A theorem for quantum operator correspondence to the solution of the Helmholtz equation*

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We propose a theorem for the quantum operator that corresponds to the solution of the Helmholtz equation, i.e.,

$$\iiint V(x_1,x_2,x_3)\,|x_1,x_2,x_3\rangle\langle x_1,x_2,x_3|\,\mathrm{d}^3x = V(X_1,X_2,X_3) = \mathrm{e}^{-\lambda^2/4}:V(X_1,X_2,X_3):,$$

where $V(x_1,x_2,x_3)$ is the solution to the Helmholtz equation $\nabla^2V+\lambda^2V=0$, the symbol : : denotes normal ordering, and X_1,X_2,X_3 are three-dimensional coordinate operators. This helps to derive the normally ordered expansion of Dirac's radius operator functions. We also discuss the normally ordered expansion of Bessel operator functions.

Keywords: normally ordered expansion, radius operators, Helmholtz equation, Bessel operator function

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1. Introduction

Due to the fundamental commutative relation $[X,P]=\mathrm{i}\hbar$, where $\hbar$ is the reduced Planck constant and set to be 1 in this paper, X is the coordinate operator, and P is the momentum operator, the correspondence between the quantum mechanical operators and their classical counterparts has been a major topic since the birth of quantum mechanics. A direct correspondence between an operator $F(X)$ and its classical correspondence $F(x)$ can be established by using Dirac's representation,[1] $\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\langle x|=1$, here $|x\rangle$ is the coordinate eigenstate of the position operator X, i.e.,

$$F(X)=\int_{-\infty}^{\infty}\mathrm{d}xF(x)\,|x\rangle\langle x|. \tag{1}$$

In Refs. [2]–[7], we have introduced the technique of integration within an ordered product (IWOP) of operators and recast the completeness relation of coordinate eigenvector $|x\rangle$ into a normally ordered Gaussian integration

$$\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle\langle x| = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}:\mathrm{e}^{-(x-X)^2}:=1,$$
$$X=(a+a^\dagger)/\sqrt{2}, \tag{2}$$

where a and $a^\dagger$ are creation and annihilation operators, respectively, satisfying $\left[a,a^\dagger\right]=1$, and a and $a^\dagger$ are permutable within : :. Thus

$$F(X)=\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}F(x):\mathrm{e}^{-(x-X)^2}:. \tag{3}$$

For example, when $F(X)=\mathrm{H}_n(X)$, the Hermite polynomial of operator X is

$$\mathrm{H}_n(X)=\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}\mathrm{H}_n(x):\mathrm{e}^{-(x-X)^2}:=2^n:X^n:, \tag{4}$$

which converts $\mathrm{H}_n(X)$ into normal ordering. For operator function $G(X,P)$, Weyl[8] proposed a correspondence rule[2] through the integration with the kernel being the Wigner operator $\Delta(x,p)$[9]

$$G(X,P)=\iint\mathrm{d}x\mathrm{d}p\Delta(x,p)\,g(x,p). \tag{5}$$

Fan[10,11] found that the normal ordering form of $\Delta(x,p)$ is

$$\Delta(x,p)=\frac{1}{\pi}:\mathrm{e}^{-(x-X)^2-(p-P)^2}:. \tag{6}$$

In this work, we will study the quantum operators that correspond to the classical solution of the Helmholtz equation. Our motivation lies in that some special functions are solutions to the Helmholtz equation, so we may derive their operator function's normally ordered expansions. Here we propose the following theorem.

Theorem 1 Suppose that $V(x_1,x_2,x_3)$ satisfies the Helmholtz equation

$$\nabla^2V+\lambda^2V=0, \tag{7}$$

then the quantum operator corresponding to $V(x_1,x_2,x_3)$ is

$$\iiint V(x_1,x_2,x_3)\,|x_1,x_2,x_3\rangle\langle x_1,x_2,x_3|\,\mathrm{d}^3x$$

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$$= V(X_1, X_2, X_3), \quad \mathrm{d}^3x = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3, \tag{8}$$

and its normally ordered expansion is

$$V(X_1, X_2, X_3) = \mathrm{e}^{-\lambda^2/4} : V(X_1, X_2, X_3) : . \tag{9}$$

We will employ the IWOP technique to derive the above theorem. As its application, we can directly derive the normally ordered expansion of Dirac's radius operators. We also find $\mathrm{J}_0(\lambda\hat{r}) = \mathrm{e}^{-\lambda^2/4} : \mathrm{J}_0(\lambda\hat{r}) :$, where J_0 is the Bessel function.

2. Proof of the theorem

Since $V(x, y, z)$ is a solution of the Helmholtz equation (7), it is obvious that $U_{\alpha,\beta,\gamma}(x_1, x_2, x_3) = V(x_1+\alpha, x_2+\beta, x_3+\gamma)$ also satisfies the Helmholtz equation

$$\nabla^2 U + \lambda^2 U = 0, \tag{10}$$

where α, β, γ are real. By introducing the spherical coordinates $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$, and $x_3 = r\cos\theta$, the Laplace operation is given by

$$\begin{aligned}\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2},\end{aligned} \tag{11}$$

where L is the angular momentum operator, and the eigenvalue of L^2 is $l(l+1)$. Now let $\mu = \cos\theta$, we can reform Eq. (10) as

$$\begin{aligned}&\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial\mu}\left(\left(1-\mu^2\right)\frac{\partial U}{\partial\mu}\right)\right.\\ &\left.+ \frac{1}{1-\mu^2}\frac{\partial^2 U}{\partial\varphi^2}\right] + \lambda^2 U = 0.\end{aligned} \tag{12}$$

From Eq. (12), we can see

$$\begin{aligned}&\int_{-1}^{1}\int_{-\pi}^{\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U_{\alpha,\beta,\gamma}}{\partial r}\right)\mathrm{d}\mu\,\mathrm{d}\varphi\\ &= \lambda^2\int_{-1}^{1}\int_{-\pi}^{\pi} U\,\mathrm{d}\mu\,\mathrm{d}\varphi\\ &\quad -\frac{1}{r^2}\int_{-1}^{1}\int_{-\pi}^{\pi}\left[\frac{\partial}{\partial\mu}\left(\left(1-\mu^2\right)\frac{\partial U}{\partial\mu}\right)\right.\\ &\quad \left.+\frac{1}{1-\mu^2}\frac{\partial^2 U}{\partial\varphi^2}\right]\mathrm{d}\mu\,\mathrm{d}\varphi.\end{aligned} \tag{13}$$

By defining the average of $U_{\alpha,\beta,\gamma}(x_1, x_2, x_3)$ over the θ–φ space

$$\begin{aligned}u_{\alpha,\beta,\gamma}(r) &\equiv \int_0^{\pi}\int_{-\pi}^{\pi} U_{\alpha,\beta,\gamma}(x_1, x_2, x_3)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\\ &= \int_{-1}^{1}\int_{-\pi}^{\pi} U_{\alpha,\beta,\gamma}(x_1, x_2, x_3)\,\mathrm{d}\mu\,\mathrm{d}\varphi\\ &= \int_{-1}^{1}\int_{-\pi}^{\pi} V(x_1+\alpha, x_2+\beta, x_3+\gamma)\,\mathrm{d}\mu\,\mathrm{d}\varphi,\end{aligned} \tag{14}$$

then from Eq. (13) we know that $u_{\alpha,\beta,\gamma}(r)$ obeys the radial equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_{\alpha,\beta,\gamma}(r)}{\partial r}\right) + \lambda^2 u_{\alpha,\beta,\gamma}(r) = 0. \tag{15}$$

Here $u_{\alpha,\beta,\gamma}(r)$ can also be viewed as the solution of Eq. (10) for $l = 0$. The solution to Eq. (15) is

$$u_{\alpha,\beta,\gamma}(r) = u_{\alpha,\beta,\gamma}(0)\frac{\sin\lambda r}{\lambda r} = 4\pi V(\alpha,\beta,\gamma)\frac{\sin\lambda r}{\lambda r}. \tag{16}$$

Note $\sqrt{2/\pi}\sin\lambda r/r$ is the normalized spherical Bessel function

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda r}{r}\frac{\sin\lambda' r}{r}r^2\,\mathrm{d}r = \delta(\lambda-\lambda'). \tag{17}$$

Assuming that $V(x', y', z')$ satisfies the Helmoltz equation (7), then using Eq. (16), we can calculate the following integration:

$$\begin{aligned}&\iiint V(x', y', z')\\ &\times\exp\{-p^2[(\alpha-x')^2 + (\beta-y')^2 + (\gamma-z')^2]\}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\\ &\overset{x'-\alpha=x'',\, y'-\beta=y'',\, z'-\gamma=z''}{=} \iiint V(x''+\alpha, y''+\beta, z''+\gamma)\\ &\times\exp\{-p^2\left[x''^2 + y''^2 + z''^2\right]\}\mathrm{d}x''\mathrm{d}y''\mathrm{d}z''\\ &\overset{\text{from } x'', y'', z'' \text{ to } r'',\, \theta'',\, \varphi''}{=} \int_0^{\infty} r''^2\mathrm{d}r''\,\mathrm{e}^{-p^2r''^2}\\ &\times\int_{-1}^{1}\int_{-\pi}^{\pi} U_{\alpha,\beta,\gamma}(x'', y'', z')\,\mathrm{d}\mu''\mathrm{d}\varphi''\\ &= \int_0^{\infty} r''^2\mathrm{d}r''\,\mathrm{e}^{-p^2r''^2}u_{\alpha,\beta,\gamma}(r'')\\ &= u_{\alpha,\beta,\gamma}(0)\int_0^{\infty} r''^2\mathrm{d}r''\,\mathrm{e}^{-p^2r''^2}\frac{\sin\lambda r''}{\lambda r''}.\end{aligned} \tag{18}$$

Due to

$$\begin{aligned}&\int_0^{\infty} r''^2\,\mathrm{e}^{-p^2r''^2}\frac{\sin\lambda r''}{\lambda r''}\mathrm{d}r''\\ &= \frac{1}{2}\int_{-\infty}^{\infty} t^2\,\mathrm{e}^{-p^2t^2}\frac{\sin\lambda t}{\lambda t}\mathrm{d}t\\ &= \frac{1}{2\lambda}\,\mathrm{Im}\int_{-\infty}^{\infty} t\,\mathrm{e}^{-p^2t^2+\mathrm{i}\lambda t}\mathrm{d}t\\ &= \frac{1}{2\lambda}\,\mathrm{Im}\frac{\partial}{\mathrm{i}\partial\lambda}\int_{-\infty}^{\infty}\mathrm{e}^{-p^2t^2+\mathrm{i}\lambda t}\mathrm{d}t\\ &= \frac{1}{2\lambda}\,\mathrm{Im}\frac{\partial}{\mathrm{i}\partial\lambda}\left(\sqrt{\frac{\pi}{p^2}}\,\mathrm{e}^{-\frac{\lambda^2}{4p^2}}\right) = \frac{\sqrt{\pi}}{4p^3}\,\mathrm{e}^{-\frac{\lambda^2}{4p^2}},\end{aligned} \tag{19}$$

we obtain the integration formula for the Helmholtz equation's solution $V(x', y', z')$

$$\begin{aligned}&\iiint V(x', y', z')\exp\{-p^2[(\alpha-x')^2\\ &+ (\beta-y')^2 + (\gamma-z')^2]\}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\\ &= u_{\alpha,\beta,\gamma}(0)\frac{\sqrt{\pi}}{4p^3}\,\mathrm{e}^{-\frac{\lambda^2}{4p^2}} = \frac{\pi^{3/2}}{p^3}\,\mathrm{e}^{-\frac{\lambda^2}{4p^2}}V(\alpha,\beta,\gamma).\end{aligned} \tag{20}$$

Now using the completeness relation of 3D coordinate representation

$$\int \mathrm{d}^3 r |\boldsymbol{r}\rangle \langle \boldsymbol{r}| = \int \frac{\mathrm{d}^3 r}{\pi^{3/2}} : \exp\{-[(x_1 - X_1)^2 + (x_2 - X_2)^2 + (x_3 - X_3)^2]\} := 1, \quad |\boldsymbol{r}\rangle \equiv |x_1, x_2, x_3\rangle, \tag{21}$$

where

$$X_i = \frac{a_i + a_i^\dagger}{\sqrt{2}}, \quad [a_i, a_j^\dagger] = \delta_{i,j}, \quad i, j = 1, 2, 3, \tag{22}$$

and using Eq. (20), we have

$$\begin{aligned} & V(X_1, X_2, X_3) \\ = & \int \frac{\mathrm{d}^3 r}{\pi^{3/2}} \iiint V(x_1, x_2, x_3) \\ & : \exp\{-[(x_1 - X_1)^2 + (x_2 - X_2)^2 + (x_3 - X_3)^2]\} : \\ = & \mathrm{e}^{-\frac{\lambda^2}{4}} : V(X_1, X_2, X_3) : . \end{aligned} \tag{23}$$

Thus the theorem is proved.

3. Application of the theorem

For the classical radius variables, it is Dirac[1] who first introduced the radius operator $\hat{r}$ in 3-dimensional coordinate space

$$r = \sqrt{X_1^2 + X_2^2 + X_3^2}, \; r|\boldsymbol{r}\rangle = r|\boldsymbol{r}\rangle, \; |\boldsymbol{r}\rangle \equiv |x_1, x_2, x_3\rangle, \tag{24}$$

the eigenvalues of r go from 0 to ∞, so

$$r^n = \int \mathrm{d}^3 r r^n |\boldsymbol{r}\rangle \langle \boldsymbol{r}|. \tag{25}$$

Since $\sin \lambda r/(\lambda r)$ is the solution of the Helmholz equation, it then follows from the theorem that

$$\begin{aligned} \frac{\sin \lambda r}{\lambda r} &= \iiint \frac{\sin \lambda r}{\lambda r} |\boldsymbol{r}\rangle \langle \boldsymbol{r}| \, \mathrm{d}^3 r \\ &= \frac{1}{\pi^{3/2}} \iiint \frac{\sin \lambda r}{\lambda r} : \exp\left[-(\boldsymbol{r} - r)^2\right] : \mathrm{d}^3 r \\ &= \mathrm{e}^{-\lambda^2/4} : \frac{\sin \lambda r}{\lambda \widehat{r}} : . \end{aligned} \tag{26}$$

This is the normally ordered expansion of $\sin(\lambda r)/(\lambda r)$, a remarkable operator identity.

On one hand, we note the Taylor expansion

$$\frac{\sin \lambda r}{\lambda r} = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n} r^{2n}}{(2n+1)!}, \tag{27}$$

on the other hand, we have

$$\begin{aligned} & \mathrm{e}^{-\lambda^2/4} : \frac{\sin \lambda r}{\lambda r} := \sum_{l=0}^{\infty} \frac{(-1)^l \lambda^{2l}}{4^l l!} : \frac{\sin \lambda r}{\lambda r} : \\ = & \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+l} \lambda^{2m+2l}}{4^l (2m+1)! l!} : r^{2m} : \\ = & \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} \frac{(-1)^n \lambda^{2n}}{(2n+1)!} \frac{(2n+1)!}{4^l (2n+1-2l)! l!} : r^{2n-2l} : \\ = & \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n}}{(2n+1)!} \sum_{l=0}^{n} \frac{(2n+1)!}{4^l (2n+1-2l)! l!} : r^{2n-2l} : . \end{aligned} \tag{28}$$

Comparing Eqs. (27) and (28), we have the normal ordering expansion of the power of the radial coordinate

$$r^{2n} = \sum_{l=0}^{n} \frac{(2n+1)!}{4^l (2n+1-2l)! l!} : r^{2n-2l} : . \tag{29}$$

The above result coincides with that in Refs. [12] and [13]. For instances,

$$\begin{aligned} r^2 &= : r^2 : + 3/2, \\ r^4 &= : r^4 : + 5 : r^2 : + 15/4. \end{aligned} \tag{30}$$

4. Two-dimensional case

In the two-dimensional case, the Helmhpoltz equation is

$$\frac{\partial^2 \Psi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_2^2} + \lambda^2 \Psi = 0, \tag{31}$$

where we have used the cylindrical coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Simply completeing the integral over z' in Eq. (20), we have

$$\begin{aligned} & \iint \Psi(x', y') \exp\{-p^2[(x - x')^2 + (y - y')^2]\} \mathrm{d}x' \mathrm{d}y' \\ = & \frac{\pi}{p^2} \mathrm{e}^{-\frac{\lambda^2}{4p^2}} \Psi(x, y). \end{aligned} \tag{32}$$

Then we have the operator expansion

$$\begin{aligned} \Psi(X_1, X_2) &= \iint \mathrm{d}x_1 \mathrm{d}x_2 \Psi(x_1, x_2) |x_1, x_2\rangle \langle x_1, x_2| \\ &= \iint \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\pi} \Psi(x_1, x_2) \\ &\quad \times : \exp\{-[(x_1 - X_1)^2 + (x_2 - X_2)^2]\} : \\ &= \mathrm{e}^{-\lambda^2/4} : \Psi(X_1, X_2) : . \end{aligned} \tag{33}$$

The basic solution of the 2D Helmholtz equation is $\mathrm{J}_n(\lambda r)(A\cos n\theta + B\sin n\theta)$, where $\mathrm{J}_n(\lambda r)$ is the Bessel function. For $n = 0$, using the IWOP technique and Eq. (32), we have the operator identity in the 2D case

$$\begin{aligned} \mathrm{J}_0(\lambda \widehat{r}) &= \iint \mathrm{J}_0(\lambda r') |x_1, x_2\rangle \langle x_1, x_2| \, \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \frac{1}{\pi} \iint \mathrm{J}_0(\lambda r') : \exp\{-[(X_1 - x_1)^2 + (X_2 - x_2)^2]\} \mathrm{d}x_1 \mathrm{d}x_2 : \\ &= \mathrm{e}^{-\lambda^2/4} : \mathrm{J}_0(\lambda \widehat{r}) : . \end{aligned} \tag{34}$$

According to the power-series definition

$$\mathrm{J}_0(\lambda \widehat{r}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l} (l!)^2} \lambda^{2l} \widehat{r}^{2l}, \tag{35}$$

on the other hand, we have

$$\begin{aligned} & \mathrm{e}^{-\lambda^2/4} : \mathrm{J}_0(\lambda\widehat{r}) : \\ = & \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l} l!} \lambda^{2l} : \sum_{s=0}^{\infty} \frac{(-1)^s}{2^{2s} s!^2} \lambda^{2s} \widehat{r}^{2s} : \\ = & \sum_{l=0}^{\infty} \frac{(-1)^l \lambda^{2l}}{2^{2l}} \sum_{s=0}^{l} : \frac{\widehat{r}^{2s}}{(l-s)!s!^2} : . \end{aligned} \tag{36}$$

Comparing Eqs. (28) and (29), we find that in the 2D case, the normally ordered expansion of the radius operator $\widehat{r}^{2l}$ is

$$\widehat{r}^{2l} = (l!)^2 \sum_{s=0}^{l} : \frac{\widehat{r}^{2s}}{(l-s)!s!^2} : . \tag{37}$$

For instance,

$$\begin{aligned} \widehat{r}^2 &= : r^2 : +1, \\ \widehat{r}^4 &= : \widehat{r}^4 : +4 : \widehat{r}^2 : +2. \end{aligned} \tag{38}$$

5. Conclusion

In summary, for those quantum-mechanical operators that correspond to the radial solution of the Helmholtz equation, we find their normally-ordered expansion formulas. Both the cases in 2-dimensional and 3-dimensional coordinate spaces are considered. In so doing, Dirac's radius-operator function theory is further developed.

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