Enhancing the precision of phase estimation by weak measurement and quantum measurement reversal*

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The enhancement of the precision of phase estimation in quantum metrology is investigated by employing weak measurement (WM) and quantum measurement reversal (QMR). We derive the exact expressions of the optimal quantum Fisher information (QFI) and success probability of phase estimation for an exactly solving model consisting of a qubit interacting with a structured reservoir. We show that the QFI can be obviously enhanced by means of the WM and QMR in different regimes. In addition, we also show that the magnitude of the decoherence involved in the WM and QMR can be a general complex number, which extends the applicable scope of the WM and QMR approach.

Keywords: precision of phase estimation, quantum fisher information, weak measurement, quantum measurement reversal

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1. Introduction

It is well known that the precision of parameter estimation plays a crucial role in quantum metrology. The quantum Cramér–Rao inequality[1] gives the best bound of precise parameter estimation, which is associated with quantum Fisher information (QFI).[2] Consequently, the QFI becomes a key concept in the quantum estimation theory. According to quantum Cramér–Rao inequality, a large QFI means a high precise of parameter estimation. Hence, the enhancement of the QFI has been a very important topic in quantum metrology.

Recently, many works have indicated that the entangled state can enhance the precision of parameter estimation.[3–8] However, due to entanglement fragility to the influence of the environment, one has to overcome the decoherence by some strategies in order to obtain the high precision of parameter estimation. In the past few decades, several theoretical schemes, including the quantum Zeno effect,[9,10] decoherence-free subspace (DFS),[11–14] quantum error correction code,[15–17] dynamical decoupling (DD)[18,19] and the non-Markovian effect of environment[20–28] have been proposed to tame the decoherence. Therefore, the improvement of the QFI under decoherence were intensely studied[29–37] using different methods. For example, the author in Ref. [34] discussed the non-Markovian effect of environment on the QFI. It has been shown that the QFI is affected dramatically by the environment memory effect. In Ref. [35], the authors studied the optimal QFI from a noisy quantum system by the DD strategy. In Ref. [36], the authors investigated the enhancement of the QFI in a noisy system by DD pulses. In Ref. [37], the authors studied the phase matching condition for enhancement of the phase sensitivity in quantum metrology. More recently, some works[38–42] investigated the effect of weak-value amplification proposed by Aharonov *et al.*[43] on the enhancement of the QFI for noisy quantum metrology and found that the weak value amplification is not really helpful to improve the estimation precision. However, here we propose a different scheme based on the weak measurement (WM) and quantum measurement reversal (QMR) approach to enhance the QFI under the decoherence. It should be noted that the new scheme we propose here is no longer a deterministic process but a higher precision of parameter estimation by adjusting the strength of the WM.

More recently, it was shown that the WM and QMR approach can effectively suppress amplitude-damping decoherence for a single qubit.[44–46] The kind of weak measurement is usually referred to as a "null weak measurement" (or called partial-collapse measurement),[44] and should not be confused with "weak value"[43] as used in Refs. [38]–[42], which corresponds to the post-selected weak measurement of an operator. Now, the WM and QMR approach have also been widely applied to various aspects of quantum information processing. It found that the WM and QMR approach can effectively protect quantum entanglements of two qubits[47–49] and qutrits[50] from amplitude-damping decoherence. More interestingly, the famous entanglement sudden death (ESD)[51] can be avoided by the WM and QMR approach.[48] In Ref. [52], the authors investigated the generation of the concurrence of assistance from tripartite to bipartite entanglement by the WM

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and QMR. In Ref. [53], the authors investigated entanglement amplification via local weak measurements. In Ref. [54], the authors discussed the improvement of the fidelity of teleportation through a noisy channel using the WM and QMR. The robust state transfer and entanglement distribution in a single linear open ended spin-1/2 isotropic Heisenberg chain using the WM and QMR was also discussed in Ref. [55]. Up to now, we have not seen any report for the application of the WM and QMR approach in the precision of parameter estimation.

In this paper, the effect of the WM and QMR approach on enhancement of the precision of phase estimation is explored for an exactly solving model consisting of a qubit interacting with a structured reservoir. We derive the exact expressions of the optimal quantum Fisher information (QFI) and success probability of phase estimation. We show that the QFI can be distinctly enhanced by means of the WM and QMR in different regimes by adjusting the strength of the WM. In addition, we also show that the magnitude of the decoherence involved in the WM and QMR can be a general complex number, which extends the applicable scope of the WM and QMR approach as considered in Refs. [44]–[50] and [52]–[55], where the magnitude of the decoherence is only the real number.

The rest of the paper is organized as follows. In Section 2, we give the scheme and detailed solutions. The results and discussion are given in Section 3. Finally, we give the conclusion of our results in Section 4.

2. Scheme and solutions

Before describing our scheme, we take some brief reviews of the QFI and the WM and QMR approach. In an unknown parameter θ estimation, the QFI gives a theoretically achievable limit on its precision. According to the quantum Cramér–Rao (QCR) theorem,[1] the mean square fluctuation of θ satisfies the following QCR inequality:

$$\Delta\theta \geq \Delta\theta_{\mathrm{QCR}} \equiv \frac{1}{\sqrt{vF_\theta}}, \tag{1}$$

where v is the number of independent measurements and F_θ is the QFI. The QFI is defined as

$$F_\theta = \mathrm{Tr}[\rho(\theta)L^2(\theta)], \tag{2}$$

where $\rho(\theta)$ is the density operator of the system, $L(\theta)$ is the symmetric logarithmic derivation (SLD), which are Hermitian operators determined by

$$\frac{\partial\rho(\theta)}{\partial\theta} = \frac{1}{2}[L(\theta)\rho(\theta)+\rho(\theta)L(\theta)]. \tag{3}$$

From Eq. (1), we can see that the large QFI means a high precision of the estimation. Therefore, how to enhance the QFI in quantum metrology becomes an interesting topic.

The WM involved in our discussion is a null weak measurement or partial-collapse measurement,[44] which can be performed by a device that indirectly monitors a qubit. If the device has no signal (namely, the null result case), the qubit was only partially collapsed and let it evolve, which is the so-called WM. If the device has a signal, we discard the result. The corresponding map of WM on the qubit can be expressed as $\mathrm{WM}(p)\,|n\rangle \to (1-p)^{n/2}\,|n\rangle$ $(n=0,1)$, where p $(0 \leq p < 1)$ is the strength of WM. Equivalently, the map of WM in the computational basis $\{|0\rangle, |1\rangle\}$ can also be written as $M^{\mathrm{w}} = \mathrm{diag}\left(1, \sqrt{1-p}\right)$. Obviously, the effect of WM is a reduction of the weight of the excited state $|1\rangle$, so that the excited state $|1\rangle$ will undergo less influence of decoherence. Conversely, the corresponding map of QMR has the following form $\mathrm{QMR}\ (p_{\mathrm{r}})\,|n\rangle \to (1-p)^{(n\oplus 1)/2}\,|n\rangle$, where $\oplus$ stands for an addition mod 2 and p_{r} $(0 \leq p_{\mathrm{r}} < 1)$ is the strength of QMR. Likewise, the map of QMR in the computational basis $\{|0\rangle,\ |1\rangle\}$ can also be represented as

$$M^{\mathrm{r}} = \begin{pmatrix} \sqrt{1-p_{\mathrm{r}}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the effect of QMR is restoration of the weight of the excited state $|1\rangle$. As the weak measurement and quantum measurement reversal are non-unitary operations, our scheme naturally has less than unity success probability.

Our scheme is shown in Fig. 1. Strictly speaking, on the basis of the original phase estimation scheme, as in Ref. [56], before going through the reservoir, the state $U_\phi\,|\psi_{\mathrm{opt}}\rangle$ implemented a WM (see left-dashed-frame) and after undergoing the reservoir in a reasonable time t, a QMR was applied to the corresponding state (see right-dashed-frame). Similar to Ref. [56], the optimal input state may be chosen as $|\psi_{\mathrm{opt}}\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$, which can maximize the QFI of the output state $U_\phi\,|\psi_{\mathrm{opt}}\rangle$[57] and the phase gate operation can be denoted by $U_\phi := |g\rangle\langle g| + \mathrm{e}^{\mathrm{i}\phi}\,|e\rangle\langle e|$, which induces the estimating parameter ϕ, where $\phi = \theta$ represents some inference parameters (see Fig. 1).

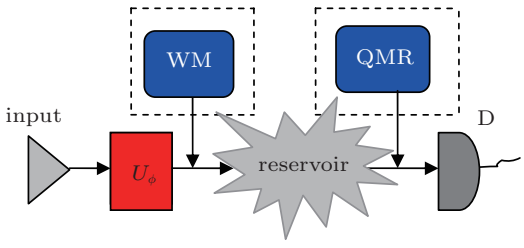


Fig. 1. (color online) Schematic diagram of phase ϕ estimation in a unitary operation. In the original phase estimation scheme[56] (i.e., without dashed-frame operations), the precision of the estimation is impacted by the decoherence of the interaction, the system and reservoir. Here, before going through the reservoir, the state $U_\phi\,|\psi_{\mathrm{opt}}\rangle$ was implemented in a WM (see left-dashed-frame) and after undergoing the reservoir in a reasonable time t, a QMR was applied to the corresponding state (see right-dashed-frame).

We consider a two-level atom interacting with a zero-temperature bosonic reservoir formed by the quantized modes

of a high-Q cavity. The Hamiltonian of the total system under the rotating-wave approximation is given by ($\hbar = 1$)

$$\hat{H} = \omega_0 \hat{\sigma}_+ \hat{\sigma}_- + \sum_{k=1}^{N} \omega_k \hat{a}_k^\dagger \hat{a}_k + \sum_{k=1}^{N} (g_k \hat{\sigma}_- \hat{a}_k^\dagger + g_k^* \hat{\sigma}_+ \hat{a}_k), \quad (4)$$

where $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ are the Pauli raising and lowering operators for the two-level atom, respectively, ω_0 is the Bohr frequency of the two-level atom, $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the annihilation and creation operators for reservoir mode k, ω_k is the frequency of the mode k of the reservoir, and g_k is the coupling constant. Here, we consider one excitation case so that the reduced density matrix for the two-level atom can be written as[58]

$$\rho^a(t) = \begin{pmatrix} \rho_{ee}^a(0)\,|h(t)|^2 & \rho_{eg}^a(0)h(t) \\ \rho_{eg}^{a*}(0)h^*(t) & 1-\rho_{ee}^a(0)\,|h(t)|^2 \end{pmatrix} \quad (5)$$

in the basis $\{|e\rangle, |g\rangle\}$, where the superscript a represents the two-level atom, $h(t)$ denotes the amplitude of the upper level $|e\rangle$ of the qubit initially prepared in $|e\rangle$ and satisfies the following integrodifferential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = -\int_0^t \mathrm{d}t_1 f(t-t_1)h(t_1), \quad (6)$$

where the kernel $f(t-t_1) = \int \mathrm{d}\omega J(\omega)\exp[\mathrm{i}(\omega_0-\omega)(t-t_1)]$ is related to the spectral density $J(\omega)$ of the reservoir. Here, we take the Lorentzian spectral density[58]

$$J(\omega) = \frac{1}{2\pi}\frac{\gamma_0\lambda^2}{(\omega_0-\delta-\omega)^2+\lambda^2}. \quad (7)$$

Here, $\delta = \omega_0 - \omega_\mathrm{c}$ is the detuning of the center frequency of the cavity ω_c and the Bohr frequency of the two-level atom ω_0, the parameter λ defines the spectral width of the coupling, which is associated with the reservoir correlation time by the relations $\tau_\mathrm{B} = \lambda^{-1}$ and the parameter γ_0 is related to the relaxation time scale τ_R by the relations $\tau_\mathrm{R} = \gamma_0^{-1}$. Therefore, the analytic expression of $h(t)$ can be obtained as

$$h(t) = \mathrm{e}^{-(\lambda-\mathrm{i}\delta)t/2}[\cosh(dt/2) + (\lambda-\mathrm{i}\delta)\sinh(dt/2)/d], \quad (8)$$

with $d = \sqrt{(\lambda-\mathrm{i}\delta)^2 - 2\gamma_0\lambda}$.

Before we calculate the QFI of using the WM and QMR, it is noted that the author in Ref. [34] has discussed the non-Markovian effect on the precision of parameter estimation. However, the exact expression of the QFI is absent in his paper. Here, we firstly give the exact expression of the QFI without the WM and QMR, which is very useful to the derivation of the QFI with the WM and QMR. In the estimation of phase parameter ϕ, for initial state $U_\phi \left|\psi_\mathrm{opt}\right\rangle$ the reduced density operator $\rho_1(t)$ in the basis of $\{|e\rangle, |g\rangle\}$ can be written as

$$\rho_1(t) = \frac{1}{2}\begin{pmatrix} |h(t)|^2 & \mathrm{e}^{\mathrm{i}(\phi+\varphi)}|h(t)| \\ \mathrm{e}^{-\mathrm{i}(\phi+\varphi)}|h(t)| & 2-|h(t)|^2 \end{pmatrix}, \quad (9)$$

where φ is an argument of complex number $h(t)$ (when $h(t)$ is real, i.e., in the resonant case $\delta = 0$, $\varphi = 0$). In order to calculate the QFI, we first diagonalize $\rho_1(t)$ as

$$\rho_1(t) = \sum_i p_i(t)\,|\psi_i(t)\rangle\,\langle\psi_i(t)|. \quad (10)$$

The corresponding eigenvalues and eigenvectors are given by

$$p_{1,2}(t) = \frac{1}{2}\left(1 \pm \sqrt{(1-|h(t)|^2)^2 + |h(t)|^2}\right), \quad (11)$$

and

$$\begin{aligned} |\psi_1(t)\rangle &= \sin\alpha(t)\,|g\rangle + \mathrm{e}^{\mathrm{i}(\phi+\varphi)}\cos\alpha(t)\,|e\rangle, \\ |\psi_2(t)\rangle &= \cos\alpha(t)\,|g\rangle - \mathrm{e}^{\mathrm{i}(\phi+\varphi)}\sin\alpha(t)\,|e\rangle, \end{aligned} \quad (12)$$

where

$$\alpha(t) = \arctan\frac{|h(t)|}{|h(t)|^2 - 1 + \sqrt{(1-|h(t)|^2)^2 + |h(t)|^2}}. \quad (13)$$

Taking into account that

$$\partial_\phi\rho_1(t) = \frac{1}{2}\phi|h(t)|\left(\mathrm{e}^{\mathrm{i}(\phi+\varphi)}\,|e\rangle\,\langle g| - \mathrm{e}^{-\mathrm{i}(\phi+\varphi)}\,|g\rangle\,\langle e|\right), \quad (14)$$

thus in the diagonal representation, the matrix elements of the symmetric logarithmic derivative are

$$L_{ij} = \frac{2\langle\psi_i|\,\partial_\phi\rho_1\,|\psi_j\rangle}{p_i + p_j}. \quad (15)$$

Then $L(t)$ is obtained explicitly as

$$L(t) = \mathrm{i}|h(t)|\left(|\psi_1(t)\rangle\,\langle\psi_2(t)| - |\psi_2(t)\rangle\,\langle\psi_1(t)|\right). \quad (16)$$

Finally, the QFI can be obtained as

$$F = |h(t)|^2. \quad (17)$$

Interestingly, from Eq. (17), we can easily see that when $h(t)$ is real, i.e., in the resonant case $\delta = 0$, $|h(t)|^2 = h(t)^2$, which reduces to the result of in Ref. [56].

Now we calculate the QFI of using the WM and QMR. Taking the initial state $U_\phi \left|\psi_\mathrm{opt}\right\rangle$, after by the sequence of the WM with strength p, decoherence, the QMR with strength p_r, the reduced density operator $\rho_2(t)$ in the basis of $\{|e\rangle, |g\rangle\}$ can be written as

$$\rho_2(t) = \frac{1}{2P}\begin{pmatrix} (1-p)|h(t)|^2 & \mathrm{e}^{\mathrm{i}(\phi+\varphi)}\sqrt{1-p}\sqrt{1-p_\mathrm{r}}|h(t)| \\ \mathrm{e}^{-\mathrm{i}(\phi+\varphi)}\sqrt{1-p}\sqrt{1-p_\mathrm{r}}|h(t)| & (1-p_\mathrm{r})\xi \end{pmatrix}, \quad (18)$$

where $\xi = [1+(1-p)(1-|h(t)|^2)]$, $P = [(1-p)|h(t)|^2+(1-p_r)\xi]/2$ is the total success probability of the combined action of the WM, decoherence, and QMR.

From Eq. (18), we can clearly see that if we have *a priori* knowledge about $|h(t)|$, we can perform the optimal reversing operation with $p_r = 1-(1-p)|h(t)|^2$. This similar condition for the real number $h(t)$ has been proposed in Refs. [44] and [46]. However, here we can find that the condition can also be applied to the complex number $h(t)$. The corresponding reduced density operator and the optimal success probability are given by, respectively

$$\rho_2^{\rm opt}(t) = \frac{(1-p)|h(t)|^2}{2P^{\rm opt}} \begin{pmatrix} 1 & e^{i(\phi+\varphi)} \\ e^{-i(\phi+\varphi)} & \xi \end{pmatrix}, \quad (19)$$

$$P^{\rm opt} = (1-p)|h(t)|^2(1+\xi)/2. \quad (20)$$

Similar to the calculation of the QFI without the WM and QMR as Eq. (9), by simple algebra, the corresponding eigenvalues and eigenvectors of Eq. (19) can obtained as

$$\eta_{1,2}(t) = \frac{(1-p)|h(t)|^2}{2P^{\rm opt}} \left(1+\xi \pm \sqrt{5-2\xi+\xi^2}\right), \quad (21)$$

and

$$\begin{aligned} |\eta_1(t)\rangle &= \sin\beta(t)\,|g\rangle + e^{i(\phi+\varphi)}\cos\beta(t)\,|e\rangle, \\ |\eta_2(t)\rangle &= \cos\beta(t)\,|g\rangle - e^{i(\phi+\varphi)}\sin\beta(t)\,|e\rangle, \end{aligned} \quad (22)$$

where

$$\beta(t) = \arctan\frac{2}{1-\xi+\sqrt{5-2\xi+\xi^2}}. \quad (23)$$

Then, $L(t)$ in this diagonal representation can be represented as

$$\begin{aligned} L(t) &= \frac{2i}{2+(1-p)(1-|h(t)|^2)} \\ &\times \left(|\eta_1(t)\rangle\langle\eta_2(t)| - |\eta_2(t)\rangle\langle\eta_1(t)|\right). \end{aligned} \quad (24)$$

The optimal QFI with WM and QMR can be obtained exactly as

$$F^{\rm opt} = \frac{4}{[2+(1-p)(1-|h(t)|^2)]^2}. \quad (25)$$

3. Results and discussion

Having Eqs. (17), (20), and (25) in mind, we can discuss the QFI with and without the WM and QMR, and the success probability. It should be noted that, from Eqs. (20) and (25), we can find that the optimal $F^{\rm opt}$ is an increasing function of the WM strength p, but conversely, the optimal success probability $P^{\rm opt}$ is a decreasing function of the WM strength p. In other words, the larger the QFI is, the less the success probability is. There is hence a trade-off between the optimal QFI and success probability, such that the desirable value of p might change depending on the practical needs involved. Especially, in the limit there are $\lim_{p\to1} F^{\rm opt} = 1$ and $\lim_{p\to1} P^{\rm opt} = 0$. The effects of p on the optimal QFI and success probability will be concretely discussed in the following.

Firstly, we consider the resonant case, i.e., $\delta = 0$. (i) $\lambda = 5\gamma_0$, which corresponds to the Markovian regime. In Fig. 2, we plot the QFI evolution with and without using the WM and QMR and the optimal success probability evolution for different p. From Fig. 2(a), we can clearly see that the QFI without using the WM and QMR (see dotted line) decreases monotonically to zero, but the QFI with using the WM and QMR (see the dashed and solid lines) decreases monotonically to a steady value, i.e., at $\gamma_0 t = 10$, when $p = 0.1$, $F^{\rm opt} = 0.476$; $p = 0.9$, $F^{\rm opt} = 0.907$. In fact, the steady value $F^{\rm opt} = 4/(3-p)^2$ is only a function of p because of $h(\infty) = 0$ in Eq. (8). From Fig. 2(b), we can see that the optimal success probability decreases monotonically with $\gamma_0 t$ to zero. Moreover, the trade-off between the optimal QFI and success probability is indicated in Figs. 2(a) and 2(b), i.e., at $\gamma_0 t = 1$, $F = 0.423$, when $p = 0.1$, $F^{\rm opt} = 0.63$ and $P^{\rm opt} = 0.479$; when $p = 0.9$, $F^{\rm opt} = 0.945$ and $P^{\rm opt} = 0.044$. Therefore, the higher QFI comes at the expense of the success probability. However, the advantage of the WM and QMR approach is the improvement of the QFI under noise. (ii) $\lambda = 0.05\gamma_0$, which corresponds to the non-Markovian regime.

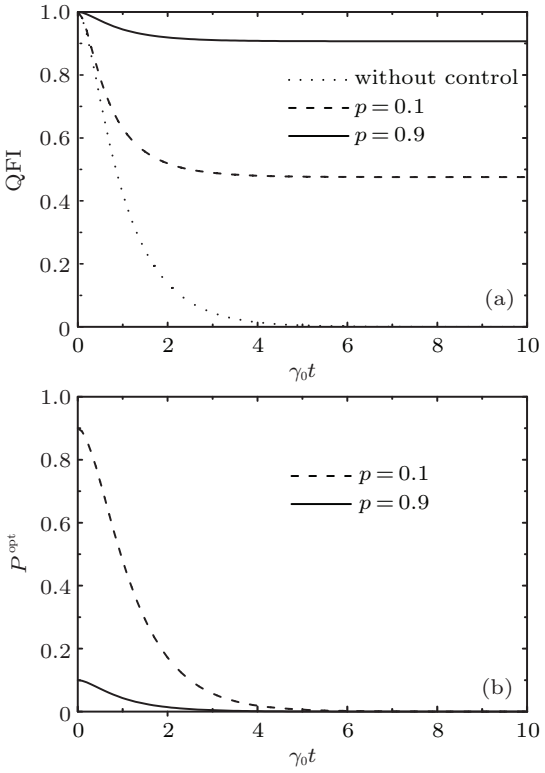


Fig. 2. The QFI with and without the WM and QMR, and the optimal success probability $P^{\rm opt}$ as a function of the dimensionless parameter $\gamma_0 t$ for different WM strengths p ($p = 0.1$ in dashed line and $p = 0.9$ in solid line). (a) The QFI with WM and QMR control and without control (dotted line). (b) The optimal success probability $P^{\rm opt}$. The other parameters are $\lambda = 5\gamma_0$ (in the Markovian regime), $\delta = 0$ (in the resonant case).

In Fig. 3, we plot the QFI evolution with and without using the WM and QMR, and the optimal success probability evolution for different p. Evidently, the dynamical behaviors of the QFI and success probability in the non-Markovian regime are quite different from that in the Markovian regime. It is shown that from Fig. 3, the QFI and success probability indicates the oscillating behavior with $\gamma_0 t$. The survival time of the QFI and success probability is longer than that in the Markovian regime. The reason is that, in the non-Markovian regime, the information or energy, which the atom loses to the reservoir, is later recovered by the atom. In this sense, the memory effect of the reservoir can improve the precision of the parameter estimation, which is pointed out in Ref. [33]. However, we can clearly see that in Fig. 3(a) the precision of the parameter estimation can be further improved by using the WM and QMR (see dashed and solid lines). We calculate some cases, i.e., at $\gamma_0 t = 20$, $F = 0.366$, and when $p = 0.1$, $F^{\rm opt} = 0.605$ and $P^{\rm opt} = 0.423$; when $p = 0.9$, $F^{\rm opt} = 0.939$ and $P^{\rm opt} = 0.038$. Obviously, the trade-off between the optimal QFI and success probability is satisfied. Thus the choice of p mainly depends on the practical needs for the balance between the optimal QFI and success probability.

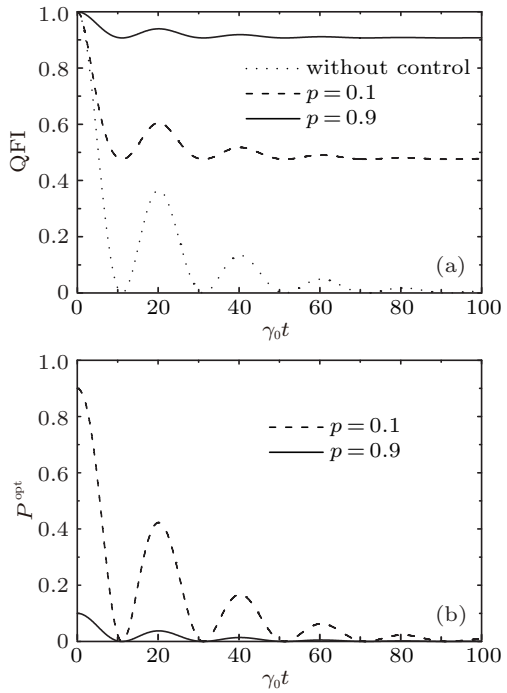


Fig. 3. The QFI with and without the WM and QMR, and the optimal success probability $P^{\rm opt}$ as a function of the dimensionless parameter $\gamma_0 t$ for different WM strengths p ($p = 0.1$ in the dashed line and $p = 0.9$ in the solid line) and without control (dotted line). (a) The QFI with WM and QMR control and without control (dotted line). (b) The optimal success probability $P^{\rm opt}$. The other parameters are $\lambda = 0.05\gamma_0$ (in the non-Markovian regime), $\delta = 0$ (in the resonant case).

Secondly, we consider the off-resonant case, i.e., $\delta = 0.5\gamma_0$ and $\lambda = 0.05\gamma_0$. From Eq. (8), we can clearly see that the magnitude of the decoherence $h(t)$ becomes the general complex number. However, we are aware of some previous works[44–50,52–55] where the involved magnitude of the decoherence is the real number rather than the complex number. So here the applicable scope of the WM and QMR approach is extended. In Fig. 4, we plot the QFI evolution with and without using the WM and QMR, and the optimal success probability evolution for different p. In comparison with Fig. 3 (i.e., the resonant and non-Markovian regime), from Fig. 4, we can find that the amplitudes of the QFI and success probability in the same time are greatly strengthened. The reason can be easily understood: when the value of the detuning δ increases, the effective coupling between the atom and reservoir decreases. So the exchange of information or energy between the atom and the reservoir is neither effective nor adequate. More interestingly, we find that the QFI with the WM and QMR can reach a very high level (see the dashed and solid lines) with a certain finite probability, i.e., at $\gamma_0 t = 80$, $F_\phi = 0.458$, and when $p = 0.1$, $F^{\rm opt} = 0.647$, and $P^{\rm opt} = 0.513$; when $p = 0.9$, $F^{\rm opt} = 0.948$, and $P^{\rm opt} = 0.047$. From the above analysis, we can learn that the WM and QMR approach in enhancing the precision of parameter estimation is very useful.

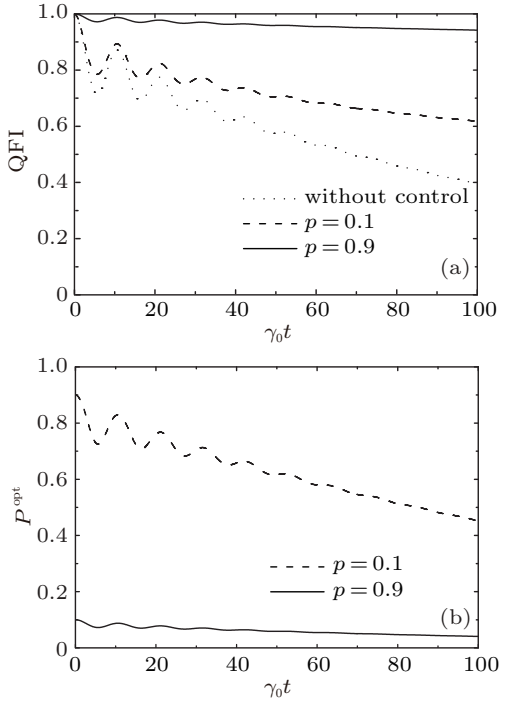


Fig. 4. The QFI and the optimal success probability $P^{\rm opt}$ as a function of the dimensionless parameter $\gamma_0 t$ for different WM strengths p ($p = 0.1$ in the dashed line and $p = 0.9$ in the solid line). (a) The QFI with WM and QMR control and without control (dotted line). (b) The optimal success probability $P^{\rm opt}$. The other parameters are $\lambda = 0.05\gamma_0$ (in the non-Markovian regime), $\delta = 0.5\gamma_0$ (in the off-resonant case).

4. Conclusions

In conclusion, we have proposed an alternative approach to enhance the precision of an unknown phase estimation in quantum metrology by means of the WM and QMR approach. We derive the exact expressions of the optimal QFI and success probability for an exactly solving model consisting of a

two-level atom interacting with a structured reservoir for the Lorentzian spectral density. By investigation, we find that the precision of the phase estimation can be distinctly enhanced by the WM and QMR in different regimes. More interestingly, we find that the magnitude of the decoherence involved in the WM and QMR approach can be a general complex number, which extends the applicable scope of the WM and QMR approach as considered in Refs. [44]–[50] and [52]–[55] where the magnitude of the decoherence is only the real number. Moreover, here we only study the structured reservoir of the Lorentzian spectral density, but a similar idea can be extended to deal with other spectral densities, i.e., $J(\omega) = \eta\omega^s\omega_c^{1-s}\mathrm{e}^{-\omega/\omega_c}$ (sub-Ohmic if $0 < s < 1$, Ohmic if $s = 1$, and super-Ohmic $s > 1$),[59] which deserves further investigation in future. Finally, we give a brief discussion of the experimental feasibility for our scheme. Here, we restrict our discussion to an optical system (a single-photon polarization qubit system is equivalent to a two-level atom system, i.e., $|e\rangle_s \leftrightarrow |1\rangle_s$, $|g\rangle_s \leftrightarrow |0\rangle_s$). Without using the WM and QMR operations, some experimental realizations of quantum phase estimation for an optical system have been demonstrated in Refs. [57], [60], and [61]. The WM and QWR operations have also been implemented for an optical system in Refs. [45], [46], and [48]. For example, in an optical system, the qubit states are encoded by the polarization of light (i.e., $|1\rangle_s$ denotes the vertical polarization state and $|0\rangle_s$ denotes the horizontal polarization state). The WM and QWR operations on the qubit are implemented by the Brewster-angle glass plates (BPs) and half-wave plates (HWPs).[45,46,48] Since the BP only probabilistically reflects the vertical polarization state $|1\rangle_s$ but completely transmits the horizontal polarization state $|0\rangle_s$, a single-photon polarization qubit found behind the BP means that the M^{w} of a WM is implemented. In addition, the M^{r} of a QWR can be rewritten as $M^{\mathrm{r}} = FM^{\mathrm{w}}F$, where

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the bit-flip operation. Therefore, a QMR can be constructed from the following three sequential operations on the qubit: bit-flip, another qubit weak measurement and bit-flip, which also has been implemented by two HWPs and a BP, as shown in Refs. [45], [46], and [48]. In a word, our scheme may be tested using current techniques in optical systems.

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