# A meshless scheme for partial differential equations based on multiquadric trigonometric B-spline quasi-interpolation*

Gao Wen-Wu(高文武)[a)b)] and Wang Zhi-Gang(王志刚)[c)†]

[a)] School of Economics, Anhui University, Hefei 230411, China
[b)] Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Sciences, Fudan University, Shanghai 200433, China
[c)] School of Mathematics and Finance, Fuyang Teachers College, Fuyang 236037, China

(Received 8 April 2014; revised manuscript received 14 May 2014; published online 24 September 2014)

Based on the multiquadric trigonometric B-spline quasi-interpolant, this paper proposes a meshless scheme for some partial differential equations whose solutions are periodic with respect to the spatial variable. This scheme takes into account the periodicity of the analytic solution by using derivatives of a periodic quasi-interpolant (multiquadric trigonometric B-spline quasi-interpolant) to approximate the spatial derivatives of the equations. Thus, it overcomes the difficulties of the previous schemes based on quasi-interpolation (requiring some additional boundary conditions and yielding unwanted high-order discontinuous points at the boundaries in the spatial domain). Moreover, the scheme also overcomes the difficulty of the meshless collocation methods (i.e., yielding a notorious ill-conditioned linear system of equations for large collocation points). The numerical examples that are presented at the end of the paper show that the scheme provides excellent approximations to the analytic solutions.

**Keywords:** quasi-interpolation, meshless collocation, periodicity, divided difference

**PACS:** 02.60.–x **DOI:** 10.1088/1674-1056/23/11/110207

## 1. Introduction

Partial differential equations play a vital role in describing physical, chemical, and engineering phenomena. They arise in various fields, such as nonlinear optics, plasma physics, particle vibrations in lattices, quantum physics, optical fiber communication, etc. Generally, the analytic solutions of these equations are either discontinuous or unavailable. Therefore, the development of efficient and accurate numerical schemes for solving differential equations is very important. Based on whether a scheme relies on a mesh or not, there are basically two types of schemes: the mesh-based schemes (i.e., finite elements,[1–3] finite differences,[4] finite volume, spectral element, etc.), and the meshless schemes (such as the element free Galerkin method,[5,6] meshless collocation methods,[7,8] and so on).

The mesh-based schemes partition the domain of a problem into smaller subdomains. This will bring in discretization errors and could take a long time for a solution. This could even be hard on non-smooth and irregular domains. On the contrary, meshless schemes only distribute scattered nodes over the domain and thus save a lot of computation time. Moreover, they provide more flexibility and adaptivity for the cases that involve adaptivity and time-dependent changes of the underlying region. Therefore, meshless schemes have attracted considerable attention. Among them, collocations with radial basis functions are popular.[9–13] However, the collocation method requires us to solve a large-scale linear system of equations, which takes a large amount of CPU time for computation. In addition, this method may yield an ill-conditioning system when the number of collocation points is large. To overcome these difficulties, people usually use quasi-interpolation (instead of collocation) to solve some differential equations numerically.[14–19]

Quasi-interpolation gives an approximant directly without solving any large-scale linear system of equations. Moreover, some quasi-interpolants (i.e., the Bernstein's approximant, the B-spline quasi-interpolant, the MQ quasi-interpolant, etc) can even possess shape preserving properties. Therefore, quasi-interpolation has been widely discussed.[20–23]

Particularly, based on MQ quasi-interpolation, a scheme for solving some time-dependent differential equations was proposed.[15,16,24,25] In the scheme, the spatial derivatives of the equations were approximated by the derivatives of MQ quasi-interpolation, while the temporal ones were approximated by divided differences.

Note that for some differential equations, the solutions may be periodic with respect to the spatial variable.[15,16,25,26] Then, it is more reasonable to approximate the spatial derivatives of the equations with the derivatives of a periodic approx-

*Project supported by the Shanghai Guidance of Science and Technology, China (Grant No. 12DZ2272800), the Natural Science Foundation of Education Department of Anhui Province, China (Grant No. KJ2013B203), and the Foundation of Introducing Leaders of Science and Technology of Anhui University, China (Grant No. J10117700057).

†Corresponding author. E-mail: zgw_1122@163.com

© 2014 Chinese Physical Society and IOP Publishing Ltd 

imant. However, the MQ quasi-interpolant is not a periodic function because its kernel (the MQ function) is not periodic. Thus, they had to extend the quasi-interpolant periodically outside of the boundaries to approximate the analytic solutions in the spatial domain. This requires some additional boundary conditions, which may be extremely complicated for some cases. Moreover, it also yields unwanted high-order discontinuous points at the boundaries and thus destroys high-order smoothness of the analytic solutions.

To overcome these difficulties, it is better to take into account the periodicity and use a quasi-interpolant whose kernel itself is also periodic in these cases.

Lyche, Schumaker, and Stanley constructed a trigonometric B-spline quasi-interpolant with periodicity $2\pi$ and discussed its approximation orders for high-order derivatives.[27] Recently, by using a multiquadric trigonometric function as the kernel, a multiquadric trigonometric B-spline quasi-interpolant was constructed in Ref. [17]. Later, approximation orders for high-order derivatives and shape preserving properties of the quasi-interpolant were discussed in Ref. [28].

Based on the multiquadric trigonometric B-spline quasi-interpolant, this paper proposes a scheme to solve some time-dependent differential equations (whose analytic solutions are periodic with respect to the spatial variable). This scheme consists of three steps. First, the temporal derivatives of the differential equations are approximated by the divided differences. Second, we approximate the spatial derivatives by the derivatives of the multiquadric trigonometric B-spline quasi-interpolant. These in turn yield an iterative formula for approximations of the analytic solutions at the discrete centers. Finally, solving the iterative formula step by step gives these approximations. This scheme is meshless and does not require us to solve any large-scale linear system of equations.

This paper is organized as follows. To make the paper self-contained, some preliminaries about multiquadric trigonometric B-spline quasi-interpolation are presented in Section 2. Based on the quasi-interpolation, a scheme for some time-dependent differential equations is proposed in Section 3. Numerical examples are provided in Section 4. A discussion and conclusions are given in Section 5.

## 2. Preliminaries

The construction of multiquadric trigonometric B-spline quasi-interpolation consists of three steps.[17]

First, the kernel (multiquadric trigonometric function) is constructed

$$\phi(x)=\sqrt{c^2+\sin^2\frac{x}{2}},$$

where $c$ is a positive shape parameter.

Second, with respect to the kernel, the multiquadric trigonometric B-splines can be given in the form

$$\begin{aligned}\psi_j(x) &= \frac{\phi(x-x_{j+1})-\cos\dfrac{x_{j+1}-x_j}{2}\phi(x-x_j)}{2\sin\dfrac{x_{j+1}-x_j}{2}\sin\dfrac{x_{j+1}-x_{j-1}}{2}}\\ &\quad-\frac{\cos\dfrac{x_j-x_{j-1}}{2}\phi(x-x_j)-\phi(x-x_{j-1})}{2\sin\dfrac{x_j-x_{j-1}}{2}\sin\dfrac{x_{j+1}-x_{j-1}}{2}}.\end{aligned}\tag{1}$$

Finally, with these $\{\psi_j(x)\}$ being the basis, the multiquadric trigonometric B-spline quasi-interpolation can be constructed as

$$Qf(x)=\sum_{j=1}^{N}\sin\frac{x_{j+1}-x_{j-1}}{2}f(x_j)\psi_j(x),\tag{2}$$

where $0=x_0<\cdots<x_N=2\pi<x_{N+1}=2\pi+x_1$.

Moreover, the quasi-interpolant approximates high-order derivatives well and thus can be applied in the numerical solutions of partial differential equations.[28]

In the following section, based on the quasi-interpolant, we shall propose a scheme to solve some time-dependent differential equations whose solutions are periodic with respect to the spatial variable.

## 3. Main scheme

This section proposes a scheme for solving some time-dependent differential equations based on multiquadric trigonometric B-spline quasi-interpolation.

Without loss of generality, we show the scheme using a simple example; however, it can be generalized to several kinds of time-dependent problems, that is

$$\begin{aligned}u_t(t,x) &= u_{xx}(t,x)+p(t)u(t,x)+\varphi(t,x),\\ &\quad (t,x)\in(0,\ T]\times[0,\ 2\pi],\end{aligned}\tag{3}$$

with the initial condition

$$u(0,x)=u_0(x),\quad 0\le x\le 2\pi,$$

and boundary conditions

$$u(t,0)=u(t,2\pi)=g(t),\quad 0\le t\le T.$$

Such a type of equations arise in many physical phenomena. Examples include heat conduction processes, chemical diffusion, thermoelasticity, control theory, and so on. For more details about the equations, we refer our readers to Ref. [25].

Now, we present the numerical scheme to solve the equations by multiquadric trigonometric B-spline quasi-interpolation.

Let $M$ and $N$ be two given positive integers. Denoted by $\{(t_n,x_k)\}$, $n=0,1,\ldots,M$, $k=0,1,\ldots,N$, the scattered centers satisfy

$$\begin{aligned}0 &= t_0<t_1<t_2<\cdots<t_{M-1}<t_M=1,\\ 0 &= x_0<x_1<x_2<\cdots<x_{N-1}<x_N\end{aligned}$$

$$= 2\pi < x_{N+1} = 2\pi + x_1.$$

Let $u_k^n$ be approximations of $u(t,x)$ at the centers $(t_n, x_k)$. The scheme to solve the equation consists of three steps.

First, discretizing the equation

$$u_t(t,x) = u_{xx}(t,x) + p(t)u(t,x) + \varphi(t,x)$$

in time with the first-order forward divided differences yields

$$\begin{aligned} u_k^{n+1} &= u_k^n + (t_{n+1} - t_n)[(u_{xx})_k^n + p(t_n)u_k^n + \varphi(t_n, x_k)], \\ & n = 0, 1, \ldots, M-1. \end{aligned}$$

Second, using the second-order derivative of the multiquadric trigonometric B-spline quasi-interpolant to approximate $(u_{xx})_k^n$ gives

$$\begin{aligned} u_k^{n+1} &= u_k^n + (t_{n+1} - t_n) \\ &\quad \times \left[ \sum_{j=1}^{N} u_j^n \sin \frac{x_{j+1} - x_{j-1}}{2} \psi_j''(x_k) \right. \\ &\quad \left. + p(t_n)u_k^n + \varphi(t_n, x_k) \right]. \end{aligned}$$

Note that $u_k^0 = u_0(x_k)$ for $k = 0, 1, \ldots, N$. This implies that the approximations $\{u_k^{n+1}\}$ at the time step $t_{n+1}$ can be obtained by those at the time step $t_n$ through the iterative formula.

Finally, solving the iterative formula step by step gives the approximations $u_k^n$ of the analytic solution $u(t,x)$ at the centers $(t_n, x_k)$.

Consequently, we have proposed a scheme for some time-dependent differential equations. This scheme does not require us to solve any linear system of equations. Moreover, for a given $x$, we only need to compute the second-order derivative $\psi_j''(x)$, $j = 1, 2, \ldots, N$, one time.

Besides, based on Eq. (1), $\psi_j''(x)$ can be given analytically as

$$\begin{aligned} \psi_j''(x) &= \frac{\phi''(x - x_{j+1}) - \cos \dfrac{x_{j+1} - x_j}{2} \phi''(x - x_j)}{2 \sin \dfrac{x_{j+1} - x_j}{2} \sin \dfrac{x_{j+1} - x_{j-1}}{2}} \\ &\quad - \frac{\cos \dfrac{x_j - x_{j-1}}{2} \phi''(x - x_j) - \phi''(x - x_{j-1})}{2 \sin \dfrac{x_j - x_{j-1}}{2} \sin \dfrac{x_{j+1} - x_{j-1}}{2}}, \end{aligned}$$

where

$$\phi''(x) = \frac{c^2(c^2+1) - \phi^4(x)}{4\phi^3(x)}.$$

Based on Theorem 3.1 in Ref. [28] and the Theorem 4.2 in Ref. [29], the truncation error of this scheme is about $\mathcal{O}(\Delta t + (\Delta x)^{2/3})$, where $\Delta t = \max\{t_{n+1} - t_n\}$, $\Delta x = \max\{x_{k+1} - x_k\}$. Moreover, according to Theorem 3.4 and Remark 3.5 in Ref. [30], the scheme is much stabler and simpler than the divided differences that are used to solve high-order differential equations with respect to the spatial variable.

Furthermore, by constructing a multivariate quasi-interpolant with the tensor product method[29] for the uniform centers, or the method in Ref. [31] for the scattered centers, this scheme can be used to solve equations with two or more spatial variables.

In the following section, we shall present some numerical examples to illustrate the validity of this scheme.

## 4. Numerical examples

In this section, we assume that the centers are equidistant; that is, $\Delta t = t_{n+1} - t_n = 1/M$, $\Delta x = x_{k+1} - x_k = 2\pi/N$.

### 4.1. Example 1

The first example is the above equation. As an example, we let

$$\begin{aligned} \varphi(t,x) &= -t(t+2)\,\mathrm{e}^{-t^2}(\cos x + \sin x), \\ u_0(x) &= \cos x + \sin x, \end{aligned}$$

$p(t) = 1 + t^2$, and $g(t) = \mathrm{e}^{-t^2}$, such that the analytic solution of the above problem is $u(t,x) = \mathrm{e}^{-t^2}(\cos x + \sin x)$. Besides, we set $\Delta t = (\Delta x)^2/4$ and $c = 0.15(\Delta x)^{1/3}$.

Moreover, to demonstrate the efficiency of the scheme, we compare our scheme with that based on the multiquadric quasi-interpolant that was proposed in Ref. [25] for solving these equations. Table 1 shows the numerical results of our scheme, while Table 2 shows the corresponding results of the scheme proposed in Ref. [25].

**Table 1.** $L_\infty$-error between the analytic solution and the numerical solution of our scheme.

| $t$ | $N = 64$ | $N = 128$ | $N = 256$ |
|---|---|---|---|
| 0.1 | $7.4424 \times 10^{-6}$ | $5.7792 \times 10^{-6}$ | $5.3258 \times 10^{-6}$ |
| 0.2 | $1.3319 \times 10^{-5}$ | $9.1892 \times 10^{-6}$ | $8.0306 \times 10^{-6}$ |
| 0.3 | $1.5607 \times 10^{-5}$ | $1.0415 \times 10^{-5}$ | $8.9827 \times 10^{-6}$ |
| 0.4 | $1.5853 \times 10^{-5}$ | $1.0405 \times 10^{-5}$ | $8.9071 \times 10^{-6}$ |
| 0.5 | $1.4945 \times 10^{-5}$ | $9.6247 \times 10^{-6}$ | $8.2711 \times 10^{-6}$ |
| 0.6 | $1.3471 \times 10^{-5}$ | $8.6697 \times 10^{-6}$ | $7.3694 \times 10^{-6}$ |
| 0.7 | $1.1735 \times 10^{-5}$ | $7.5373 \times 10^{-6}$ | $6.3817 \times 10^{-6}$ |
| 0.8 | $9.9901 \times 10^{-6}$ | $6.3912 \times 10^{-6}$ | $5.4125 \times 10^{-6}$ |
| 0.9 | $8.3960 \times 10^{-6}$ | $5.3541 \times 10^{-6}$ | $4.5182 \times 10^{-6}$ |
| 1.0 | $6.9329 \times 10^{-6}$ | $4.4088 \times 10^{-6}$ | $3.7248 \times 10^{-6}$ |

Based on these two tables, we can find that our scheme provides more accurate approximations than that based on the multiquadric quasi-interpolation in Ref. [25]. Moreover, it also costs less in terms of computation time. The computation time of our scheme is about 27.545611 s, 559.532856 s, and 8957.363822 s, under $N = 64$, $N = 128$, and $N = 256$, respectively, while the corresponding time of the scheme proposed in Ref. [25] is about 39.301494 s, 600.588628 s, and 9576.600206 s, respectively.

**Table 2.** $L_\infty$-error between the analytic solution and the numerical solution of the scheme proposed in Ref. [25].

| $t$ | $N=64$ | $N=128$ | $N=256$ |
|---|---|---|---|
| 0.1 | $7.9387\times10^{-5}$ | $5.7794\times10^{-6}$ | $4.7435\times10^{-6}$ |
| 0.2 | $1.7355\times10^{-4}$ | $1.3521\times10^{-5}$ | $8.7708\times10^{-6}$ |
| 0.3 | $2.6804\times10^{-4}$ | $2.1622\times10^{-5}$ | $1.6933\times10^{-5}$ |
| 0.4 | $3.6094\times10^{-4}$ | $2.9724\times10^{-5}$ | $2.4931\times10^{-5}$ |
| 0.5 | $4.5193\times10^{-4}$ | $3.7781\times10^{-5}$ | $3.2761\times10^{-5}$ |
| 0.6 | $5.4080\times10^{-4}$ | $4.5789\times10^{-5}$ | $4.0420\times10^{-5}$ |
| 0.7 | $6.2740\times10^{-4}$ | $5.3747\times10^{-5}$ | $4.7906\times10^{-5}$ |
| 0.8 | $7.1158\times10^{-4}$ | $6.1656\times10^{-5}$ | $5.5219\times10^{-5}$ |
| 0.9 | $7.9319\times10^{-4}$ | $6.9514\times10^{-5}$ | $6.2358\times10^{-5}$ |
| 1.0 | $8.7211\times10^{-4}$ | $7.7320\times10^{-5}$ | $6.9323\times10^{-5}$ |

### 4.2. Example 2

The second example is the telegraph equation,[23] which is

$$u_{tt}(t,x)+u_t(t,x)=u_{xx}(t,x)+u(t,x),$$
$$(t,x)\in[0,\ 1]\times[0,\ 2\pi), \tag{4}$$

with the initial conditions $u_t(0,x)=-\sin x$ and $u(0,x)=u_0(x)=\sin x$ for $0\le x<2\pi$. The analytic solution is $u(t,x)=\mathrm{e}^{-t}\sin x$.

Discretizing the temporal derivatives of the equation

$$u_{tt}(t,x)+u_t(t,x)=u_{xx}(t,x)+u(t,x),$$

with centered divided differences yields

$$u_k^{n+1}=\frac{[2+(\Delta t)^2]u_k^n-(1-0.5\Delta t)u_k^{n-1}+(\Delta t)^2(u_{xx}(t,x))_k^n}{1+0.5\Delta t}. \tag{5}$$

Observe that

$$u_k^0=u_0(x_k)=\sin x_k,\quad k=0,1,\ldots,N.$$

Moreover, to get $\{u_k^1\}$, we use the first-order forward divided difference to approximate $u_t(0,x)$; that is, $u_k^1=u_k^0(x)+t_1u_t(0,x_k)=(1-t_1)\sin x_k$, $k=0,1,\ldots,N$.

We also compare our scheme with the centered divided differences. Table 3 shows the numerical results of our scheme under $\Delta t=0.01$ and $c=0.15(\Delta x)^{1/3}$, while Table 4 demonstrates those of centered divided differences.

**Table 3.** $L_\infty$-error between the analytic solution and the numerical one of our scheme.

| $t$ | $N=64$ | $N=128$ | $N=256$ |
|---|---|---|---|
| 0.1 | $6.1527\times10^{-5}$ | $4.3294\times10^{-5}$ | $2.9319\times10^{-5}$ |
| 0.2 | $2.7362\times10^{-4}$ | $1.9253\times10^{-4}$ | $1.3039\times10^{-4}$ |
| 0.3 | $6.0861\times10^{-4}$ | $4.2823\times10^{-4}$ | $2.9000\times10^{-4}$ |
| 0.4 | 0.0010 | $7.3262\times10^{-4}$ | $4.9612\times10^{-4}$ |
| 0.5 | 0.0015 | 0.0011 | $7.3846\times10^{-4}$ |
| 0.6 | 0.0021 | 0.0015 | 0.0010 |
| 0.7 | 0.0027 | 0.0019 | 0.0013 |
| 0.8 | 0.0034 | 0.0024 | 0.0016 |
| 0.9 | 0.0040 | 0.0028 | 0.0019 |
| 1.0 | 0.0047 | 0.0033 | 0.0022 |

The two tables demonstrate that the centered divided difference scheme yields more accurate approximations than our scheme. In addition, the centered divided difference scheme costs less computation time. The computation time of the centered divided difference scheme is about 0.105903 s, 0.172105 s, and 0.253865 s, under $N=64$, $N=128$, and $N=256$, respectively, while the one of our scheme is about 6.435605 s, 25.446588 s, and 105.592014 s, respectively. However, our scheme is stabler than the centered divided difference scheme, as shown in Tables 5 and 6.

**Table 4.** $L_\infty$-error between the analytic solution and the numerical solution of centered divided differences.

| $t$ | $N=64$ | $N=128$ | $N=256$ |
|---|---|---|---|
| 0.1 | $2.7465\times10^{-6}$ | $7.0794\times10^{-7}$ | $1.9815\times10^{-7}$ |
| 0.2 | $1.2214\times10^{-5}$ | $3.1482\times10^{-6}$ | $8.812\times10^{-7}$ |
| 0.3 | $2.7164\times10^{-5}$ | $7.0019\times10^{-6}$ | $1.9599\times10^{-6}$ |
| 0.4 | $4.6469\times10^{-5}$ | $1.1978\times10^{-5}$ | $3.3527\times10^{-6}$ |
| 0.5 | $6.9164\times10^{-5}$ | $1.7828\times10^{-5}$ | $4.9901\times10^{-6}$ |
| 0.6 | $9.4430\times10^{-5}$ | $2.4340\times10^{-5}$ | $6.8129\times10^{-6}$ |
| 0.7 | $1.2157\times10^{-4}$ | $3.1336\times10^{-5}$ | $8.7710\times10^{-6}$ |
| 0.8 | $1.5000\times10^{-4}$ | $3.8664\times10^{-5}$ | $1.0822\times10^{-5}$ |
| 0.9 | $1.7923\times10^{-4}$ | $4.6198\times10^{-5}$ | $1.2931\times10^{-5}$ |
| 1.0 | $2.0886\times10^{-4}$ | $5.3833\times10^{-5}$ | $1.5068\times10^{-5}$ |

**Table 5.** $L_\infty$-error between the analytic solution and the numerical solution of our scheme under the case that the initial conditions contain Gaussian white noise with variance 0.01.

| $t$ | $N=128$ | $N=256$ | $N=512$ | $N=1024$ |
|---|---|---|---|---|
| 0.2 | 0.0088 | 0.0174 | 0.0265 | 0.0246 |
| 0.4 | 0.0132 | 0.0154 | 0.0200 | 0.0253 |
| 0.6 | 0.0174 | 0.0147 | 0.0229 | 0.0262 |
| 0.8 | 0.0145 | 0.0158 | 0.0193 | 0.0232 |
| 1.0 | 0.0114 | 0.0134 | 0.0176 | 0.0252 |

**Table 6.** $L_\infty$-error between the analytic solution and the numerical solution of centered divided differences under the case that the initial conditions contain Gaussian white noise with variance 0.01.

| $t$ | $N=128$ | $N=256$ | $N=512$ | $N=1024$ |
|---|---|---|---|---|
| 0.2 | 0.0164 | 0.0190 | 0.0326 | Nay |
| 0.4 | 0.0168 | 0.0184 | 0.0229 | Nay |
| 0.6 | 0.0183 | 0.0143 | 0.0233 | Nay |
| 0.8 | 0.0124 | 0.0153 | 0.0246 | Nay |
| 1.0 | 0.0140 | 0.0127 | 0.0217 | Nay |

## 5. Concluding remarks

This paper proposes a numerical scheme for some partial differential equations whose analytic solutions are periodic with respect to the spatial variable. Unlike previous schemes, which were based on quasi-interpolation,[15,16,25] this scheme takes into account the periodicity of the analytic solution. Thus, it requires no additional boundary conditions and does not yield unwanted high-order discontinuous points at the boundaries in the spatial domain. Moreover, the scheme

also overcomes the difficulty of the meshless collocation methods (i.e., yielding a notorious ill-conditioned linear system of equations for large collocation points). Our numerical results demonstrate that the scheme possesses fair approximation accuracy.

## References


[1] Ciarlet P 1978 *The Finite Element Method for Elliptic Problems* (North-Holland)

[2] Chen L, Ma H P and Cheng Y M 2013 *Chin. Phys. B* **22** 050202

[3] Wu X and Qiu Y F 2013 *Chin. Phys. Lett.* **30** 080203

[4] Dehghan M 2006 *Math. Comput. Simul.* **71** 16

[5] Liu G R and Gu Y T 2004 *Eng. Anal. Bound. Elem.* **28** 475

[6] Liu D, Chen Q and Wang Y F 2013 *J. Comput. Eng.* **2013** 432192

[7] Wang F, Lin G, Zheng B J and Hu Z Q 2013 *Chin. Phys. B* **22** 060206

[8] Ge H X and Cheng R J 2014 *Chin. Phys. B* **23** 040203

[9] Kansa E 1990 *Comput. Math. Appl.* **19** 127

[10] Kansa E 1990 *Comput. Math. Appl.* **19** 147

[11] Wu Z M 1992 *Approx. Theory Appl.* **8** 1

[12] Fasshauer G 1991 *Adv. Comput. Math.* **11** 139

[13] Fasshauer G 2007 *Meshfree Approximation Methods with Matlab* (Singapore: World Science Publishing Company)

[14] Hon Y C and Mao X Z 1998 *Appl. Math. Comput.* **95** 37

[15] Chen R H and Wu Z M 2006 *Numer. Meth. Part. Diff. Equ.* **22** 776

[16] Chen R H and Wu Z M 2007 *Appl. Math. Comput.* **186** 1502

[17] Gao W W and Wu Z M 2012 *J. Comput. Appl. Math.* **236** 3256

[18] Hon Y C 2002 *Comput. Math. Appl.* **43** 513

[19] Zhu C G and R H Wang 2009 *App. Math. Comput.* **208** 260

[20] Beatson R and Powell M 1992 *Constr. Approx.* **8** 275

[21] Beatson R and Dyn N 1996 *J. Approx. Theory* **87** 1

[22] Buhmann M 1988 *IMA J. Numer. Anal.* **8** 365

[23] Wu Z M 1994 *Acta. Math. Appl. Sin.* **10** 441

[24] Ma L M and Wu Z M 2009 *Chin. Phys. B* **18** 3099

[25] Ma L M and Wu Z M 2010 *Chin. Phys. B* **19** 010201

[26] Dehghan M and Shokri A 2007 *Numer. Meth. Part. Diff. Equ.* **24** 1080

[27] Lyche T, Schumaker and Stanley S 1998 *J. Approx. Theory* **95** 280

[28] Gao W W and Wu Z M 2014 *J. Comput. Appl. Math.* **271** 20

[29] Wu Z M and Zhang S L 2013 *Eng. Anal. Bound. Elem.* **37** 1052.

[30] Ma L M and Wu Z M 2010 *Sci. Chin. Math.* **53** 985

[31] Wu Z M and Liu J P 2005 *Adv. Comput. Math.* **23** 201