Static and adaptive feedback control for synchronization of different chaotic oscillators with mutually Lipschitz nonlinearities*

Muhammad Riaz[a)b)], Muhammad Rehan[a)†], Keum-Shik Hong[c)], Muhammad Ashraf[b)], and Haroon Ur Rasheed[a)]

[a)] Department of Electrical Engineering, Pakistan Institute of Engineering and Applied Sciences (PIEAS), Islamabad, Pakistan
[b)] Department of Electronics Engineering, Mohammad Ali Jinnah University, Islamabad, Pakistan
[c)] Department of Cogno-Mechatronics Engineering and School of Mechanical Engineering, Pusan National University; 2 Busandaehak-ro, Geumjeong-gu, Busan 609-735, Republic of Korea

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This paper addresses the control law design for synchronization of two different chaotic oscillators with mutually Lipschitz nonlinearities. For analysis of the properties of two different nonlinearities, an advanced mutually Lipschitz condition is proposed. This mutually Lipschitz condition is more general than the traditional Lipschitz condition. Unlike the latter, it can be used for the design of a feedback controller for synchronization of chaotic oscillators of different dynamics. It is shown that any two different Lipschitz nonlinearities always satisfy the mutually Lipschitz condition. Applying the mutually Lipschitz condition, a quadratic Lyapunov function and uniformly ultimately bounded stability, easily designable and implementable robust control strategies utilizing algebraic Riccati equation and linear matrix inequalities, are derived for synchronization of two distinct chaotic oscillators. Furthermore, a novel adaptive control scheme for mutually Lipschitz chaotic systems is established by addressing the issue of adaptive cancellation of unknown mismatch between the dynamics of different chaotic systems. The proposed control technique is numerically tested for synchronization of two different chaotic Chua's circuits and for obtaining identical behavior between the modified Chua's circuit and the Rössler system.

Keywords: control theory and feedback, synchronization, mutually Lipschitz nonlinearity, adaptive control system

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1. Introduction

Synchronization of chaos, observed in naturally occurring processes, has a significant impact on biological, chemical, and physical systems, and an important role in applied science fields, such as medicine and engineering.[1–11] Two or more synchronous chaotic systems either arise naturally due to strong coupling effects, or are intentionally brought about by application of a control law. In the former case, a control law is derived to investigate the properties of synchronous oscillators; in the latter, a control law formulation is required in order to achieve chaos synchronization for different applications in the areas of secure communication, aerospace technology, information processing, image processing, optics, and medical therapies.[10–23]

Numerous adaptive, evolutionary, intelligent, optimal and robust control methodologies based on neural networks, state feedback and fuzzy logic for synchronization of identical chaotic systems and attainment of asymptotic (or exponential) stability, finite-time stability, robustness, disturbance rejection, desired steady-state performance, improved transient response, and noise handling have been investigated.[9–11,24–30] Control strategies have been applied to cope with various circumstances, such as input saturation, slope bounds, time delays, and unknown dynamics, and also to deal with different dynamics including Lure oscillators, Rössler systems, Chua's circuits, FitzHugh–Nagumo networks, and Lipschitz structures.[10,30–34] However, synthesis of control schemes for synchronization of different chaotic identities are lacking in the literature; and in fact, this problem requires significant research attention owing to various applications of the chaos synchronization phenomenon.

Among the research work on chaos synchronization of unlike dynamical systems, a few exceptional examples require specific mention. In Refs. [35]–[37], adaptive control schemes were developed, and adaptation laws were formulated, to cope with the problem of the synchronization of two different chaotic oscillators for unknown parameters. Sliding-mode control strategies for synchronization of distinct chaotic systems under disturbances, slope-restricted input nonlinearity, and different types of uncertainties have been addressed.[38–40] Recently, a control methodology was developed for chaos synchronization of non-identical fractional-order systems with different numbers of states.[41] An adaptive synchronization approach entailing a unified chaotic oscillator and a cellular network for development of an asymmetric image cryptosystem utilizing Lyapunov stability theory is intro-

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†Corresponding author. E-mail: rehanqau@gmail.com
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duced in a recent work.[42] However, these control strategies are sufficiently computationally complex for real-time implementation. Moreover, further research is needed to classify various types of chaotic oscillators according to their dynamical characteristics and to design synchronization controllers derived from those properties.

In this paper, the problem of the synthesis of static and adaptive state feedback controllers for synchronization of two different chaotic systems under bounded disturbance is addressed. The dynamics of the chaotic systems are assumed to satisfy the mutually Lipschitz condition provided herein, which is more general than the traditional Lipschitz condition.[43–45] The conventional Lipschitz condition, used often to derive control laws for synchronization of identical chaotic oscillators,[33,46,47] is inapplicable to different chaotic oscillators. The proposed mutually Lipschitz condition offers the advantage of addressing two different nonlinear functions. The properties of individual nonlinearities satisfying the mutually Lipschitz condition are investigated in order to examine its parameters. The mutually Lipschitz condition is applied along with the Lyapunov stability and uniformly ultimately bounded stability theories to derive the simplest state feedback control law for chaos synchronization. An algebraic Riccati equation based control methodology is formulated, and further, a less conservative robust control strategy utilizing linear matrix inequalities (LMIs) is established for chaos synchronization against disturbances. The proposed control strategy, owing to utilization of the mutually Lipschitz condition, is uncomplicated in design and straightforward in implementation, as compared with the conventional schemes for synchronization of unlike oscillators. Additionally, a novel adaptive control strategy is proposed for cancellation of mismatch between chaotic-system nonlinearities and avoidance of a large controller gain. It should be emphasized that a new adaptive control scheme can be applied for chaos synchronization without any requirement for dynamic inversion of the input matrix of the slave system. Even if the complete dynamics of the chaotic systems are unknown or uncertain, the resultant control schemes can be readily applied for chaos synchronization by utilizing the knowledge of the parameters of the mutually Lipschitz condition. To the best of our knowledge, the proposed mutually Lipschitz condition is provided for the first time, and furthermore this condition is analyzed and applied for robust and adaptive synchronization control of different chaotic systems. Simulation results for the proposed control methodology are applied both to the synchronization of two different Chua's circuits and to the synchronization of the Rössler system with the modified Chua's circuit.

This paper is organized as follows. Section 2 describes the chaotic systems. Section 3 addresses the mutually Lipschitz condition, and Section 4 derives the proposed control strategies for chaos synchronization. Section 5 presents and analyzes the numerical simulation results. Section 6 draws conclusions. Standard notation is used in this paper. For a vector $\boldsymbol{z}$, $\|z\|$ denotes the Euclidean norm. For entries y_i $(i = 1,2,...,n)$, $\mathrm{diag}(y_1,y_2,...,y_n)$ represents a diagonal matrix. A positive-definite (or a negative-definite) matrix $\boldsymbol{Q}$ is represented by $\boldsymbol{Q} > 0$ (or $\boldsymbol{Q} < 0$). The transpose of a matrix $\boldsymbol{Q}$ is denoted by $\boldsymbol{Q}^{\mathrm{T}}$.

2. System description

Consider a chaotic system, deemed as the master system and described by

$$\frac{\mathrm{d}x_{\mathrm{m}}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{m}} + \boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) + \boldsymbol{d}_{\mathrm{m}}, \quad \boldsymbol{x}_{\mathrm{m}}(0) = \boldsymbol{x}_{\mathrm{m0}}, \tag{1}$$

where $\boldsymbol{x}_{\mathrm{m}} \in \boldsymbol{R}^{\mathrm{n}}$ and $\boldsymbol{d}_{\mathrm{m}} \in \boldsymbol{R}^{\mathrm{n}}$ denote the state of the system and the input disturbance to the oscillator, respectively, the vector $\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) \in \boldsymbol{R}^{\mathrm{n}}$ represents a time-varying function, and $\boldsymbol{x}_{\mathrm{m}}(0) = \boldsymbol{x}_{\mathrm{m0}}$ is the initial condition of the chaotic oscillator. The slave chaotic system is given by

$$\frac{\mathrm{d}x_{\mathrm{s}}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{s}} + \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}}) + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{d}_{\mathrm{s}}, \quad \boldsymbol{x}_{\mathrm{s}}(0) = \boldsymbol{x}_{\mathrm{s0}}, \tag{2}$$

where $\boldsymbol{x}_{\mathrm{s}} \in \boldsymbol{R}^{\mathrm{n}}$, $\boldsymbol{u} \in \boldsymbol{R}^{m}$, and $\boldsymbol{d}_{\mathrm{s}} \in \boldsymbol{R}^{\mathrm{n}}$ denote the state of the system, the control input to the oscillator, and the disturbance effects, respectively. Clearly, the functions $\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) \in \boldsymbol{R}^{\mathrm{n}}$ and $\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}}) \in \boldsymbol{R}^{\mathrm{n}}$, capable of representing both the linear and the nonlinear components, are taken to be different for the two oscillators. The linear parts in Eqs. (1) and (2) are represented by the same matrix $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$. The variations in the nonlinear parts for the master and the slave oscillators can be included in the functions $\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})$ and $\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})$, respectively. The input matrix $\boldsymbol{B} \in \boldsymbol{R}^{n\times m}$ has constant entries. Defining $\boldsymbol{e} = \boldsymbol{x}_{\mathrm{m}} - \boldsymbol{x}_{\mathrm{s}}$, the synchronization error dynamics can be written as

$$\begin{aligned} &\frac{\mathrm{d}e}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}}) - \boldsymbol{B}\boldsymbol{u} + \boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}, \\ &\boldsymbol{e}(0) = \boldsymbol{x}_{\mathrm{m0}} - \boldsymbol{x}_{\mathrm{s0}}. \end{aligned} \tag{3}$$

Assumption 1 Disturbances $\|\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}\| \le \boldsymbol{d}_{\max}$ and $\boldsymbol{d}_{\mathrm{s}}$ are bounded such that $\|\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}\|^2 \le \boldsymbol{d}_{\max}$. Note that if $\|\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}\| \le \boldsymbol{d}_{\max}$ and $\boldsymbol{d}_{\mathrm{s}}$ are bounded in the Euclidean norm sense, the relation $\|\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}\|^2 \le \boldsymbol{d}_{\max}$ can be easily verified.

The purpose of the present study is to develop static feedback and adaptive control strategies for synchronization of two different chaotic oscillators (1) and (2) under bounded disturbances. For this reason, in the next section we define, for two nonlinearities $\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})$ and $\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})$, the mutually Lipschitz condition that can be used in the derivation of a suitable control law.

3. Mutually Lipschitz nonlinearities

First, two mutually Lipschitz nonlinearities are defined as follows.

Definition 1 Nonlinear functions $\boldsymbol{f}(t,\boldsymbol{x}) \in \boldsymbol{R}^{\mathrm{n}}$ and $\boldsymbol{g}(t,\bar{\boldsymbol{x}}) \in \boldsymbol{R}^{\mathrm{n}}$ for all $\boldsymbol{x},\bar{\boldsymbol{x}} \in \boldsymbol{R}^{\mathrm{p}}$ are said to be (globally) mutually Lipschitz, if

$$\|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\|^2 \le l_{\max}^2 \|\boldsymbol{x}-\bar{\boldsymbol{x}}\|^2 + \phi_{\max} \tag{4}$$

for scalars $l_{\max} \ge 0$ and $\phi_{\max} \ge 0$.

Definition 2 Nonlinear functions $\boldsymbol{f}(t,\boldsymbol{x}) \in \boldsymbol{R}^{\mathrm{n}}$ and $\boldsymbol{g}(t,\bar{\boldsymbol{x}}) \in \boldsymbol{R}^{\mathrm{n}}$ are said to be locally mutually Lipschitz, if condition (4) is satisfied for all $\boldsymbol{x},\bar{\boldsymbol{x}} \in \boldsymbol{\Omega} \subset \boldsymbol{R}^{\mathrm{p}}$ with scalars $l_{\max} \ge 0$ and $\phi_{\max} \ge 0$.

Condition (4) is called the mutually Lipschitz condition. The dimensions of $\boldsymbol{x}$ and $\boldsymbol{f}(t,\boldsymbol{x})$ are taken to be different for the general case; however, usually, $n = p$ can be assumed as seen in the present case. It can be verified that condition (4) is more general than the traditional Lipschitz condition. Mutually Lipschitz condition (4) reduces to the Lipschitz condition with Lipschitz constant $l_{\max}$, if $\boldsymbol{f}(t,\boldsymbol{x}) = \boldsymbol{g}(t,\boldsymbol{x})$ is chosen for $\phi_{\max} = 0$ and $l_{\max} \ne 0$.

After defining the mutually Lipschitz nonlinearities, a task essential to identification of the two mutually Lipschitz functions, $\boldsymbol{f}(t,\boldsymbol{x})$ and $\boldsymbol{g}(t,\bar{\boldsymbol{x}})$, is to determine the characteristics of the nonlinearities. Another major concern is to find a method for computing parameters $l_{\max}$ and $\phi_{\max}$. These problems are investigated in the remainder of this section.

Proposition 1 Suppose that nonlinear functions $\boldsymbol{f}(t,\boldsymbol{x}) \in \boldsymbol{R}^{\mathrm{n}}$ and $\boldsymbol{g}(t,\bar{\boldsymbol{x}}) \in \boldsymbol{R}^{\mathrm{n}}$ for all $\boldsymbol{x},\bar{\boldsymbol{x}} \in \boldsymbol{R}^{\mathrm{p}}$ belong to Lipschitz nonlinearities with Lipschitz constants l_{f} and l_{g}, respectively. Then the inequalities

$$\begin{aligned} &\|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\|^2 \\ \le\ & (1+\varepsilon) l_{\mathrm{f}}^2 \|\boldsymbol{x}-\bar{\boldsymbol{x}}\|^2 + \left(1+\varepsilon^{-1}\right)\delta_1, \end{aligned} \tag{5}$$

$$\begin{aligned} &\|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\|^2 \\ \le\ & (1+\varepsilon) l_{\mathrm{g}}^2 \|\boldsymbol{x}-\bar{\boldsymbol{x}}\|^2 + \left(1+\varepsilon^{-1}\right)\delta_2, \end{aligned} \tag{6}$$

are satisfied for any positive scalar ε, where $\delta_1 = \|\boldsymbol{f}(t,\bar{\boldsymbol{x}})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\|^2$ and $\delta_2 = \|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{g}(t,\boldsymbol{x})\|^2$.

Proof The left side of Eq. (5) can be written as

$$\begin{aligned} &\|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\|^2 \\ =\ & \|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\bar{\boldsymbol{x}})\|^2 \\ & + 2\left(\boldsymbol{f}(t,\bar{\boldsymbol{x}})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\right)^{\mathrm{T}}\left(\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\bar{\boldsymbol{x}})\right) \\ & + \|\boldsymbol{f}(t,\bar{\boldsymbol{x}})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\|^2. \end{aligned} \tag{7}$$

It can be easily verified that

$$\begin{aligned} &2\left(\boldsymbol{f}(t,\bar{\boldsymbol{x}})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\right)^{\mathrm{T}}\left(\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\bar{\boldsymbol{x}})\right) \\ \le\ & \varepsilon\|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\bar{\boldsymbol{x}})\|^2 + \varepsilon^{-1}\|\boldsymbol{f}(t,\bar{\boldsymbol{x}})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\|^2. \end{aligned} \tag{8}$$

The traditional Lipschitz condition reveals

$$\|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{f}(t,\bar{\boldsymbol{x}})\| \le l_{\mathrm{f}}\|\boldsymbol{x}-\bar{\boldsymbol{x}}\|, \tag{9}$$

$$\|\boldsymbol{g}(t,\boldsymbol{x})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\| \le l_{\mathrm{g}}\|\boldsymbol{x}-\bar{\boldsymbol{x}}\|. \tag{10}$$

Incorporating Eqs. (8) and (9) into Eq. (7) implies inequality (5). Inequality (6) can be obtained in the same way.

Proposition 2 Two nonlinear functions $\boldsymbol{f}(t,\boldsymbol{x}) \in \boldsymbol{R}^{\mathrm{n}}$ and $\boldsymbol{g}(t,\bar{\boldsymbol{x}}) \in \boldsymbol{R}^{\mathrm{n}}$ for all $\boldsymbol{x},\bar{\boldsymbol{x}} \in \boldsymbol{R}^{\mathrm{p}}$ belonging to Lipschitz nonlinearities are mutually Lipschitz.

Proof Note that the Euclidean norm of any Lipschitz function, say $\boldsymbol{f}(t,\boldsymbol{x})$ or $\boldsymbol{g}(t,\boldsymbol{x})$, is bounded for all $\boldsymbol{x} \in \boldsymbol{R}^{\mathrm{p}}$; therefore, δ_1 and δ_2 remain bounded. Inequality (5) implies mutually Lipschitz condition (4) for $l_{\max} = l_{\mathrm{f}}\sqrt{(1+\varepsilon)}$ and $\phi_{\max} = \left(1+\varepsilon^{-1}\right)\max(\delta_1)$. Mutually Lipschitz condition (4) can also be obtained from inequality (6) for $l_{\max} = l_{\mathrm{g}}\sqrt{(1+\varepsilon)}$ and $\phi_{\max} = \left(1+\varepsilon^{-1}\right)\max(\delta_2)$.

Remark 1 There are two options, inequalities (5)- and (6)-dependent, for selecting parameters $l_{\max}$ and $\phi_{\max}$. The parameter $l_{\max}$, like the Lipschitz constant, is required to be small in a control law design. Therefore, $l_{\max} = \min\left(l_{\mathrm{f}}\sqrt{1+2\varepsilon}, l_{\mathrm{g}}\sqrt{1+2\varepsilon}\right)$ can be a good choice, and the corresponding value of $\phi_{\max}$ can be selected. Parameter $\phi_{\max}$ can be chosen for a bounded region of $\boldsymbol{x}$ (or $\bar{\boldsymbol{x}}$), as it can be difficult to determine for the entire space. Further, selection of $\phi_{\max}$ for a local bounded region is quite reasonable in the case of chaos synchronization, because the states of (the master and the slave) chaotic systems remain in bounded regions and these regions can be estimated easily via numerical simulations.

Remark 2 It has been validated that the two different Lipschitz nonlinearities are always mutually Lipschitz. Similar results provided in Propositions 1 and 2 can be derived for the locally mutually Lipschitz nonlinearities, if $\boldsymbol{x},\bar{\boldsymbol{x}} \in \boldsymbol{\Omega}$. In that case, inequalities (5) and (6) are satisfied for the locally Lipschitz nonlinearities $\boldsymbol{f}(t,\boldsymbol{x})$ and $\boldsymbol{g}(t,\boldsymbol{x})$ satisfying Eqs. (9) and (10) in a local region $\boldsymbol{x},\bar{\boldsymbol{x}} \in \boldsymbol{\Omega} \subset \boldsymbol{R}^{\mathrm{p}}$. Consequently, the control strategies developed for synchronization of the globally mutually Lipschitz nonlinear systems can be readily applied to the locally mutually Lipschitz nonlinear systems.

Now, to devise control strategies for synchronization of chaotic systems (1) and (2), the following assumption is made.

Assumption 2 Nonlinear functions $\boldsymbol{f}(t,\boldsymbol{x})$ and $\boldsymbol{g}(t,\bar{\boldsymbol{x}})$ are mutually Lipschitz.

How the knowledge of the proposed mutually Lipschitz condition (4) can be incorporated to synchronize two different systems (1) and (2) will be explored in the following section.

4. Feedback control

The structure of the proposed static state feedback controller is chosen as

$$\boldsymbol{u} = \boldsymbol{K}\boldsymbol{e}, \tag{11}$$

where $\boldsymbol{K} \in \boldsymbol{R}^{m\times n}$ is the controller gain to be determined. Incorporating Eq. (11) into Eq. (3) yields

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t} = (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})\boldsymbol{e} + \boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}}) + \boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}, \\ &\boldsymbol{e}(0) = \boldsymbol{x}_{\mathrm{m}0} - \boldsymbol{x}_{\mathrm{s}0}. \end{aligned} \tag{12}$$

Now a matrix inequality based approach to the resolution of the synchronization dilemma is provided for mutually Lipschitz nonlinear systems (1) and (2) in the form of the following theorem.

Theorem 1 Consider mutually Lipschitz nonlinear systems (1) and (2) satisfying Assumptions 1 and 2, and suppose that there exist a positive-definite symmetric matrix $\boldsymbol{P} \in \boldsymbol{R}^{n\times n}$ and a positive scalar λ such that algebraic Riccati inequality

$$\begin{aligned} &\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} \\ &- \boldsymbol{P}\boldsymbol{B}\boldsymbol{K} + 2\boldsymbol{P}^2 + l_{\max}^2\boldsymbol{I} + \lambda\boldsymbol{I} < 0 \end{aligned} \tag{13}$$

is satisfied. Then the control law (11) ensures uniformly ultimately bounded synchronization error $\boldsymbol{e} = \boldsymbol{x}_{\mathrm{m}} - \boldsymbol{x}_{\mathrm{s}}$ in region $\|\boldsymbol{e}\|^2 \le (\phi_{\max} + d_{\max})/\lambda$. Furthermore, the error can be minimized by selecting a large value of λ.

Proof Consider a quadratic Lyapunov function

$$\boldsymbol{V}(t,\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}. \tag{14}$$

The time derivative (14) along (11) is

$$\begin{aligned} \dot{\boldsymbol{V}}(t,\boldsymbol{e}) \;=\; & \boldsymbol{e}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\right)\boldsymbol{e} \\ & + \left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} \\ & + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right) \\ & + (\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}). \end{aligned} \tag{15}$$

Incorporating the inequalities

$$\begin{aligned} &2\left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} \\ &\le \left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right)^{\mathrm{T}}\left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right) \\ &\quad + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{P}\boldsymbol{e}, \end{aligned} \tag{16}$$

$$\begin{aligned} &2(\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} \\ &\le (\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}})^{\mathrm{T}}(\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}) + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{P}\boldsymbol{e}, \end{aligned} \tag{17}$$

into Eq. (15) leads to

$$\begin{aligned} \dot{\boldsymbol{V}}(t,\boldsymbol{e}) \;\le\; & \boldsymbol{e}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K} + 2\boldsymbol{P}^2\right)\boldsymbol{e} \\ & + \left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right)^{\mathrm{T}}\left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}}) - \boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right) \\ & + (\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}})^{\mathrm{T}}(\boldsymbol{d}_{\mathrm{m}} - \boldsymbol{d}_{\mathrm{s}}). \end{aligned} \tag{18}$$

Applying Assumptions 1 and 2 results in

$$\begin{aligned} &\dot{\boldsymbol{V}}(t,\boldsymbol{e}) \\ &\le \boldsymbol{e}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K} + 2\boldsymbol{P}^2 + l_{\max}^2\boldsymbol{I}\right)\boldsymbol{e} \\ &\quad + \phi_{\max} + d_{\max}. \end{aligned} \tag{19}$$

Inequality (13) reveals $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K} + 2\boldsymbol{P}^2 + l_{\max}^2\boldsymbol{I} < -\lambda\boldsymbol{I}$, which along with (19) implies

$$\dot{\boldsymbol{V}}(t,\boldsymbol{e}) < -\lambda\|\boldsymbol{e}\|^2 + \phi_{\max} + d_{\max}. \tag{20}$$

Based on Eq. (20), $\dot{\boldsymbol{V}}(t,\boldsymbol{e}) < 0$, if $\|\boldsymbol{e}\|^2 > (\phi_{\max} + d_{\max})/\lambda$. Hence, the error converges to a sphere $\|\boldsymbol{e}\|^2 \le (\phi_{\max} + d_{\max})/\lambda$, the size of which can be minimized by maximizing λ.

There are two issues pertinent to the approach developed in Theorem 1. First, selection of matrices $\boldsymbol{P}$ and $\boldsymbol{K}$ is not an easy task; second, this might not offer optimization of parameter λ to achieve maximum rejection of unwanted disturbance signals. Therefore, an LMI-based approach addressing these issues is presented in the following theorem.

Theorem 2 Consider mutually Lipschitz nonlinear systems (1) and (2) satisfying Assumptions 1 and 2, and suppose that there exist a symmetric matrix $\boldsymbol{Q} \in \boldsymbol{R}^{n\times n}$, a matrix $\boldsymbol{M} \in \boldsymbol{R}^{m\times n}$, and a scalar μ. By solving the optimization $\min\mu$,

$$\boldsymbol{Q} > 0, \;\; \mu > \mu_{\mathrm{o}} \ge 0, \tag{21}$$

$$\begin{bmatrix} \boldsymbol{Q}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A}\boldsymbol{Q} - \boldsymbol{M}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} - \boldsymbol{B}\boldsymbol{M} + 2\boldsymbol{I} & \boldsymbol{Q}l_{\max} & \boldsymbol{Q} \\ * & -\boldsymbol{I} & 0 \\ * & * & -\mu\boldsymbol{I} \end{bmatrix} < 0, \tag{22}$$

the parameter $\boldsymbol{K}$ of control law (11) can be obtained as $\boldsymbol{K} = \boldsymbol{M}\boldsymbol{Q}^{-1}$. Furthermore, control law $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{e}$ ensures uniformly ultimately bounded synchronization error $\boldsymbol{e} = \boldsymbol{x}_{\mathrm{m}} - \boldsymbol{x}_{\mathrm{s}}$ in region $\|\boldsymbol{e}\|^2 \le \mu(\phi_{\max} + d_{\max})$.

Proof Pre- and post-multiplication of algebraic Riccati inequality (13) by $\boldsymbol{P}^{-1}$ and furthermore substitution of $\boldsymbol{P}^{-1} = \boldsymbol{Q}$, $\lambda^{-1} = \mu$, and $\boldsymbol{M} = \boldsymbol{K}\boldsymbol{Q}$, obtains

$$\begin{aligned} &\boldsymbol{Q}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A}\boldsymbol{Q} - \boldsymbol{M}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} - \boldsymbol{B}\boldsymbol{M} \\ &+ 2\boldsymbol{I} + l_{\max}^2\boldsymbol{Q}^2 + \mu^{-1}\boldsymbol{Q}^2 < 0. \end{aligned} \tag{23}$$

Applying the Schur complement yields LMI (22). Region $\|\boldsymbol{e}\|^2 \le (\phi_{\max} + d_{\max})/\lambda$, in which the error remains uniformly ultimately bounded, is equivalent to $\|\boldsymbol{e}\|^2 \le \mu(\phi_{\max} + d_{\max})$ by application of $\lambda^{-1} = \mu$, and $\max\lambda$ is changed to $\min\mu$. Parameter μ_{o} is then introduced to set a lower bound on μ. This completes the proof of Theorem 2.

Remark 3 Significantly, by application of the proposed mutually Lipschitz condition, the simplest state feedback control law can be derived as in Theorems 1 and 2 for synchronization of two different chaotic systems. Furthermore, robustness of the resultant control strategy against disturbances and $\phi_{\max}$ can be achieved by maximizing λ (or minimizing μ). It is also significant that a control law can be readily designed even if nonlinear functions $\boldsymbol{f}(t,\boldsymbol{x})$ and $\boldsymbol{g}(t,\boldsymbol{x})$ are unknown because knowledge of $l_{\max}$ is sufficient for that purpose.

Remark 4 The traditional control strategies for synchronization of different chaotic systems are,[35–42] to a considerable extent, computationally complex in their implementation. Contrastingly, design of the proposed control strategy, by virtue of the mutually Lipschitz condition and the uncomplicated state feedback control law, is straightforward for optimal results via LMIs, as well as intuitively implementable.

Remark 5 Motivated by the Lipschitz-like condition,[48–50] a more general form of (4), called the mutually Lipschitz-like condition, is

$$\|\boldsymbol{f}(t,\boldsymbol{x})-\boldsymbol{g}(t,\bar{\boldsymbol{x}})\|^2 \leq \|\boldsymbol{L}_{\max}(\boldsymbol{x}-\bar{\boldsymbol{x}})\|^2+\phi_{\max}, \tag{24}$$

where $\boldsymbol{L}_{\max} \in \boldsymbol{R}^{n\times n}$. Condition (24) reduces to (4) for $\boldsymbol{L}_{\max}^{\mathrm{T}}\boldsymbol{L}_{\max} = l_{\max}^2\boldsymbol{I}$. It can be validated that two nonlinearities $\boldsymbol{f}(t,\boldsymbol{x})$ and $\boldsymbol{g}(t,\boldsymbol{x})$ satisfying (24) always assure (4), because a suitable value of $l_{\max}$ can be chosen such that $\boldsymbol{L}_{\max}^{\mathrm{T}}\boldsymbol{L}_{\max} \leq l_{\max}^2\boldsymbol{I}$. In designing synchronization controller (11) for systems (1) and (2), parameters $l_{\max}^2$ and $l_{\max}$ in Theorems 1 and 2 can be replaced by $\boldsymbol{L}_{\max}^{\mathrm{T}}\boldsymbol{L}_{\max}$ and $\boldsymbol{L}_{\max}^{\mathrm{T}}$ (or $\boldsymbol{L}_{\max}$), respectively. For instance, LMI (22) can be replaced by

$$\begin{bmatrix} \boldsymbol{Q}\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{A}\boldsymbol{Q}-\boldsymbol{M}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}-\boldsymbol{B}\boldsymbol{M}+2\boldsymbol{I} & \boldsymbol{Q}\boldsymbol{L}_{\max}^{\mathrm{T}} & \boldsymbol{Q} \\ * & -\boldsymbol{I} & 0 \\ * & * & -\mu\boldsymbol{I} \end{bmatrix} < 0 \tag{25}$$

in Theorem 2 in designing a feedback controller. If matrix $\boldsymbol{L}_{\max}$ is known, LMI (25) can be less conservative to design a feasible synchronization controller for chaotic oscillators (1) and (2).

The approach developed in Theorems 1 and 2 can be used effectively for synchronization of chaotic systems with different, low-mismatch nonlinearities $\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})$ and $\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})$. However, if δ_1 and δ_2 are higher, $\phi_{\max}$ will be larger, resulting in a highgain controller. Such a highgain controller can be conservative in the presence of actuator saturation and measurement noise. To deal with large differences in chaotic system dynamics, we develop an adaptive control strategy based on adaptive cancellation of unknown terms that can be implemented straightforwardly. The proposed adaptive controller is given by

$$\boldsymbol{u} = \boldsymbol{K}\boldsymbol{e}+\boldsymbol{\Phi}(t), \tag{26}$$

where $\boldsymbol{\Phi} \in \boldsymbol{R}^m$ is an adaptive parameter for compensation of disturbance and mismatch between nonlinearities. Incorporating Eqs. (11) and (3) yields

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t} = (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})\boldsymbol{e}+\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})-\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})-\boldsymbol{B}\boldsymbol{\Phi}+\boldsymbol{d}_{\mathrm{m}}-\boldsymbol{d}_{\mathrm{s}}, \\ &\boldsymbol{e}(0) = \boldsymbol{x}_{\mathrm{m}0}-\boldsymbol{x}_{\mathrm{s}0}. \end{aligned} \tag{27}$$

Theorem 3 Consider mutually Lipschitz nonlinear systems (1) and (2) satisfying Assumptions 1 and 2, and suppose that there exist a symmetric matrix $\boldsymbol{Q} \in \boldsymbol{R}^{n\times n}$, a matrix $\boldsymbol{M} \in \boldsymbol{R}^{m\times n}$, and a scalar μ, such that LMIs $\boldsymbol{Q} > 0$, $\mu > 0$, and inequality (22) are satisfied. Then there exists an adaptive controller of the form (26) with adaptation law

$$\dot{\boldsymbol{\Phi}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}-\frac{(\phi_{\max}+d_{\max})\boldsymbol{\Phi}}{\|\boldsymbol{\Phi}\|^2+\sigma}, \tag{28}$$

where σ is a positive infinitesimally small scalar employed for singularity avoidance, such that synchronization error $\boldsymbol{e}$ converges to zero. Parameter $\boldsymbol{K}$ of control law (11) can then be obtained as $\boldsymbol{K} = \boldsymbol{M}\boldsymbol{Q}^{-1}$.

Proof Consider a Lyapunov function given by

$$\boldsymbol{V}(t,\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}+\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}. \tag{29}$$

Time derivative (29) along (27) is given by

$$\begin{aligned} \dot{\boldsymbol{V}}(t,\boldsymbol{e}) = {} & \boldsymbol{e}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\right)\boldsymbol{e} \\ & +\left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})-\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} \\ & +\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})-\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right) \\ & -\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\boldsymbol{\Phi}-\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}+\left(\boldsymbol{d}_{\mathrm{m}}-\boldsymbol{d}_{\mathrm{s}}\right)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} \\ & +\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\left(\boldsymbol{d}_{\mathrm{m}}-\boldsymbol{d}_{\mathrm{s}}\right)+\boldsymbol{\Phi}^{\mathrm{T}}\dot{\boldsymbol{\Phi}}+\dot{\boldsymbol{\Phi}}^{\mathrm{T}}\boldsymbol{\Phi}. \end{aligned} \tag{30}$$

Applying Eqs. (16) and (17), we obtain

$$\begin{aligned} \dot{\boldsymbol{V}}(t,e) \leq {} & \boldsymbol{e}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}+2\boldsymbol{P}^2\right)\boldsymbol{e} \\ & +\left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})-\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right)^{\mathrm{T}}\left(\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})-\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})\right) \\ & -\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\boldsymbol{\Phi}-\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}+\left(\boldsymbol{d}_{\mathrm{m}}-\boldsymbol{d}_{\mathrm{s}}\right)^{\mathrm{T}}\left(\boldsymbol{d}_{\mathrm{m}}-\boldsymbol{d}_{\mathrm{s}}\right) \\ & +\boldsymbol{\Phi}^{\mathrm{T}}\dot{\boldsymbol{\Phi}}+\dot{\boldsymbol{\Phi}}^{\mathrm{T}}\boldsymbol{\Phi}. \end{aligned} \tag{31}$$

Assumptions 1 and 2 reveal that

$$\begin{aligned} & \dot{\boldsymbol{V}}(t,\boldsymbol{e}) \\ \leq {} & \boldsymbol{e}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}+2\boldsymbol{P}^2+l_{\max}^2\boldsymbol{I}\right)\boldsymbol{e} \\ & +\phi_{\max}+d_{\max}-\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\boldsymbol{\Phi}-\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} \\ & +\boldsymbol{\Phi}^{\mathrm{T}}\dot{\boldsymbol{\Phi}}+\dot{\boldsymbol{\Phi}}^{\mathrm{T}}\boldsymbol{\Phi}. \end{aligned} \tag{32}$$

By exploiting the proposed adaptation law, we obtain

$$\begin{aligned} & \dot{\boldsymbol{V}}(t,\boldsymbol{e}) \\ \leq {} & \boldsymbol{e}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}+2\boldsymbol{P}^2+l_{\max}^2\boldsymbol{I}\right)\boldsymbol{e} \\ & +\phi_{\max}+d_{\max}-\frac{(\phi_{\max}+d_{\max})\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}}{\|\boldsymbol{\Phi}\|^2+\sigma}. \end{aligned} \tag{33}$$

The fact that positive scalar σ is infinitesimally small implies

$$\dot{\boldsymbol{V}}(t,\boldsymbol{e}) \leq \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{K} + 2\boldsymbol{P}^2+l_{\max}^2\boldsymbol{I})\boldsymbol{e}. \tag{34}$$

For convergence of synchronization error $\boldsymbol{e}$ to the origin, we require that inequality (13) should be satisfied, which further reveals LMI (22) after congruence transform, $\boldsymbol{Q}=\boldsymbol{P}^{-1}$, as well as the Schur complement. This completes the proof of Theorem 3.

Remark 6 Practically, parameter σ can be selected as a small positive scalar rather than an infinitesimally small number. In that case, inequality (34) entails

$$\dot{\boldsymbol{V}}(t,\boldsymbol{e}) \leq \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{K} + 2\boldsymbol{P}^2+l_{\max}^2\boldsymbol{I})\boldsymbol{e}+\frac{(\phi_{\max}+d_{\max})\,\sigma}{\|\boldsymbol{\Phi}\|^2+\sigma}, \tag{35}$$

which along with (13) implies

$$\dot{\boldsymbol{V}}(t,\boldsymbol{e}) < -\lambda\,\|\boldsymbol{e}\|^2+\frac{(\phi_{\max}+d_{\max})\,\sigma}{\|\boldsymbol{\Phi}\|^2+\sigma}. \tag{36}$$

Hence the synchronization error remains uniformly ultimately bounded in set

$$\|\boldsymbol{e}\|^2 \leq \frac{(\phi_{\max}+d_{\max})\,\sigma}{\lambda\left(\|\boldsymbol{\Phi}\|^2+\sigma\right)}. \tag{37}$$

Note that $(\phi_{\max}+d_{\max})\,\sigma/\left(\|\boldsymbol{\Phi}\|^2+\sigma\right) \leq \phi_{\max}+d_{\max}$. Therefore, the size of the sphere in which the synchronization error will converge is smaller than in the case of the robust control strategy provided in Theorem 2. Thus, a large gain controller, at the cost of additional computation for the adaptive term, might not be required.

Remark 7 The adaptive control strategy proposed in Theorem 3 is advantageous, because it ensures adaptive cancellation of unknown disturbances and mismatch between nonlinearities, specifically by utilizing their bounds. It should be noted that adaptive parameter $\boldsymbol{\Phi}$, unlike the traditional adaptive approaches (e.g., Refs. [35] and [36]), has been used to cancel unknown terms without requiring invertible input matrix $\boldsymbol{B}$.

5. Simulation results

Two simulation examples of controlled synchronization of different chaotic oscillators are provided below.

5.1. Synchronization of two different Chua's circuits

Consider the following two uncertain and different chaotic Chua's circuits:[33,49]

$$\boldsymbol{A}=\begin{bmatrix}-2.548 & 9.1 & 0\\ 1 & -1 & 1\\ 0 & 14.2 & 0\end{bmatrix},$$

$$\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})=\begin{bmatrix}\alpha_1\left(|x_{\mathrm{m}}+\alpha_2|+|x_{\mathrm{m}}-\alpha_3|\right)\\ 0\\ 0\end{bmatrix}, \tag{38}$$

$$\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})=\begin{bmatrix}\beta_1\left(|x_{\mathrm{s}}+\beta_2|+|x_{\mathrm{s}}-\beta_3|\right)\\ 0\\ 0\end{bmatrix},$$

$$d_{\mathrm{m}}=\begin{bmatrix}0.8\sin 70t\\ 0.15\sin 90t\\ 1.2\sin 130t\end{bmatrix}, \tag{39}$$

$$\boldsymbol{d}_{\mathrm{s}}=-\begin{bmatrix}0.7\sin 75t\\ 0.12\sin 100t\\ 1.1\sin 138t\end{bmatrix}. \tag{40}$$

Clearly, functions $\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})$ and $\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})$ are different, and their parameters α_i and β_i $(i=1,2,3)$ are assumed to be unknown and bounded, such that $\alpha_1,\beta_1 \in \begin{bmatrix}0 & 2\end{bmatrix}$ and $\alpha_2,\alpha_3,\beta_2,\beta_3 \in \begin{bmatrix}0 & 1.2\end{bmatrix}$. The feedback gain matrix is determined as

$$\boldsymbol{K}=\begin{bmatrix}53.04 & 8.3 & -10.35\\ 2.07 & 43 & 2.38\\ 13.78 & -15.58 & 44\end{bmatrix}, \tag{41}$$

by selecting $\mu_{\mathrm{o}}=0.01$ and $L_{\max}=\mathrm{diag}(3,0,0)$, and employing the optimization in Theorem 2. The proposed controller is applied to synchronize the Chua's circuits with parameters $\alpha_1=1.9286$, $\beta_1=1.8482$, $\alpha_2=1$, $\beta_2=1.1$, $\alpha_3=1.1$, and $\beta_3=1$. The synchronization error plots shown in Fig. 1 validate the convergence of the synchronization errors to a small region near the origin. Therefore, the proposed methodology outlined in Theorem 1 can be considered to be applicable to different chaotic systems with low mismatch between their dynamics.

5.2. Synchronization of Rössler system and modified Chua's circuit

Consider a modified Chua's circuit as the master system and a Rössler system as the slave system with dynamics,[35] given by

$$\boldsymbol{f}(t,\boldsymbol{x}_{\mathrm{m}})=\begin{bmatrix}10\left(x_{\mathrm{m2}}-\frac{2x_{\mathrm{m1}}^3-x_{\mathrm{m1}}}{7}\right)\\ x_{\mathrm{m1}}-x_{\mathrm{m2}}+x_{\mathrm{m3}}\\ -\frac{100}{7}x_{\mathrm{m2}}\end{bmatrix},$$

$$\boldsymbol{g}(t,\boldsymbol{x}_{\mathrm{s}})=\begin{bmatrix}-x_{\mathrm{s2}}-x_{\mathrm{s3}}\\ x_{\mathrm{s1}}+0.2x_{\mathrm{s2}}\\ 0.2+x_{\mathrm{s3}}\left(x_{\mathrm{s1}}+5.7\right)\end{bmatrix}, \tag{42}$$

and $\boldsymbol{A}=0$. The Lipschitz constant l_{f} is calculated numerically by determining the supremum of the maximum eigenvalues of $(\partial\boldsymbol{f}(t,\boldsymbol{x})/\partial\boldsymbol{x})^{\mathrm{T}}(\partial\boldsymbol{f}(t,\boldsymbol{x})/\partial\boldsymbol{x})$ for $x \in \begin{bmatrix}-2 & 2\end{bmatrix}$. By selecting $\varepsilon=0.1$, the mutually Lipschitz constant is determined to be $l_{\max}=23.72$. By solving the LMIs in Theorem 2, the feedback gain matrix $K=\mathrm{diag}(77.61,77.61,77.61)$ is calculated.

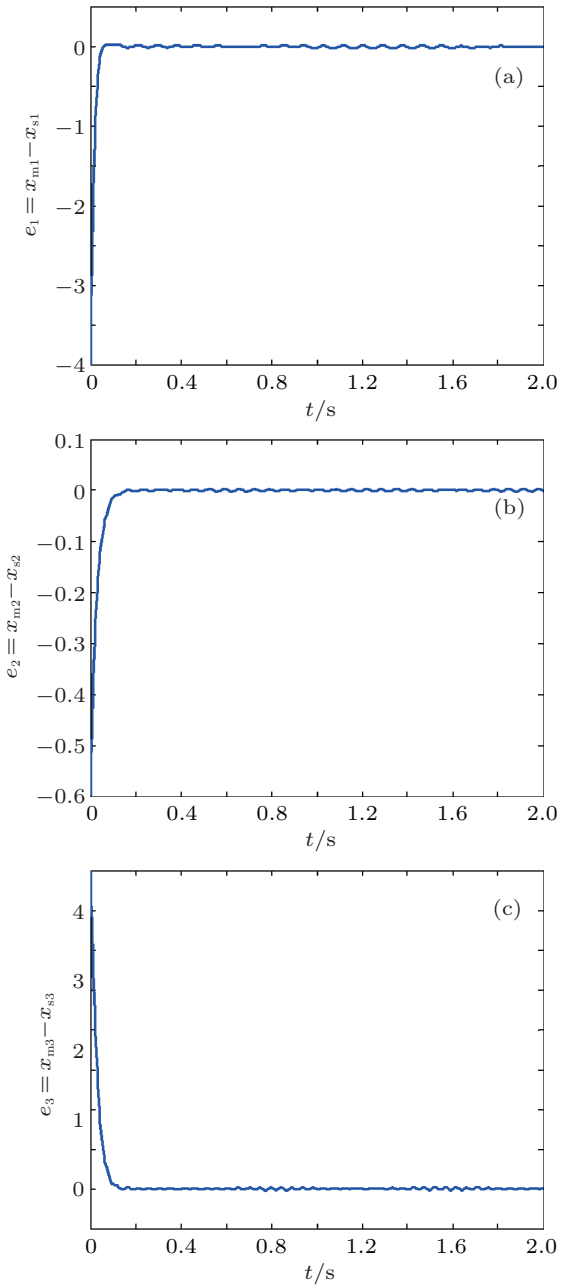


Fig. 1. (color online) Synchronization error plots for two different Chua's circuits in Example 1: (a) e_1; (b) e_2; and (c) e_3.

Figures 2 and 3 show the chaotic behaviors of the master and slave systems without a controller and with the proposed controller, respectively. Figure 2 indicates that in the absence of a controller, the master and the slave systems have their own distinct behaviors. By contrast, figure 3 indicates that with the proposed control methodology, the phase portrait of the Rössler system follows the modified Chua's circuit. Figure 4 provides the relevant synchronization error plots. For time $t < 200$ s, the control signal u is selected as zero, and the synchronization errors exhibit an oscillatory behavior. By application of the proposed controller for $t \geq 200$ s, the synchronization errors converge to a bounded region. In a test of the proposed adaptive control strategy's obtainment of a lowgain controller for $l_{\max} = 2$, the controller gain is

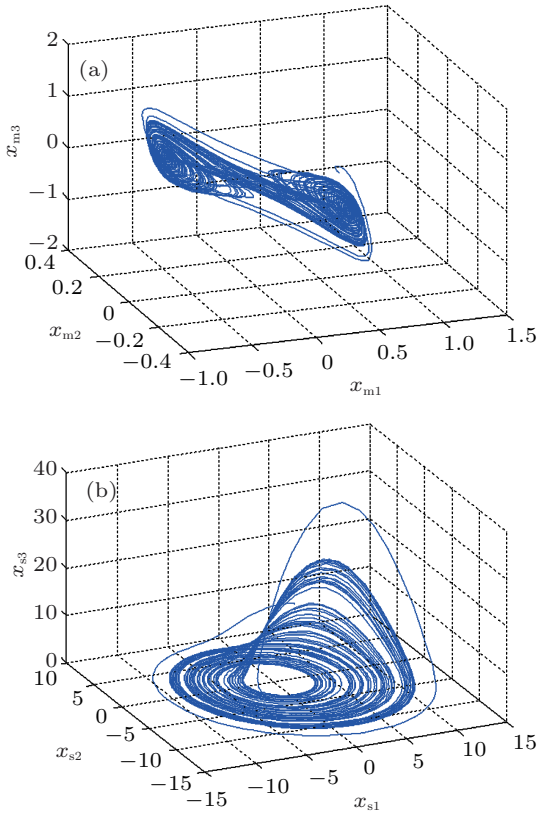


Fig. 2. (color online) Phase portraits of the chaotic systems in Example 2 without application of a controller: (a) phase portrait of the modified Chua's circuit; (b) phase portrait of the Rössler system.

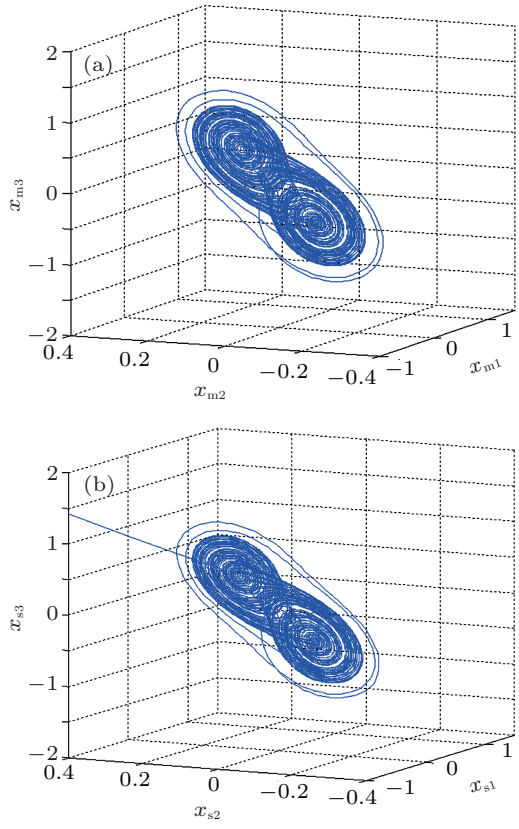


Fig. 3. (color online) Phase portraits of the chaotic systems in Example 2 by application of the proposed control law: (a) phase portrait of the modified Chua's circuit; (b) phase portrait of the RÖssler system.

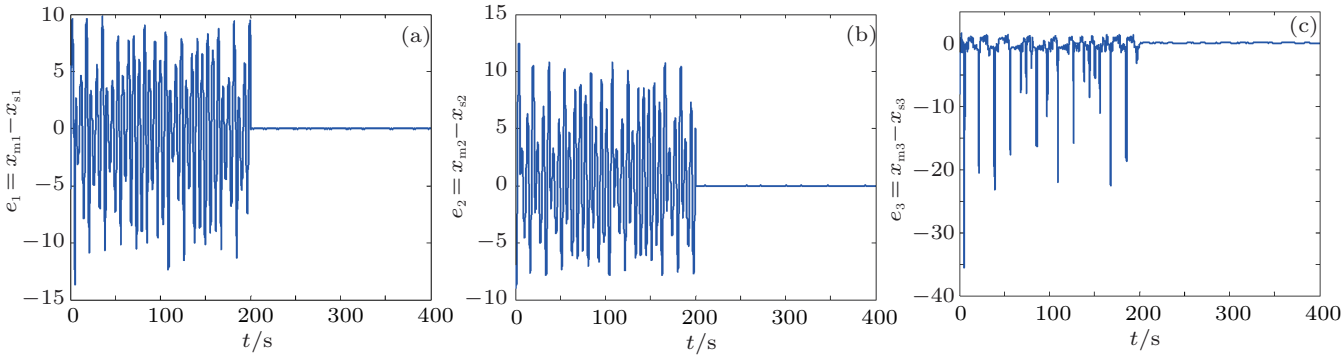


Fig. 4. (color online) Synchronization error plots for the Chua's circuit and the Rössler system in Example 2 using the proposed static state feedback controller: (a) e_1; (b) e_2; and (c) e_3.

$\boldsymbol{K} = \mathrm{diag}(18.96, 18.96, 18.96)$, which is about four times lower than the robust controller gain. Figure 5 plots the results of the proposed adaptive control strategy for $\sigma = 0.05$ and $t = 200$ s. It is noteworthy that the performance of the synchronization error convergence to a bounded region is similar to that in the case of the non-adaptive controller.

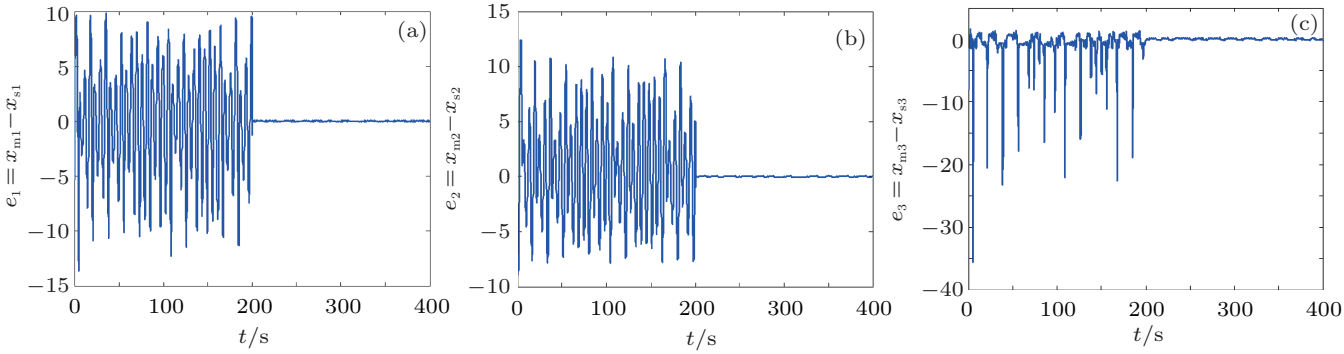


Fig. 5. (color online) Synchronization error plots for the Chua's circuit and the Rössler system in Example 2 using the proposed adaptive feedback control strategy: (a) e_1; (b) e_2; and (c) e_3.

6. Conclusions

This paper addresses uncomplicated static state feedback and adaptive dynamic controller synthesis methodologies for synchronization of different chaotic oscillators containing mutually Lipschitz nonlinearities. Algebraic Riccati-inequality-based and LMI-based formulations are applied to the computation of the proposed controller gain matrix for synchronization of mutually Lipschitz nonlinear systems. The proposed design conditions, developed by using a quadratic Lyapunov function, the mutually Lipschitz condition, and the uniformly ultimately bounded stability theory, offer robustness against disturbance and dynamical perturbations. For synchronization of chaotic systems and cancellation of the nonlinearities arising from mismatch between their dynamics, an adaptive scheme is investigated. The entailed control schemes proposed, uncomplicated in design as well as in implementation, can be applied for synchronization of different chaotic oscillators of unknown dynamics. Two simulation examples entailing synchronization of some of the well-known chaotic oscillators are provided as a demonstration of the effectiveness of the approach.

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