begin{matrix}S^{\alpha}_{(b-)}\\ C^{\alpha}_{(b-)}\end{matrix}\right\}f(t)\\ &=\frac{1}{\Gamma(\alpha)}\int_t^b f(\tau)\left\{\begin{matrix}\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\\ \cos\left((\alpha-1)(t-\tau)\right)\end{matrix}\right\}\mathrm{d}\tau,\quad(2)\end{aligned}$$

where $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}\exp(-t)\mathrm{d}t$ is the Euler-Gamma function.

The integrals (1) and (2) are fractional integrals extended by periodic laws presented by El-Nabulsi. Since the sine function and the cosine function above are equal in value, we consider the fractional variational problem under the situation of cosine.

Assume that the configuration of a mechanical system is determined by n generalized coordinates q_k $(k=1,\cdots,n)$, where L is the Lagrangian of the system. Introduce the generalized momentum and the Hamiltonian as

$$p_k=\frac{\partial L}{\partial\dot{q}_k},\quad H=p_k\dot{q}_k-L.\qquad(3)$$

For a non-conservative Hamilton system, the variational problem based on a fractional integral extended by periodic laws may be defined as follows.

Find the stationary points of the integral functional

$$\begin{aligned}S(\gamma)=&\frac{1}{\Gamma(\alpha)}\int_a^b[p_k\dot{q}_k-H(\tau,\boldsymbol{q},\boldsymbol{p})]\\ &\times\cos\left((\alpha-1)(t-\tau)\right)\mathrm{d}\tau\end{aligned}\qquad(4)$$

with the fixed boundary conditions

$$q_k(a)=q_{k,a},q_k(b)=q_{k,b},\quad(k=1,\ldots,n),\qquad(5)$$

where γ is a curve, $\dot{q}_k=\mathrm{d}q_k/\mathrm{d}\tau$, Γ is the Euler gamma function, $0<\alpha\le1$, τ is the intrinsic time, t is the observer time, $\tau\neq t$, and the Lagrangian H is a C^2 function with respect to all its arguments.

The variational problem above is called the fractional variational problem based on fractional integral extended by periodic laws. Functional (4) can be called the El-Nabulsi–Hamitlon action. When $\alpha=1$, the problem is reduced to the standard problem of the calculus of variations for a conservative Hamilton system.

According to the theory of calculus of variations,[51] the necessary condition that the functional (4) has an extreme value at $q_k=q_k(\tau)$ is that its variation equals zero, that is, $\delta SC^{\alpha}_{(a+)}=0$, so we have

$$\begin{aligned}\delta SC^{\alpha}_{(a+)}=&\frac{1}{\Gamma(\alpha)}\int_a^b\left(\dot{q}_k\delta p_k+p_k\delta\dot{q}_k-\frac{\partial H}{\partial q_k}\delta q_k-\frac{\partial H}{\partial p_k}\delta p_k\right)\\ &\times\cos\left((\alpha-1)(t-\tau)\right)\mathrm{d}\tau=0.\end{aligned}\qquad(6)$$

According to

$$\begin{aligned}&\int_a^b p_k\delta\dot{q}_k\left(\cos\left((\alpha-1)(t-\tau)\right)\right)\mathrm{d}\tau\\ &=\left[p_k\cos\left((\alpha-1)(t-\tau)\right)\delta q_k\right]\Big|_a^b\\ &\quad-\int_a^b[\dot{p}_k\cos\left((\alpha-1)(t-\tau)\right)\\ &\quad-p_k(1-\alpha)\sin\left((\alpha-1)(t-\tau)\right)]\delta q_k\mathrm{d}\tau,\end{aligned}\qquad(7)$$

and the boundary conditions (5), we have

$$\delta q_k|_{\tau=a}=\delta q_k|_{\tau=b}=0,\quad(k=1,2,\ldots,n).\qquad(8)$$

Using Eqs. (7) and (8), equation (6) can be written as

$$\int_a^b \left\{ \left[-\dot{p}_k - \frac{\partial H}{\partial q_k} + (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) p_k \right] \delta q_k + \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k \right\} \cos\left((\alpha-1)(t-\tau)\right) \mathrm{d}\tau = 0. \quad (9)$$

Taking the partial derivative of the second formula of (3) with respect to p_k, we obtain

$$\frac{\partial H}{\partial p_k} = \dot{q}_k, \quad (k = 1, 2, \ldots, n). \quad (10)$$

Substituting Eq. (10) into Eq. (9) and considering the independence of δq_k $(k = 1, \ldots, n)$ and the arbitrariness of the integral interval $[a, b]$, we have

$$-\dot{p}_k - \frac{\partial H}{\partial q_k} + (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) p_k = 0, \quad (k = 1, 2, \ldots, n). \quad (11)$$

Combining Eq. (10) with Eq. (11), we obtain El-Nabulsi–Hamilton's canonical equations for the fractional variational problem based on fractional integral extended by periodic laws, that is,

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \dot{p}_k = -\frac{\partial H}{\partial q_k} + (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) p_k, \quad (k = 1, 2, \ldots, n). \quad (12)$$

Suppose that the system is subjected to g ideal nonholonomic constraints of Chetaev's type

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad (\beta = 1, \ldots, g). \quad (13)$$

The restriction of constraints (13) on the virtual displacements δq_k is[1]

$$\frac{\partial f_\beta}{\partial \dot{q}_k} \delta q_k = 0, \quad (\beta = 1, \ldots, g). \quad (14)$$

Because δq_k $(k = 1, \ldots, n)$ are not all independent of each other, by using the usual Lagrange multiplier method, from Eq. (9) and the condition (14), we can obtain

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k}, \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} + (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) p_k \\ &\quad + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_k}, \quad (k = 1, 2, \ldots, n), \end{aligned} \quad (15)$$

where λ_β are the constraint multipliers.

Example 1 Let us consider a system whose Lagrangian is

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right). \quad (16)$$

The nonholonomic constraint is

$$f = \dot{q}_2 - \tau \dot{q}_1 = 0. \quad (17)$$

The generalized momentum and the Hamiltonian are

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1, \quad p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2,$$
$$H = p_1 \dot{q}_1 + p_2 \dot{q}_2 - L = \frac{1}{2}\left(p_1^2 + p_2^2\right). \quad (18)$$

The second formula of Eq. (15) yields

$$\dot{p}_1 = (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) p_1 - \lambda\tau,$$
$$\dot{p}_2 = (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) p_2 + \lambda. \quad (19)$$

From Eqs. (16) and (19), we can find the constraint multiplier

$$\lambda = -\frac{1}{\tau^2+1}[(1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) \times (\dot{q}_2 - \tau\dot{q}_1) - \dot{q}_1]. \quad (20)$$

Then, equation (15) can be rewritten as

$$\begin{aligned} \dot{q}_1 &= \frac{\partial H}{\partial p_1} = p_1, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2, \\ \dot{p}_1 &= (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) \\ &\quad \times \left[p_1 + \frac{\tau(p_2 - \tau p_1)}{\tau^2+1} \right] - \frac{\tau}{\tau^2+1} p_1, \\ \dot{p}_2 &= (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right) \\ &\quad \times \left(p_2 + \frac{p_2 - \tau p_1}{\tau^2+1} \right) + \frac{p_1}{\tau^2+1}. \end{aligned} \quad (21)$$

If $\alpha = 1$, equation (21) is reduced to the standard Hamilton canonical equation

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2,$$
$$\dot{p}_1 = -\frac{\tau}{\tau^2+1} p_1, \quad \dot{p}_2 = \frac{p_1}{\tau^2+1}. \quad (22)$$

3. Variation of El-Nabulsi–Hamilton action

Introduce the infinitesimal transformations of the r-parameter finite transformation group

$$\bar{\tau} = \tau + \Delta\tau, \quad \bar{q}_k(\bar{\tau}) = q_k(\tau) + \Delta q_k,$$
$$\bar{p}_k(\bar{\tau}) = p_k(\tau) + \Delta p_k, \quad (k = 1, \ldots, n), \quad (23)$$

or their expansion formulae

$$\bar{\tau} = \tau + \varepsilon_\sigma \xi_0^\sigma(\tau, \boldsymbol{q}, \boldsymbol{p}),$$
$$\bar{q}_k(\bar{\tau}) = q_k(\tau) + \varepsilon_\sigma \xi_k^\sigma(\tau, \boldsymbol{q}, \boldsymbol{p}),$$
$$\bar{p}_k(\bar{\tau}) = p_k(\tau) + \varepsilon_\sigma \eta_k^\sigma(\tau, \boldsymbol{q}, \boldsymbol{p}), \quad (k = 1, 2, \ldots, n), \quad (24)$$

where ε_σ $(\sigma = 1, \ldots, r)$ are infinitesimal parameters, and ξ_0^σ, ξ_k^σ, η_k^σ are the generators or the generating functions of the infinitesimal transformations. Under the infinitesimal transformations of group (23), the curve γ will be transformed into a neighbor curve $\bar{\gamma}$, the action $S(\gamma)$ will be transformed into $S(\bar{\gamma})$, and we have

$$S(\bar{\gamma}) = \frac{1}{\Gamma(\alpha)} \int_{\bar{a}}^{\bar{b}} \left[\bar{p}_k(\bar{\tau}) \dot{\bar{q}}_k(\bar{\tau}) - H(\bar{\tau}, \bar{q}_j(\bar{\tau}), \bar{p}_j(\bar{\tau})) \right]$$

$$\times \cos\left(\left(\alpha-1\right)\left(t-\bar{\tau}\right)\right)\mathrm{d}\bar{\tau}. \tag{25}$$

Hence, we have

$$\begin{aligned}
&S(\bar{\gamma})-S(\gamma)\\
=&\frac{1}{\Gamma(\alpha)}\int_{\bar{a}}^{\bar{b}}\left[\bar{p}_k(\bar{\tau})\dot{\bar{q}}_k(\bar{\tau})-H(\bar{\tau},\bar{q}_j(\bar{\tau}),\bar{p}_j(\bar{\tau}))\right]\\
&\times\cos\left(\left(\alpha-1\right)\left(t-\bar{\tau}\right)\right)\mathrm{d}\bar{\tau}\\
&-\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\left[p_k(\tau)\dot{q}_k(\tau)-H(\tau,q_j(\tau),p_j(\tau))\right]\\
&\times\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\mathrm{d}\tau\\
=&\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\Big\{\left[\left(p_k+\Delta p_k\right)\left(\dot{q}_k+\Delta\dot{q}_k\right)\right.\\
&\left.-H(\tau+\Delta\tau,q_j+\Delta q_j,p_j+\Delta p_j)\right]\left(1+\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\right)\\
&\times\cdot\cos\left(\left(\alpha-1\right)\left(t-\tau-\Delta\tau\right)\right)-\left[p_k\dot{q}_k-H(\tau,q_j,p_j)\right]\\
&\times\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\Big\}\mathrm{d}\tau.
\end{aligned}\tag{26}$$

Denoting ΔS as the main linear part of the difference $S(\bar{\gamma})-S(\gamma)$ relative to ε_σ, we have

$$\begin{aligned}
\Delta S=&\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\Big\{\Big[-\frac{\partial H}{\partial\tau}\Delta\tau-\frac{\partial H}{\partial q_k}\Delta q_k\\
&+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\Delta p_k+p_k\Delta\dot{q}_k\\
&+\left(p_k\dot{q}_k-H\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\Big]\\
&\times\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)+\left(p_k\dot{q}_k-H\right)\left(\alpha-1\right)\\
&\times\sin\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\Delta\tau\Big\}\mathrm{d}\tau.
\end{aligned}\tag{27}$$

According to the relation between the non-isochronous variation Δ and the isochronous variation δ[1]

$$\Delta F=\delta F+\dot{F}\Delta\tau, \tag{28}$$

where F is an arbitrary differentiable function, we have

$$\begin{aligned}
&\delta q_k=\Delta q_k-\dot{q}_k\Delta\tau,\quad \delta p_k=\Delta p_k-\dot{p}_k\Delta\tau,\\
&\Delta\dot{q}_k=\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta q_k-\dot{q}_k\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau.
\end{aligned}\tag{29}$$

From formulae (29), formula (27) can be expressed as

$$\begin{aligned}
\Delta S=&\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\Big\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(p_k\Delta q_k-H\Delta\tau\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\right]\\
&+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\delta p_k\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\\
&+\left[-\dot{p}_k-\frac{\partial H}{\partial q_k}+\left(1-\alpha\right)\tan\left(\left(\alpha-1\right)\left(t-\tau\right)\right)p_k\right]\delta q_k\\
&\times\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\Big\}\mathrm{d}\tau.
\end{aligned}\tag{30}$$

From formulae (24), the formula above can be further expressed as

$$\begin{aligned}
\Delta S=&\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\varepsilon_\sigma\Big\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(p_k\xi_k^\sigma-H\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\right]\\
&+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\left(\eta_k^\sigma-\dot{p}_k\xi_0^\sigma\right)\\
&\times\cos\left(\left(\alpha-1\right)\ \left(t-\tau\right)\right)\\
&+\left[-\dot{p}_k-\frac{\partial H}{\partial q_k}+\left(1-\alpha\right)\tan\left(\left(\alpha-1\right)\left(t-\tau\right)\right)p_k\right]\\
&\times\left(\xi_k^\sigma-\dot{q}_k\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\Big\}\mathrm{d}\tau.
\end{aligned}\tag{31}$$

Formulae (27) and (31) are basic formulae for the variation of El-Nabulsi–Hamilton action (4).

4. El-Nabulsi–Noether symmetric transformations

In this section, we will establish the definitions and criteria of El-Nabulsi–Noether symmetrical transformations and quasi-symmetrical transformations for fractional variational problem based on fractional integral extended by periodic laws.

Definition 2 If the El-Nabulsi–Hamilton action (4) is an invariant of the infinitesimal transformations of group (23), that is, for each of the infinitesimal transformations, the formula

$$\Delta S=0 \tag{32}$$

holds, then the infinitesimal transformations are called the El-Nabulsi–Noether symmetrical transformations based on fractional integral extended by periodic laws.

From Definition 2 and formula (27), we can obtain the following criterion.

Criterion 1 For the infinitesimal transformations of group (23), if the condition

$$\begin{aligned}
&-\frac{\partial H}{\partial\tau}\Delta\tau-\frac{\partial H}{\partial q_k}\Delta q_k+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\Delta p_k+p_k\Delta\dot{q}_k\\
&+\left(p_k\dot{q}_k-H\right)\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau+\left(\alpha-1\right)\tan\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\Delta\tau\right]\\
=&\ 0
\end{aligned}\tag{33}$$

is satisfied, then the transformations (33) are the El-Nabulsi–Noether symmetrical transformations in the sense of Definition 2.

Condition (33) can also be expressed as r equations

$$\begin{aligned}
&-\frac{\partial H}{\partial\tau}\xi_0^\sigma-\frac{\partial H}{\partial q_k}\xi_k^\sigma+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\eta_k^\sigma\\
&+p_k\left(\dot{\xi}_k^\sigma-\dot{q}_k\xi_0^\sigma\right)+\left(p_k\dot{q}_k-H\right)\\
&\times\left[\dot{\xi}_0^\sigma+\left(\alpha-1\right)\tan\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\xi_0^\sigma\right]=0,\\
&\left(\sigma=1,2,\ldots,r\right).
\end{aligned}\tag{34}$$

When $r=1$, equation (34) is called the El-Nabulsi–Noether identity, i.e.,

$$-\frac{\partial H}{\partial\tau}\xi_0-\frac{\partial H}{\partial q_k}\xi_k+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\eta_k$$

$$+p_k\left(\dot{\xi}_k-\dot{q}_k\xi_0\right)+\left(p_k\dot{q}_k-H\right)\times\left[\dot{\xi}_0+(\alpha-1)\tan\left((\alpha-1)(t-\tau)\right)\xi_0\right]=0. \quad (35)$$

From Definition 2 and formula (31), we can obtain the following criterion.

Criterion 2 For the infinitesimal transformations of group (24), if the conditions

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(p_k\xi_k^\sigma-H\xi_0^\sigma\right)\cos\left((\alpha-1)(t-\tau)\right)\right]\\&+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\left(\eta_k^\sigma-\dot{p}_k\xi_0^\sigma\right)\cos\left((\alpha-1)(t-\tau)\right)\\&+\left[-\dot{p}_k-\frac{\partial H}{\partial q_k}+(1-\alpha)\tan\left((\alpha-1)(t-\tau)\right)p_k\right]\\&\times\left(\xi_k^\sigma-\dot{q}_k\xi_0^\sigma\right)\cos\left((\alpha-1)(t-\tau)\right)=0,\\&(\sigma=1,2,\ldots,r)\end{aligned} \quad (36)$$

are satisfied, then the transformations are the El-Nabulsi–Noether symmetrical transformations, according to Definition 2.

Next, we establish the definition and criteria of the El-Nabulsi–Noether quasi-symmetrical transformations for fractional variational problem based on fractional integral extended by periodic laws.

Suppose that H' is another Hamiltonian and the infinitesimal transformations of group (23) accurate to the first-order infinitesimal satisfy the condition

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_a^b\left[p_k(\tau)\dot{q}_k(\tau)-H\left(\tau,q_j(\tau),p_j(\tau)\right)\right]\\&\times\cos\left((\alpha-1)(t-\tau)\right)\mathrm{d}\tau\\=&\frac{1}{\Gamma(\alpha)}\int_{\bar{a}}^{\bar{b}}\left[\bar{p}_k(\bar{\tau})\dot{\bar{q}}_k(\bar{\tau})-H'\left(\bar{\tau},\bar{q}_j(\bar{\tau}),\bar{p}_j(\bar{\tau})\right)\right]\\&\times\cos\left((\alpha-1)(t-\bar{\tau})\right)\mathrm{d}\bar{\tau},\end{aligned} \quad (37)$$

then this invariance is called the quasi-invariance of the El-Nabulsi–Hamilton action (4) under the infinitesimal transformations of group (23). The functions H' and H determined by Eq. (37) satisfy the same differential equations of motion, then the transformations are called the El-Nabulsi–Noether quasi-symmetrical transformations for a fractional variational problem based on the fractional integral extended by periodic laws, and we have

$$H'=H-\frac{\mathrm{d}}{\mathrm{d}\tau}G(\tau,\boldsymbol{q},\boldsymbol{p})\sec\left((\alpha-1)(t-\tau)\right). \quad (38)$$

Substituting (38) into (37), we have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_a^b\left[p_k(\tau)\dot{q}_k(\tau)-H\left(\tau,q_j(\tau),p_j(\tau)\right)\right]\\&\times\cos\left((\alpha-1)(t-\tau)\right)\mathrm{d}\tau\\=&\frac{1}{\Gamma(\alpha)}\int_a^b\left\{\left[\bar{p}_k(\bar{\tau})\dot{\bar{q}}_k(\bar{\tau})-H\left(\bar{\tau},\bar{q}_j(\bar{\tau}),\bar{p}_j(\bar{\tau})\right)\right]\right.\\&\left.\times\cos\left((\alpha-1)(t-\bar{\tau})\right)+\frac{\mathrm{d}}{\mathrm{d}\tau}G\left(\bar{\tau},\bar{q}_j(\bar{\tau}),\bar{p}_j(\bar{\tau})\right)\right\}\mathrm{d}\bar{\tau}.\end{aligned} \quad (39)$$

The left hand of formula (39) is a first-order infinitesimal under the transformations (23). Therefore, the right hand should be an infinitesimal of the same order, and G could be replaced by ΔG, and thus

$$\Delta G\left(\bar{\tau},\bar{q}_k(\bar{\tau})\right)=\Delta G\left(\tau,q_k(\tau),\dot{q}_k(\tau)\right). \quad (40)$$

Hence, we have the following definition.

Definition 3 If the El-Nabulsi–Hamilton action (4) is quasi-invariant under the infinitesimal transformations of group (23), i.e., for each of the infinitesimal transformations,

$$\Delta S=-\frac{1}{\Gamma(\alpha)}\int_a^b\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)\,\mathrm{d}\tau \quad (41)$$

holds, where $G=G(\tau,\boldsymbol{q},\dot{\boldsymbol{q}})$. Then the infinitesimal transformations are called the El-Nabulsi–Noether quasi-symmetrical transformations based on a fractional integral extended by periodic laws.

From Definition 3 and formula (27), we can obtain the following criterion.

Criterion 3 For the infinitesimal transformations of group (23), if the condition

$$\begin{aligned}&-\frac{\partial H}{\partial\tau}\Delta\tau-\frac{\partial H}{\partial q_k}\Delta q_k+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\Delta p_k\\&+p_k\Delta\dot{q}_k+\left(p_k\dot{q}_k-H\right)\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\right.\\&\left.+(\alpha-1)\tan\left((\alpha-1)(t-\tau)\right)\Delta\tau\right]\\=&-\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)\sec\left((\alpha-1)(t-\tau)\right)\end{aligned} \quad (42)$$

is satisfied, then the infinitesimal transformations are the El-Nabulsi–Noether quasi-symmetrical transformations according to Definition 2.

Condition (42) can also be expressed as r equations

$$\begin{aligned}&-\frac{\partial H}{\partial\tau}\xi_0^\sigma-\frac{\partial H}{\partial q_k}\xi_k^\sigma+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\eta_k^\sigma\\&+p_k\left(\dot{\xi}_k^\sigma-\dot{q}_k\dot{\xi}_0^\sigma\right)+\left(p_k\dot{q}_k-H\right)\\&\times\left[\dot{\xi}_0^\sigma+(\alpha-1)\tan\left((\alpha-1)(t-\tau)\right)\xi_0^\sigma\right]\\=&-\dot{G}^\sigma\sec\left((\alpha-1)(t-\tau)\right),\quad(\sigma=1,2,\ldots,r).\end{aligned} \quad (43)$$

When $r=1$, formula (43) is also called the El-Nabulsi–Noether identity

$$\begin{aligned}&-\frac{\partial H}{\partial\tau}\xi_0-\frac{\partial H}{\partial q_k}\xi_k+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\eta_k\\&+p_k\left(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0\right)+\left(p_k\dot{q}_k-H\right)\\&\times\left[\dot{\xi}_0+(\alpha-1)\tan\left((\alpha-1)(t-\tau)\right)\xi_0\right]\\=&-\dot{G}\sec\left((\alpha-1)(t-\tau)\right).\end{aligned} \quad (44)$$

From Definition 3 and formula (31), we can get the following criterion.

Criterion 4 For the infinitesimal transformations of group (24), if the conditions

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(p_k\xi_k^\sigma - H\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\right]\\&+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\left(\eta_k^\sigma-\dot{p}_k\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\\&+\left[-\dot{p}_k-\frac{\partial H}{\partial q_k}+\left(1-\alpha\right)\tan\left(\left(\alpha-1\right)\left(t-\tau\right)\right)p_k\right]\\&\times\left(\xi_k^\sigma-\dot{q}_k\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)=-\dot{G}^\sigma,\\&\left(\sigma=1,2,\ldots,r\right)\end{aligned}\tag{45}$$

are satisfied, where

$$\Delta G=\varepsilon_\sigma G^\sigma,\tag{46}$$

then the transformations are the El-Nabulsi–Noether quasi-symmetrical transformations in the sense of Definition 3.

By using Criterions 1 and 2, we can judge the fractional Noether symmetry under the El-Nabulsi dynamical model. By means of Criterions 3 and 4, we can judge the fractional Noether quasi-symmetry under the El-Nabulsi dynamical model.

5. Noether's theorem for a non-conservative Hamilton system with holonomic constraints based on the El-Nabulsi dynamical model

First, we give the definition of a conserved quantity of a holonomic system (12).

Definition 4 A function $I\left(\tau,\boldsymbol{q},\boldsymbol{p}\right)$ is said to be a conserved quantity of the holonomic system (12) under the El-Nabulsi model based on the fractional integral extended by periodic laws if and only if

$$\frac{\mathrm{d}}{\mathrm{d}\tau}I\left(\tau,q,p\right)=0\tag{47}$$

along all the solution curves of the El-Nabulsi–Hamilton canonical equation (12).

For the holonomic system (12), if we can find an El-Nabulsi–Noether symmetrical transformation or an El-Nabulsi–Noether quasi-symmetrical transformation, then we can find a conserved quantity correspondingly. We have the following theorem.

Theorem 1 For the holonomic system (12), if the infinitesimal transformations of group (23) are the El-Nabulsi–Noether symmetrical transformations under Definition 2, then the system exists r linearly independent conserved quantities, which are

$$\begin{aligned}&I^\sigma=\left(p_k\xi_k^\sigma-H\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)=c^\sigma,\\&\left(\sigma=1,\ldots,r\right).\end{aligned}\tag{48}$$

Proof According to the assumption, from Definition 2, we have $\Delta S=0$, that is,

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_a^b\varepsilon_\sigma\Big\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(p_k\xi_k^\sigma-H\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\right]\\&+\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\left(\eta_k^\sigma-\dot{p}_k\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\\&+\left[-\dot{p}_k-\frac{\partial H}{\partial q_k}+\left(1-\alpha\right)\tan\left(\left(\alpha-1\right)\left(t-\tau\right)\right)p_k\right]\\&\times\left(\xi_k^\sigma-\dot{q}_k\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\Big\}\mathrm{d}\tau=0.\end{aligned}\tag{49}$$

Substituting Eq. (12) into the formula above, and considering the arbitrariness of integral interval $[a,b]$ and the independence of parameters ε_σ, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(p_k\xi_k^\sigma-H\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\right]=0.\tag{50}$$

Integrating it, we obtain formula (48). Then the proof is completed.

Theorem 2 For the holonomic system (12), if the infinitesimal transformations of group (23) are the El-Nabulsi–Noether quasi-symmetrical transformations under Definition 3, then the system exists with r linear independent conserved quantities, such as

$$\begin{aligned}I^\sigma\ =\ &\left(p_k\xi_k^\sigma-H\xi_0^\sigma\right)\cos\left(\left(\alpha-1\right)\left(t-\tau\right)\right)+G^\sigma=c^\sigma,\\&\left(\sigma=1,\ldots,r\right).\end{aligned}\tag{51}$$

Theorem 1 and 2 can be called Noether's theorems for a non-conservative Hamilton system with holonomic constraints based on the El-Nabulsi dynamical model extended by periodic laws. According to Noether's theorem, for the holonomic non-conservative system, if we can find an El-Nabulsi–Noether symmetrical transformation or an El-Nabulsi–Noether quasi-symmetrical transformation, then we can get a conserved quantity.

Example 2 The Lagrangian of the planar Kepler problem is

$$L=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right)+\mu\left(q_1^2+q_2^2\right)^{-1/2},\quad\left(q_1^2+q_2^2\neq0\right).\tag{52}$$

We try to study its Noether symmetries and conserved quantities.

From Eq. (52), we have

$$\begin{aligned}&p_1=\frac{\partial L}{\partial\dot{q}_1}=\dot{q}_1,\quad p_2=\frac{\partial L}{\partial\dot{q}_2}=\dot{q}_2,\\&H=p_1\dot{q}_1+p_2\dot{q}_2-L=\frac{1}{2}\left(p_1^2+p_2^2\right)-\mu\left(q_1^2+q_2^2\right)^{-1/2}.\end{aligned}\tag{53}$$

First, we find the El-Nabulsi–Noether quasi-symmetrical transformations. The El-Nabulsi–Noether identity (44) gives

$$\begin{aligned}&-\mu\left(q_1^2+q_2^2\right)^{-3/2}\left(q_1\xi_1+q_2\xi_2\right)+p_1\left(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0\right)\\&+p_2\left(\dot{\xi}_2-\dot{q}_2\dot{\xi}_0\right)+\left(p_1\dot{q}_1+p_2\dot{q}_2-H\right)\\&\times\left[\dot{\xi}_0+\left(\alpha-1\right)\tan\left(\left(\alpha-1\right)\left(t-\tau\right)\right)\xi_0\right]\end{aligned}$$

$$= -\dot{G}\cos\left((\alpha-1)(t-\tau)\right). \tag{54}$$

Equation (54) has the following solutions:

$$\begin{aligned}
&\xi_0^1 = 0, \quad \xi_1^1 = -q_2, \quad \xi_2^1 = q_1, \quad G^1 = 0, \qquad (55)\\
&\xi_0^2 = \sec\left((\alpha-1)(t-\tau)\right),\\
&\xi_1^2 = p_1\sec\left((\alpha-1)(t-\tau)\right),\\
&\xi_2^2 = p_2\sec\left((\alpha-1)(t-\tau)\right),\\
&G^2 = -\frac{1}{2}\left(p_1^2+p_2^2\right) - \mu\left(q_1^2+q_2^2\right)^{-1/2}. \qquad (56)
\end{aligned}$$

The generator (55) corresponds to an El-Nabulsi–Noether symmetrical transformation of the system, and the generator (56) corresponds to an El-Nabulsi–Noether quasi-symmetrical transformation of the system.

From the generator (55), according to Theorem 1, we have

$$I^1 = (q_1p_2 - p_1q_2)\cos\left((\alpha-1)(t-\tau)\right) = \text{const}. \tag{57}$$

Formula (52) is a conserved quantity led by the El-Nabulsi–Noether symmetry (52) of the system.

From the generator (56), according to Theorem 2, we obtain

$$I^2 = 0. \tag{58}$$

Therefore, the infinitesimal transformation corresponding to the generator (56) is trivial.

6. Noether's theorem for a non-conservative Hamilton system with nonholonomic constraints based on the El-Nabulsi dynamical model

First, we give the definition of the El-Nabulsi–Noether quasi-symmetrical transformations of the nonholonomic systems (13) and (15).

Due to

$$\delta q_k = \Delta q_k - \dot{q}_k\Delta\tau = \varepsilon_\sigma\left(\xi_k^\sigma - \dot{q}_k\xi_0^\sigma\right), \tag{59}$$

substituting Eq. (59) into Eq. (14) and considering the independence of ε_σ, we have

$$\frac{\partial f_\beta}{\partial\dot{q}_k}\left(\xi_k^\sigma - \dot{q}_k\xi_0^\sigma\right) = 0, \quad (\beta = 1,\ldots,g;\sigma = 1,\ldots,r), \tag{60}$$

which are restriction equations. Thus we have the following definition.

Definition 5 If the infinitesimal transformations of group (23) are the El-Nabulsi–Noether quasi-symmetrical transformations and satisfy the restriction equations (60), then the transformations are called the El-Nabulsi–Noether quasi-symmetrical transformations for the nonholonomic systems (13) and (15).

Secondly, we give the definition of conserved quantity for nonholonomic systems (13) and (15).

Definition 6 A function $I(\tau,\boldsymbol{q},\boldsymbol{p})$ is said to be a conserved quantity of the nonholonomic systems (13) and (15) under the El-Nabulsi model based on the fractional integral extended by periodic law if and only if

$$\frac{\mathrm{d}}{\mathrm{d}\tau}I(\tau,\boldsymbol{q},\boldsymbol{p}) = 0 \tag{61}$$

along all the solution curves of the El-Nabulsi–Hamilton canonical equations of the nonholonomic systems (13) and (15).

Finally, we establish Noether's theorem for the non-conservative Hamilton system with nonholonomic constraints.

Theorem 3 For the nonholonomic systems (13) and (15), if the infinitesimal transformations of group (24) are the El-Nabulsi–Noether quasi-symmetrical transformations under Definition 5, then the system exists with r linear independent conserved quantities

$$\begin{aligned}
&I^\sigma = (p_k\xi_k^\sigma - H\xi_0^\sigma)\cos\left((\alpha-1)(t-\tau)\right) + G^\sigma = c^\sigma,\\
&(\sigma = 1,\ldots,r), \qquad (62)
\end{aligned}$$

Proof According to the assumption, by Definition 3, we have

$$\Delta S = -\frac{1}{\Gamma(\alpha)}\int_a^b \frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)\,\mathrm{d}\tau, \tag{63}$$

or

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_a^b \varepsilon_\sigma\Big\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(p_k\xi_k^\sigma - H\xi_0^\sigma)\cos\left((\alpha-1)(t-\tau)\right) + G^\sigma\right]\\
&+\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)(\eta_k^\sigma - \dot{p}_k\xi_0^\sigma)\cos\left((\alpha-1)(t-\tau)\right)\\
&+\left[-\dot{p}_k - \frac{\partial H}{\partial q_k} + (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right)p_k\right]\\
&\times\left(\xi_k^\sigma - \dot{q}_k\xi_0^\sigma\right)\cos\left((\alpha-1)(t-\tau)\right)\Big\}\mathrm{d}\tau = 0. \qquad (64)
\end{aligned}$$

Since the infinitesimal transformations satisfy the restriction conditions (60), we have

$$\frac{1}{\Gamma(\alpha)}\int_a^b \varepsilon_\sigma\lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_k}\left(\xi_k^\sigma - \dot{q}_k\xi_0^\sigma\right)\mathrm{d}\tau = 0. \tag{65}$$

Adding Eqs. (64) and (65) together, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_a^b \varepsilon_\sigma\Big\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(p_k\xi_k^\sigma - H\xi_0^\sigma)\cos\left((\alpha-1)(t-\tau)\right) + G^\sigma\right]\\
&+\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)(\eta_k^\sigma - \dot{p}_k\xi_0^\sigma)\cos\left((\alpha-1)(t-\tau)\right)\\
&+\left(-\dot{p}_k - \frac{\partial H}{\partial q_k} + (1-\alpha)\tan\left((\alpha-1)(t-\tau)\right)p_k + \lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_k}\right)\\
&\times\left(\xi_k^\sigma - \dot{q}_k\xi_0^\sigma\right)\cos\left((\alpha-1)(t-\tau)\right)\Big\}\mathrm{d}\tau = 0. \qquad (66)
\end{aligned}$$

Substituting equations (15) into formula (66), and considering the independence of ε_σ and the arbitrariness of integral interval $[a,b]$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(p_k\xi_k^\sigma - H\xi_0^\sigma\right)\cos\left((\alpha-1)(t-\tau)\right)+G^\sigma\right]=0,$$
$$(\sigma=1,\ldots,r) \tag{67}$$

Integrating it, we obtain formula (57), and the theorem is proved.

Theorem 3 can be called Noether's theorems for a non-conservative Hamilton system with nonholonomic constraints based on the El-Nabulsi dynamical model extended by periodic laws. From Noether's theorem, we can find a conserved quantity from a known Noether symmetry.

If the nonholonomic constraints (13) are absent, then Theorem 3 degenerates to Theorem 2; if the nonholonomic constraints (13) also satisfy $G^\sigma=0$, then Theorem 3 degenerates to Theorem 1.

Example 3 Let us consider a system whose Lagrangian of the system is

$$L=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right), \tag{68}$$

and its motion is subjected to a nonholonomic constraint

$$f=\dot{q}_1+b\tau\dot{q}_2-bq_2+\tau=0,\quad (b=\text{const.}). \tag{69}$$

From Eq. (68), we have

$$p_1=\frac{\partial L}{\partial\dot{q}_1}=\dot{q}_1,\quad p_2=\frac{\partial L}{\partial\dot{q}_2}=\dot{q}_2,$$
$$H=p_1\dot{q}_1+p_2\dot{q}_2-L=\frac{1}{2}\left(p_1^2+p_2^2\right). \tag{70}$$

First, we try to find the El-Nabulsi–Noether quasi-symmetrical transformations satisfying the restriction conditions. The El-Nabulsi–Noether identity (44) gives

$$p_1\left(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0\right)+p_2\left(\dot{\xi}_2-\dot{q}_2\dot{\xi}_0\right)+\left(p_1\dot{q}_1+p_2\dot{q}_2-H\right)$$
$$\times\left[\dot{\xi}_0+(\alpha-1)\tan\left((\alpha-1)(t-\tau)\right)\xi_0\right]$$
$$=-\dot{G}\sec\left((\alpha-1)(t-\tau)\right) \tag{71}$$

and the restriction condition (60) gives

$$\xi_1-p_1\xi_0+b\tau\left(\xi_2-p_2\xi_0\right)=0. \tag{72}$$

Equations (71) and (72) have the following solutions:

$$\xi_0^1=\sec\left((\alpha-1)(t-\tau)\right),\quad \xi_1^1=p_1\sec\left((\alpha-1)(t-\tau)\right),$$
$$\xi_2^1=p_2\sec\left((\alpha-1)(t-\tau)\right),\quad G^1=-\frac{1}{2}\left(p_1^2+p_2^2\right), \tag{73}$$
$$\xi_0^2=0,\quad \xi_1^2=-b\tau\sec\left((\alpha-1)(t-\tau)\right),$$
$$\xi_2^2=\sec\left((\alpha-1)(t-\tau)\right),$$
$$G^2=bq_1-\int\left(b\tau p_1-p_2\right)(\alpha-1)$$
$$\times\tan\left((\alpha-1)(t-\tau)\right)\mathrm{d}\tau. \tag{74}$$

The generators (73) and (74) both correspond to El-Nabulsi–Noether quasi-symmetric transformations of the nonholonomic system. According to Theorem 3, the conserved quantity (59) gives

$$I^1=0, \tag{75}$$
$$I^2=-b\tau p_1+p_2+bq_1-\int\left(b\tau p_1-p_2\right)(\alpha-1)$$
$$\times\tan\left((\alpha-1)(t-\tau)\right)\mathrm{d}\tau=\text{const.} \tag{76}$$

Therefore, the infinitesimal transformation corresponding to generator (73) is trivial. When $\alpha=1$, the conserved quantity (76) gives

$$I=-b\tau\dot{q}_1+\dot{q}_2+bq_1=\text{const.} \tag{77}$$

This is a classical conserved quantity.[1]

7. Conclusions

Fractional calculus generalized the classical calculus and it has many important applications in various fields of science and engineering. In the last few years, its importance in the calculus of variations and optimal control has been perceived, and a fractional variational theory began to be developed by several different authors. One of the oldest and most interesting classes of variational problems is the symmetry problem. Noether's theorem establishes the relation between the existence of symmetries and the existence of conservation laws, and is one of the most beautiful results of the calculus of variation mechanics and optimal control. In this paper, the fractional variational problem based on the fractional integral extended by periodic laws, as well as Noether's theorem for the non-conservative Hamilton system with holonomic constraints or nonholonomic constraints are discussed and established completely. The method and results of this paper are of universal significance: they can be further applied in various types of constrained mechanical systems, and standard Noether's theorems for conservative Hamilton systems are special cases of this paper. Moreover, we can further study perturbation and adiabatic invariants of Noether symmetry in non-conservative systems.

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