Leaderless and leader-following consensus of linear multi-agent systems*

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This paper studies the consensus problems for multi-agent systems with general linear and nonlinear dynamics. The leaderless and leader-following consensus problems are investigated respectively. Contraction theory is employed to generate some sufficient conditions for testing the agents reaching consensus. Under these conditions and certain assumptions, the trajectories of multi-agent systems in directed topology will converge to each other. Finally, two numerical examples are given to illustrate the effectiveness of the proposed results.

Keywords: linear multi-agent systems, contraction theory, virtual system, partial contraction

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1. Introduction

Coordination control of multi-agent systems has been widely studied for many years. Generally speaking, we always separate it into three categories: consensus problem, rendezvous problem, and formation control.[1,2] Specially, the consensus problem as the fundamental problem for coordination control has divided into many research branches, such as tracking control, synchronization, flocking motion, and so on. However, tracing the essence of consensus problem, a requirement that all nodes converge to a common value is the ultimate objective. The most famous work can be found in Refs. [3]–[5].

Recently, much attention has been focused on studying the consensus of linear multi-agent systems. Compared with the first-order[6,7] or second-order[8,9] multi-agent dynamics, the linear dynamics is a more general case in which each agent can be at any high order. So far, all kinds of results for solving the consensus of linear multi-agent systems have emerged. Wang *et al.*[10] found that the consensus of linear multi-agent systems can be achieved on condition that the adjacency graph is frequently connected and the linear model is completely controllable. Ni and Cheng[11] used algebraic graph theory, the Riccati inequality, and the Lyapunov inequality to consider the leader-following consensus problem of linear multi-agent systems. Based on the partial stability method, Xi *et al.*[12] solved the output consensus problem of high-order linear swarm systems. Zhao *et al.*[13] proved that the consensus of linear multi-agent systems can be achieved if the underlying topology contains a directed spanning tree. Li *et al.*[14] discussed the consensus problems of discrete and continuous linear multi-agent systems by establishing reduced-order observer-based consensus protocols. Wen *et al.*[15] studied the consensus problem for linear dynamics with discontinuous observations. However, the methods proposed in the above results, making linear multi-agent systems reach consensus, were established on the assumption that the coefficient matrices of the linear dynamics were stabilizable and detectable. To the best of our knowledge, the problem of how to ensure each agent of a linear system tends to reach consensus without this assumption has not been comprehensively studied.

On the other hand, the theory of the synchronization of complex networks has generated some fruitful results. For example, De Lellis *et al.*[16] exploited adaptive control to regulate the time-varying coupling gains such that the synchronization of networks can be achieved. Some results on pinning control of different complex networks can be found in Refs. [17] (Lurie dynamical network), [18] (scale-free dynamical network), and [19] (discrete-time network). Li and Chen[20] obtained the sufficient conditions for the global synchronization and asymptotical stability of complex networks based on the Lyapunov stability theorem. De Lellis and di Bernardo[21] listed all methods including Lyapunov techniques,[22] master stability function,[23] and contraction analysis to deal with the synchronization problem of complex networks. However, the most common method is to consider the stability of some error systems, so the Lyapunov stability technique has been the dominant position for many years. The good property of contraction theory has been expressed since it was proposed in Ref. [24]. Actually, contraction analysis

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is motivated by the element property that considering the stability does not require us to know the initial conditions and temporary disturbances. Wang and Slotine[25] extended the contraction analysis to a more general case: partial contraction theory. For the relations between the network and the graph theory, some results on the synchronization of complex networks can be extended to multi-agent systems. From this unified viewpoint, Li *et al.*[26] proposed a distributed observer-type consensus protocol and gave the consensus region for discussing the consensus problem of linear multi-agent systems. Li *et al.*[27] used an adaptive law via adjusting the time-varying coupling weights to consider the distributed consensus problems for multi-agent systems with general linear and Lipschitz nonlinear dynamics. Li *et al.*[28] considered the consensus problem of linear multi-agent systems without or with a leader via designing two different adaptive dynamic protocols. Liu *et al.*[29] constructed two different protocols to analyze the synchronization problem of multi-agent systems with a leader in an undirected topology. Zhang *et al.*[30] studied cooperative tracking control for general linear systems and showed that a networked system can achieve synchronization on arbitrary graphs containing a spanning tree in unbounded synchronization regions, by only adjusting the coupling gain c. Scardovi and Sepulchre[31] investigated the synchronization problem of discrete identical linear networks. Ding *et al.*[32] provided a novel Lyapunov–Krasovskii functional to discuss the leader-following consensus for a distributed multi-agent system with time delay. However, all these results are established on the Lyapunov stability theory for discussing the consensus or synchronization problem of multi-agent systems.

Motivated by the above work, we mainly use partial contraction to consider the consensus problem of multi-agent systems originated from Ref. [27]. Without assuming the coefficient matrices of these systems are stabilizable and detectable, the consensus problem of multi-agent (actual) systems is replaced by studying the contracting behaviors of some virtual systems. At the same time, for the Lipschitz nonlinear dynamics, the limitation on the function $f(x)$ is eliminated in the leaderless consensus problem. By employing contraction theory, some conditions for the consensus problems are given. These conditions mainly depend on the topology of the network, system parameters, and the coupling strength.

The structure of this paper is organized as follows. In Section 2, some preliminaries about the contraction theory and graph theory are introduced. Then, the main results of this paper are shown in Section 3. In Section 4, we verify the effectiveness of our method. Finally, conclusions are drawn in Section 5.

2. Preliminaries

For the sake of better understanding, some basic knowledge about the contraction analysis and graph theory is introduced in this section.

2.1. Contraction theory

2.1.1. Contraction analysis

Consider a nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \tag{1}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ is a state vector, and $\boldsymbol{f}(\boldsymbol{x}, t)$ is assumed to be an $n \times 1$ continuously differentiable vector function. Defining $\delta\boldsymbol{x}$ as a virtual displacement between two different neighboring trajectories, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{x}^{\mathrm{T}}\delta\boldsymbol{x}) \leq 2\delta\boldsymbol{x}^{\mathrm{T}}\delta\dot{\boldsymbol{x}} \\ &= 2\delta\boldsymbol{x}^{\mathrm{T}}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\delta\boldsymbol{x} \leq 2\lambda_{\max}(\boldsymbol{x}, t)\delta\boldsymbol{x}^{\mathrm{T}}\delta\boldsymbol{x}, \end{aligned} \tag{2}$$

where $\lambda_{\max}$ is the largest eigenvalue of the symmetric part of the Jacobian $J = \partial \boldsymbol{f}/\partial \boldsymbol{x}$. By path integration at fixed time in inequality (2), $\|\delta\boldsymbol{x}\| \leq \|\delta\boldsymbol{x}_0\| \mathrm{e}^{\int_0^t \lambda_{\max}(\boldsymbol{x}, t)\,\mathrm{d}t}$ can be easily obtained. Hence, if $\lambda_{\max}(\boldsymbol{x}, t)$ is uniformly strictly negative, any infinitesimal length $\|\delta\boldsymbol{x}\|$ will converge exponentially to zero. This means that all solutions of system (1) converge exponentially to a single trajectory, independent of the initial conditions.[25]

Definition 1[24] Uniformly negative definiteness (UND) of Jacobian $\partial \boldsymbol{f}(\boldsymbol{x}, t)/\partial \boldsymbol{x}$ means that there exists a scalar $\alpha > 0, \forall \boldsymbol{x}, \forall t \geq 0$ s.t. $\partial \boldsymbol{f}/\partial \boldsymbol{x} \leq -\alpha I < 0$. As all matrix inequalities will refer to the symmetric part of the square matrix involved, we can further write $\frac{1}{2}(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}^{\mathrm{T}}) \leq -\alpha I < 0$.

2.1.2. Partial contraction analysis

Partial contraction theory is a very important method for studying synchronization behaviors of dynamics. Compared to the contraction theory, it is more straightforward and more practical. Therefore, in this paper we mainly use partial contraction theory to obtain our results.

Lemma 1[25] Consider a nonlinear system of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}, t), \tag{3}$$

and assume that the auxiliary system

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{x}, t) \tag{4}$$

is contracting with respect to $\boldsymbol{y}$. If a particular solution of the auxiliary $\boldsymbol{y}$-system verifies a smooth specific property, then all trajectories of the original $\boldsymbol{x}$-system verify this property exponentially. The original system is said to be partially contracting.

2.2. Graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted digraph denoted a multi-agent network of order N, with a set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ which is a set of ordered pairs of nodes. In this graph, the node v_i stands for an agent i and an edge (v_i, v_j) denotes that agent j can obtain information from agent i. The set of neighbors of node v_i is defined as $N_i = \{v_j | (v_j, v_i) \in \mathcal{E}\}$. A graph $\mathcal{G}$ is called undirected, if $(v_i, v_j) \in \mathcal{E}$ means $(v_j, v_i) \in \mathcal{E}$. Meanwhile, self-edges (v_i, v_i) are not allowed in this paper. For a given graph, its corresponding adjacency matrix $\mathcal{A} = (a_{ij})_{N\times N}$ is defined as $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$, and is equal to 0 otherwise. The Laplacian matrix $L = (l_{ij})_{N\times N}$ of graph $\mathcal{G}$ has the property of $l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. The degree of node v_i is defined as $\deg(i) = \sum_{j=1, j\neq i}^{N} a_{ij}$, $i = 1, 2, \ldots, N$.

A (directed) path between nodes v_i and v_j is a sequence of (ordered) edges in the form $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_m}, v_j)$. A directed graph is strongly connected if there is a directed path from every node to every other node. An undirected graph is connected if there is a path between any pair of distinct nodes. A graph contains a directed spanning tree if there exists at least one node such that it can reach any other nodes with directed paths. The reader can refer to Ref. [33] for other basic notions on graph theory.

3. Main results

3.1. Consensus of multi-agent systems in leaderless case

3.1.1. Consensus of multi-agent systems with general linear dynamics

Consider a group of N identical agents, each of them is described by a general linear model

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}\boldsymbol{u}_i, \quad i = 1, 2, \ldots, N, \tag{5}$$

where $\boldsymbol{x}_i \in \boldsymbol{R}^n$ is agent i's state, and $\boldsymbol{u}_i \in \boldsymbol{R}^m$ is agent i's control input. $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant matrices with compatible dimensions.

To solve the consensus problem, a static consensus protocol proposed in Ref. [26] is presented as follows:

$$\boldsymbol{u}_i = c\boldsymbol{K}\sum_{j=1}^{N} a_{ij}(\boldsymbol{x}_i - \boldsymbol{x}_j), \quad i = 1, 2, \ldots, N, \tag{6}$$

where $c > 0$ is the coupling strength, $\boldsymbol{K} \in \boldsymbol{R}^{m\times n}$ is the feedback gain matrix to be designed, and $\mathcal{A} = (a_{ij})_{N\times N}$ is the adjacency matrix of the graph $\mathcal{G}$.

Based on the definition of Laplacian matrix L, we can know that

$$\boldsymbol{u}_i = c\boldsymbol{K}\sum_{j=1}^{N} a_{ij}(\boldsymbol{x}_i - \boldsymbol{x}_j) = c\boldsymbol{K}\sum_{j=1}^{N} l_{ij}\boldsymbol{x}_j. \tag{7}$$

Then system (5) can be simplified into the following equation:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}c\boldsymbol{K}\sum_{j=1}^{N} l_{ij}\boldsymbol{x}_j. \tag{8}$$

Definition 2 System (5) is said to achieve consensus if there is a state feedback $\boldsymbol{u}_i$ such that

$$\lim_{t\to\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| = 0, \quad \forall i, j. \tag{9}$$

Our objective of this paper is to select a suitable $\boldsymbol{K}$ in Eq. (8) such that equation (9) holds. Before presenting our results, some assumptions must be given first.

Assumption 1 The graph associated with the multi-agent network has a directed spanning tree. This can efficiently ensure $l_{ii} \neq 0$.

For the discussion of the consensus of the linear multi-agent system

$$\begin{cases} \dot{\boldsymbol{x}}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}\boldsymbol{u}_i, \\ \boldsymbol{y}_i = \boldsymbol{C}\boldsymbol{x}_i, \quad i = 1, 2, \ldots, N, \end{cases} \tag{10}$$

the following instructions are necessarily given to illustrate the superiority of our method.

Remark 1 In the existing results, to make system (10) reach consensus, some sufficient conditions must be given before the consensus algorithm begins to perform. If the consensus is only related to the state variable, $(\boldsymbol{A}, \boldsymbol{B})$ is required to be stabilizable. Moreover, if the output variable $\boldsymbol{y}_i(t)$ is also considered in the consensus problem, such as the output feedback control and observer-based consensus, and so on, the requirements that $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable and $(\boldsymbol{A}, \boldsymbol{C})$ is observable (detectable) must be satisfied at the same time. However, these sufficient conditions can be removed by using our method. On the other hand, a sufficient condition for the communication graph is indispensable, as in the previous work. That is, in the undirected case, the graph is assumed to be connected. In the directed case, the graph is assumed to have a spanning tree. Therefore, all our results for the consensus problems are established on Assumption 1.

Following the principles of the contraction analysis, our main results are subsequently given in this section.

Theorem 1 If Assumption 1 is satisfied and there exists a suitable $\boldsymbol{K}$ such that the symmetric part of the matrix

$$\boldsymbol{J}_1 = \boldsymbol{A} + \boldsymbol{B}c\boldsymbol{K}l_{ii} \tag{11}$$

is UND, then the agents in system (8) can reach consensus.

Proof From Lemma 1, to consider the convergence of system (8), we should find an available virtual system to replace system (8). Thus, we first construct a virtual system as follows:

$$\dot{\boldsymbol{y}}_i = \boldsymbol{A}\boldsymbol{y}_i + \boldsymbol{B}c\boldsymbol{K}l_{ii}\boldsymbol{y}_i + \boldsymbol{B}c\boldsymbol{K}\sum_{j=1, j\neq i}^{N} l_{ij}\boldsymbol{x}_j. \tag{12}$$

It is obvious that $\boldsymbol{y}_i = \boldsymbol{x}_i$ is a particular solution of system (12) for all $i = 1, 2, \ldots, N$. Denoting $\delta \boldsymbol{y}_i$ as the virtual displacement between different trajectories of system (12), we have

$$\delta \dot{\boldsymbol{y}}_i = \boldsymbol{A}\delta \boldsymbol{y}_i + \boldsymbol{BcK} l_{ii} \delta \boldsymbol{y}_i. \tag{13}$$

Then the differential of the square distance between neighboring trajectories $\delta \boldsymbol{y}_i^{\mathrm{T}} \delta \boldsymbol{y}_i$ can be expressed as follows:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\delta \boldsymbol{y}_i^{\mathrm{T}} \delta \boldsymbol{y}_i) &= 2\delta \boldsymbol{y}_i^{\mathrm{T}} \delta \dot{\boldsymbol{y}}_i \\ &= 2\delta \boldsymbol{y}_i^{\mathrm{T}} (\boldsymbol{A}\delta \boldsymbol{y}_i + \boldsymbol{BcK} l_{ii} \delta \boldsymbol{y}_i) \\ &= 2\delta \boldsymbol{y}_i^{\mathrm{T}} (\boldsymbol{A} + \boldsymbol{BcK} l_{ii}) \delta \boldsymbol{y}_i \\ &\leq 2\lambda_{1\max} \delta \boldsymbol{y}_i^{\mathrm{T}} \delta \boldsymbol{y}_i. \end{aligned} \tag{14}$$

Here $\lambda_{1\max}$ is the largest eigenvalue of $J_{1\mathrm{s}}$ and $J_{1\mathrm{s}} = \frac{1}{2}(J_1^{\mathrm{T}} + J_1)$. If $J_{1\mathrm{s}}$ is UND, $\lambda_{1\max} < 0$. From inequality (14), we can easily know that system (13) is globally asymptotically stable. This means that the virtual system (12) is contracting. Hence, system (8) also has the contracting behaviors, that is, system (8) can reach consensus.

Remark 2 The construction of the virtual system has more choices, such as

$$\begin{aligned} \dot{\boldsymbol{y}}_i = {} & \boldsymbol{A}\boldsymbol{y}_i + \boldsymbol{BcK} \sum_{j=1}^{N} a_{ij}(\boldsymbol{y}_i - \boldsymbol{y}_j) \\ & - K_0 \sum_{j=1}^{N} \boldsymbol{y}_j + K_0 \sum_{j=1}^{N} \boldsymbol{x}_j. \end{aligned}$$

This virtual system can also be used to check the convergence of the actual system. For detailed situations refer to Ref. [25].

3.1.2. Consensus of multi-agent systems with nonlinearity

In this section, we consider the consensus problem for a multi-agent system consisting of N identical nonlinear agents. The dynamics of each agent is described by

$$\dot{x}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{D}\boldsymbol{f}(x_i) + \boldsymbol{B}\boldsymbol{u}_i, \quad i = 1, 2, \ldots, N, \tag{15}$$

where $x_i \in \boldsymbol{R}^n, u_i \in \boldsymbol{R}^m$ are the state and the control input of agent i, respectively. $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{D}$ are known constant matrices with compatible dimensions.

Remark 3 Here, the nonlinear function $\boldsymbol{f}(\boldsymbol{x})$ is more general. The limitation on the function in Ref. [27] can be eliminated.

Theorem 2 If Assumption 1 is satisfied and there exists a suitable $\boldsymbol{K}$ such that the symmetric part of the matrix

$$\boldsymbol{J}_2 = \boldsymbol{A} + \boldsymbol{D}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}_i} + \boldsymbol{BcK} l_{ii} \tag{16}$$

is UND, then the agents in (15) can reach consensus.

Proof As before, the control input $u_i = c\boldsymbol{K} \sum_{j=1}^{N} l_{ij} \boldsymbol{x}_j$, then equation (15) can be changed into

$$\dot{x}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{D}\boldsymbol{f}(x_i) + \boldsymbol{BcK} \sum_{j=1}^{N} l_{ij} \boldsymbol{x}_j. \tag{17}$$

Following the same steps as in Theorem 1, the virtual system of system (17) is generated as

$$\dot{\boldsymbol{y}}_i = \boldsymbol{A}\boldsymbol{y}_i + \boldsymbol{D}\boldsymbol{f}(\boldsymbol{y}_i) + \boldsymbol{BcK} l_{ii} \boldsymbol{y}_i + \boldsymbol{BcK} \sum_{j=1, j\neq i}^{N} l_{ij} \boldsymbol{x}_j. \tag{18}$$

Therefore, $\boldsymbol{y}_i = \boldsymbol{x}_i$ for all $i = 1, 2, \ldots, N$ is a particular solution of system (18). Defining the virtual displacement for state $\boldsymbol{y}_i$ in system (18) as $\delta \boldsymbol{y}_i$, we have

$$\delta \dot{\boldsymbol{y}}_i = \boldsymbol{A}\delta \boldsymbol{y}_i + \boldsymbol{D}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}_i} \delta \boldsymbol{y}_i + \boldsymbol{BcK} l_{ii} \delta \boldsymbol{y}_i. \tag{19}$$

Taking the derivative of the square distance between two neighboring trajectories, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\delta \boldsymbol{y}_i^{\mathrm{T}} \delta \boldsymbol{y}_i) &= 2\delta \boldsymbol{y}_i^{\mathrm{T}} \delta \dot{\boldsymbol{y}}_i \\ &= 2\delta \boldsymbol{y}_i^{\mathrm{T}} \left(\boldsymbol{A}\delta \boldsymbol{y}_i + \boldsymbol{D}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}_i} \delta \boldsymbol{y}_i + \boldsymbol{BcK} l_{ii} \delta \boldsymbol{y}_i \right) \\ &= 2\delta \boldsymbol{y}_i^{\mathrm{T}} \left(A + \boldsymbol{D}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}_i} + \boldsymbol{BcK} l_{ii} \right) \delta \boldsymbol{y}_i \\ &\leq 2\lambda_{2\max} \delta \boldsymbol{y}_i^{\mathrm{T}} \delta \boldsymbol{y}_i, \end{aligned} \tag{20}$$

where $\lambda_{2\max}$ is the largest eigenvalue of $J_{2\mathrm{s}}$ and $J_{2\mathrm{s}} = \frac{1}{2}(J_2^{\mathrm{T}} + J_2)$. If $J_{2\mathrm{s}}$ is UND, $\lambda_{2\max} < 0$, system (19) is globally asymptotically stable. This indicates that the virtual system (18) has contracting behavior, so system (15) also has the property, reaching consensus.

3.2. Consensus of multi-agent systems with a leader

3.2.1. Consensus of multi-agent systems with general linear dynamics

Suppose that a leader which has the following dynamics is introduced into the existing multi-agent group (8):

$$\dot{\boldsymbol{x}}_0 = \boldsymbol{A}\boldsymbol{x}_0. \tag{21}$$

Then we use the following control law for agent i:

$$\boldsymbol{u}_i = c\boldsymbol{K} \sum_{j=1}^{N} a_{ij}(\boldsymbol{x}_i - \boldsymbol{x}_j) + c\boldsymbol{K} d_i(\boldsymbol{x}_i - \boldsymbol{x}_0), \tag{22}$$

such that all the agents can reach consensus with the leader asymptotically, i.e., $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \cdots = \boldsymbol{x}_n(t) = \boldsymbol{x}_0(t)$, as $t \to +\infty$. Here, $\boldsymbol{K}$ is a feedback matrix to be designed, and d_i is defined to be 1 when there is a link between agent i and the leader, and 0 otherwise. It is supposed that the following assumption must be satisfied.

Assumption 2 For a given graph, having a directed spanning tree, its root can get information from the leader.

Denote $\boldsymbol{e}_i = \boldsymbol{x}_i - \boldsymbol{x}_0$, we have

$$\dot{\boldsymbol{e}}_i = \boldsymbol{A}\boldsymbol{e}_i + \boldsymbol{B}c\boldsymbol{K}\sum_{j=1}^{N} l_{ij}\boldsymbol{e}_j + \boldsymbol{B}c\boldsymbol{K}d_i\boldsymbol{e}_i. \tag{23}$$

Just as discussed in Section 3.1, we have the following conclusion.

Theorem 3 Based on Assumption 2, if there exists a suitable $\boldsymbol{K}$ such that the symmetric part of the matrix

$$\tilde{\boldsymbol{J}}_1 = \boldsymbol{A} + \boldsymbol{B}c\boldsymbol{K}l_{ii} + \boldsymbol{B}c\boldsymbol{K}d_i \tag{24}$$

is UND, then the agents in (23) can reach consensus.

Proof Just as in Theorem 1, we constitute a virtual system as follows:

$$\begin{aligned}\dot{\tilde{\boldsymbol{y}}}_i &= \boldsymbol{A}\tilde{\boldsymbol{y}}_i + \boldsymbol{B}c\boldsymbol{K}l_{ii}\tilde{\boldsymbol{y}}_i \\ &\quad + \boldsymbol{B}c\boldsymbol{K}\sum_{j=1,j\neq i}^{N} l_{ij}\boldsymbol{e}_j + \boldsymbol{B}c\boldsymbol{K}d_i\tilde{\boldsymbol{y}}_i.\end{aligned} \tag{25}$$

Apparently, $\tilde{\boldsymbol{y}}_i = \boldsymbol{e}_i$ for all $i = 1,2,\ldots,N$ is a particular solution of (25). Considering the virtual displacement $\delta\tilde{\boldsymbol{y}}_i$ for system (25), we have

$$\delta\dot{\tilde{\boldsymbol{y}}}_i = \boldsymbol{A}\delta\tilde{\boldsymbol{y}}_i + \boldsymbol{B}c\boldsymbol{K}_{ii}\delta\tilde{\boldsymbol{y}}_i + \boldsymbol{B}c\boldsymbol{K}d_i\delta\tilde{\boldsymbol{y}}_i. \tag{26}$$

The square distance between the neighboring solution trajectories is defined as $\delta\tilde{\boldsymbol{y}}_i^{\mathrm{T}}\delta\tilde{\boldsymbol{y}}_i$. From Eq. (26), we can easily obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\tilde{\boldsymbol{y}}_i^{\mathrm{T}}\delta\tilde{\boldsymbol{y}}_i) &= 2\delta\tilde{\boldsymbol{y}}_i^{\mathrm{T}}(A + \boldsymbol{B}c\boldsymbol{K}l_{ii} + \boldsymbol{B}c\boldsymbol{K}d_i)\,\delta\tilde{\boldsymbol{y}}_i \\ &\leq 2\tilde{\lambda}_{1\max}\delta\tilde{\boldsymbol{y}}_i^{\mathrm{T}}\delta\tilde{\boldsymbol{y}}_i,\end{aligned} \tag{27}$$

where $\tilde{\lambda}_{1\max}$ is the largest eigenvalue of $\tilde{\boldsymbol{J}}_{1\mathrm{s}}$ and $\tilde{\boldsymbol{J}}_{1\mathrm{s}} = \frac{1}{2}(\tilde{\boldsymbol{J}}_1^{\mathrm{T}} + \tilde{\boldsymbol{J}}_1)$. If $\tilde{\boldsymbol{J}}_{1\mathrm{s}}$ is UND, $\tilde{\lambda}_{1\max} < 0$. The convergence of system (23) can be guaranteed.

Remark 4 In Ref. [34], the leader-following consensus can also be solved by Lyapunov stability theory. Following the conclusion, the matrix $\boldsymbol{I}_N \otimes \boldsymbol{A} - \boldsymbol{H} \otimes \boldsymbol{B}\boldsymbol{K}$ must be Hurwitz. However, the method given by Theorem 3 considerably reduces the amount of calculation and decreases the dimension of the matrix. Overall, the results in the above subsection are established on the collection of all agents, while we only consider a single agent to obtain the whole motion of all agents.

3.2.2. Consensus of multi-agent systems with nonlinearity

Here the leader has the following form:

$$\dot{\boldsymbol{x}}_0 = \boldsymbol{A}\boldsymbol{x}_0 + \boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0). \tag{28}$$

In contrast to Section 3.1.2, the nonlinear function $\boldsymbol{f}(\boldsymbol{x}_i)$ satisfies the following assumption.

Assumption 3

$$k^- \leq \frac{\boldsymbol{f}(\alpha) - \boldsymbol{f}(\beta)}{\alpha - \beta} \leq k^+, \forall \alpha, \beta \in \boldsymbol{R}^n, \tag{29}$$

where k^- and k^+ are known constants. Then the error dynamics can be described as follows:

$$\dot{\boldsymbol{e}}_i = \boldsymbol{A}\boldsymbol{e}_i + \boldsymbol{D}\boldsymbol{\eta}(\boldsymbol{e}_i) + \boldsymbol{B}c\boldsymbol{K}\sum_{j=1}^{N} l_{ij}\boldsymbol{e}_j + \boldsymbol{B}c\boldsymbol{K}d_i\boldsymbol{e}_i, \tag{30}$$

where $\boldsymbol{\eta}(\boldsymbol{e}_i) = \boldsymbol{f}(\boldsymbol{x}_i) - \boldsymbol{f}(\boldsymbol{x}_0)$. The nonlinear function $\eta(x)$ also satisfies

$$\begin{aligned}k^- &\leq \frac{\boldsymbol{\eta}(\alpha) - \boldsymbol{\eta}(\beta)}{\alpha - \beta} \\ &= \frac{\boldsymbol{f}(\alpha) - \boldsymbol{f}(\boldsymbol{x}_0) + \boldsymbol{f}(\boldsymbol{x}_0) - \boldsymbol{f}(\beta)}{\alpha - \beta} \leq k^+.\end{aligned} \tag{31}$$

Without consideration, we naturally have the following theorem.

Theorem 4 Based on Assumptions 2 and 3, if there exists a suitable K such that the symmetric part of the matrix

$$\tilde{\boldsymbol{J}}_2 = A + \boldsymbol{D}\frac{\partial\boldsymbol{\eta}}{\partial\tilde{\boldsymbol{y}}_i} + \boldsymbol{B}c\boldsymbol{K}l_{ii} + \boldsymbol{B}c\boldsymbol{K}d_i. \tag{32}$$

is UND, then the agents in (30) can reach consensus.

Proof Construct a virtual system of system (30) as follows:

$$\begin{aligned}\dot{\tilde{\boldsymbol{y}}}_i &= \boldsymbol{A}\tilde{\boldsymbol{y}}_i + \boldsymbol{D}\boldsymbol{\eta}(\tilde{\boldsymbol{y}}_i) + \boldsymbol{B}c\boldsymbol{K}l_{ii}\tilde{\boldsymbol{y}}_i \\ &\quad + \boldsymbol{B}c\boldsymbol{K}\sum_{j=1,j\neq i}^{N} l_{ij}\boldsymbol{e}_j + \boldsymbol{B}c\boldsymbol{K}d_i\tilde{\boldsymbol{y}}_i.\end{aligned} \tag{33}$$

Then the virtual displacement of system (33) in differential form can be presented as

$$\delta\dot{\tilde{\boldsymbol{y}}}_i = \boldsymbol{A}\delta\tilde{\boldsymbol{y}}_i + \boldsymbol{D}\frac{\partial\boldsymbol{\eta}}{\partial\tilde{\boldsymbol{y}}_i} + \boldsymbol{B}c\boldsymbol{K}l_{ii}\delta\tilde{\boldsymbol{y}}_i + \boldsymbol{B}c\boldsymbol{K}d_i\delta\tilde{\boldsymbol{y}}_i. \tag{34}$$

Therefore, the square distance between neighboring trajectories has the following differential form:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\tilde{\boldsymbol{y}}_i^{\mathrm{T}}\delta\tilde{\boldsymbol{y}}_i) &= 2\delta\tilde{\boldsymbol{y}}_i^{\mathrm{T}}\left(\boldsymbol{A} + \boldsymbol{D}\frac{\partial\boldsymbol{\eta}}{\partial\tilde{\boldsymbol{y}}_i} + \boldsymbol{B}c\boldsymbol{K}l_{ii} + \boldsymbol{B}c\boldsymbol{K}d_i\right)\delta\tilde{\boldsymbol{y}}_i \\ &\leq 2\tilde{\lambda}_{2\max}\delta\tilde{\boldsymbol{y}}_i^{\mathrm{T}}\delta\tilde{\boldsymbol{y}}_i,\end{aligned} \tag{35}$$

where $\tilde{\lambda}_{2\max}$ is the largest eigenvalue of $\tilde{\boldsymbol{J}}_{2\mathrm{s}}$ and $\tilde{\boldsymbol{J}}_{2\mathrm{s}} = \frac{1}{2}(\tilde{\boldsymbol{J}}_2^{\mathrm{T}} + \tilde{\boldsymbol{J}}_2)$. If $\tilde{\boldsymbol{J}}_{2\mathrm{s}}$ is UND, $\tilde{\lambda}_{2\max} < 0$. The result is easily obtained.

4. Numerical examples

In this section, some simulation examples are given to check the utility of the theoretical results in Section 3, and the communication graphs are shown in Fig. 1.

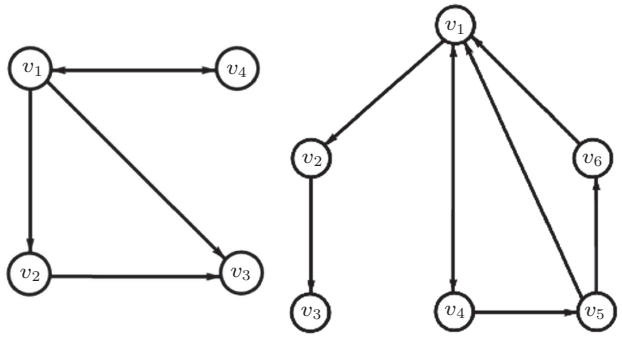


Fig. 1. (a) Communication graph in example 1. (b) Communication graph in example 2.

Example 1 Consider system (5) with parameters $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Without loss of generality, the communication graph is shown in Fig. 1(a), then the corresponding Laplacian matrix can be written as

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Applying Theorem 1, we can deduce the following results:

$$\begin{aligned} \boldsymbol{J}_1 &= \boldsymbol{A} + \boldsymbol{B}c\boldsymbol{K}l_{ii} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} cl_{ii} \\ 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}^{\mathrm{T}} \\ &= \begin{pmatrix} 2+cl_{ii}k_1 & cl_{ii}k_2 \\ 1 & -1 \end{pmatrix}, \\ \boldsymbol{J}_{1\mathrm{s}} &= \begin{pmatrix} 2+cl_{ii}k_1 & \dfrac{1+cl_{ii}k_2}{2} \\ \dfrac{1+cl_{ii}k_2}{2} & -1 \end{pmatrix}. \end{aligned}$$

To ensure that $\boldsymbol{J}_{1\mathrm{s}}$ is UND, the following conditions must be satisfied:

$$2+cl_{ii}k_1 < 0; \quad -(2+cl_{ii}k_1) - \frac{(1+cl_{ii}k_2)^2}{4} > 0.$$

Therefore, any values of the feedback gain matrix $\boldsymbol{K}$ satisfying the above conditions would ensure the contracting behaviors of the virtual system. Then the agents in system (5) will achieve consensus. After calculation, we have

$$cl_{ii}k_1 < -2 - \frac{(1+cl_{ii}k_2)^2}{4}. \tag{36}$$

Therefore, the selection of the feedback gain matrix K depends on the coupling strength and the degree of the nodes. Since $c > 0$, we choose

$$l_{ii} = \min(\deg(i)) = 1.$$

The initial values for system (5) are selected randomly as $\boldsymbol{x}_1(0) = (0.9223, 0.5612)^{\mathrm{T}}$, $\boldsymbol{x}_2(0) = (0.6523, 0.7727)^{\mathrm{T}}$, $\boldsymbol{x}_3(0) = (0.1062, 0.0011)^{\mathrm{T}}$, and $\boldsymbol{x}_4(0) = (0.5418, 0.0069)^{\mathrm{T}}$. With $c = 1$, $k_1 = -3$, $k_2 = -1$, the errors between agent i $(i = 2,3,4)$ and agent 1 of system (5) are shown in Fig. 2.

Moreover, we introduce a leader into this system with $d_1 = 1$ and any others $d_i = 0 (i = 2,3,4)$. From Theorem 2, we have

$$\tilde{\boldsymbol{J}}_{1\mathrm{s}} = \begin{pmatrix} 2+mk_1 & \dfrac{1+mk_2}{2} \\ \dfrac{1+mk_2}{2} & -1 \end{pmatrix},$$

where $m = cl_{ii} + cd_i$. All trajectories of system (5) will be consistent with the trajectory of the leader, provided that the matrix $\tilde{\boldsymbol{J}}_{1\mathrm{s}}$ is UND. After calculation, we know that the following conditions can guarantee the UND property of the matrix $\tilde{\boldsymbol{J}}_{1\mathrm{s}}$.

$$\text{i)}\ \ 2+mk_1 < 0, \ \ \text{ii)} -(2+mk_1) - \frac{(1+mk_2)^2}{4} > 0.$$

As usual, l_{ii} and d_i are taken as the minimum values. With $c = 1, k_1 = -4, k_2 = -1$, the errors between the system's states and the leader are shown in Fig. 3. Here, the initial states are taken as

$$\begin{aligned} &\boldsymbol{x}_1(0) = (0.0149, 0.2882)^{\mathrm{T}}, \quad \boldsymbol{x}_2(0) = (0.8167, 0.9855)^{\mathrm{T}}, \\ &\boldsymbol{x}_3(0) = (0.0174, 0.8194)^{\mathrm{T}}, \quad \boldsymbol{x}_4(0) = (0.6211, 0.5602)^{\mathrm{T}}, \\ &\boldsymbol{x}_0(0) = (0.2440, 0.8220)^{\mathrm{T}}. \end{aligned}$$

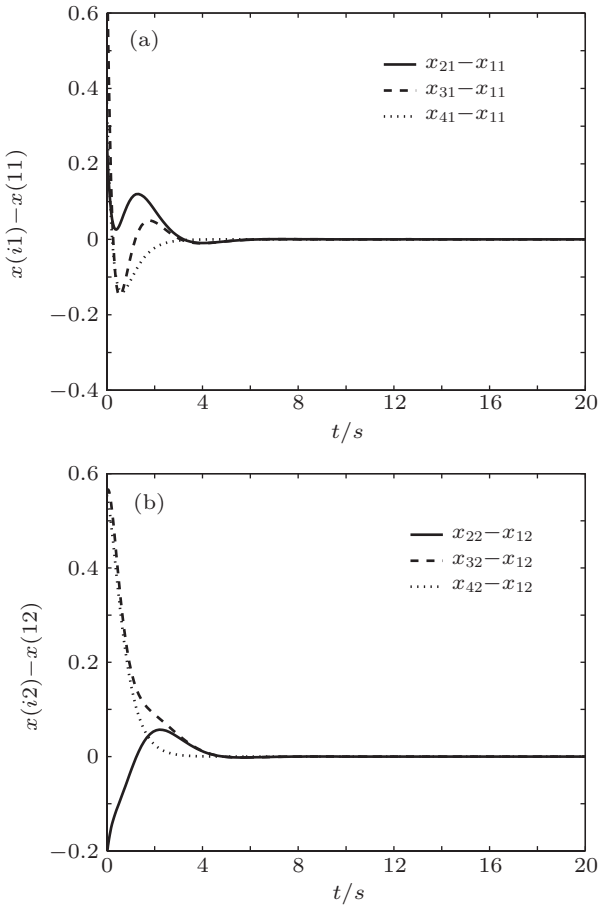


Fig. 2. Relative state trajectories of agent i $(i = 2,3,4)$ to agent 1.

Example 2 Consider system (15) with parameters

$$\boldsymbol{A}=\begin{pmatrix}-1&1&0\\-1&-1&1\\0&0&-1\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}0\\1\\0\end{pmatrix},$$

$$\boldsymbol{D}=\boldsymbol{I}_3,\quad \boldsymbol{f}(\boldsymbol{x}_i)=\begin{pmatrix}0\\0\\0.3\sin\boldsymbol{x}_{i3}\end{pmatrix}.$$

Here $\boldsymbol{x}_i=(\boldsymbol{x}_{i1},\boldsymbol{x}_{i2},\boldsymbol{x}_{i3})^{\mathrm{T}}\in\boldsymbol{R}^3$. Then the communication graph is given in Fig. 1(b) and its corresponding Laplacian matrix is shown as follows:

$$\boldsymbol{L}=\begin{pmatrix}3&0&0&-1&-1&-1\\-1&1&0&0&0&0\\-1&-1&2&0&0&0\\-1&0&0&1&0&0\\0&0&0&-1&1&0\\0&0&0&0&-1&1\end{pmatrix}.$$

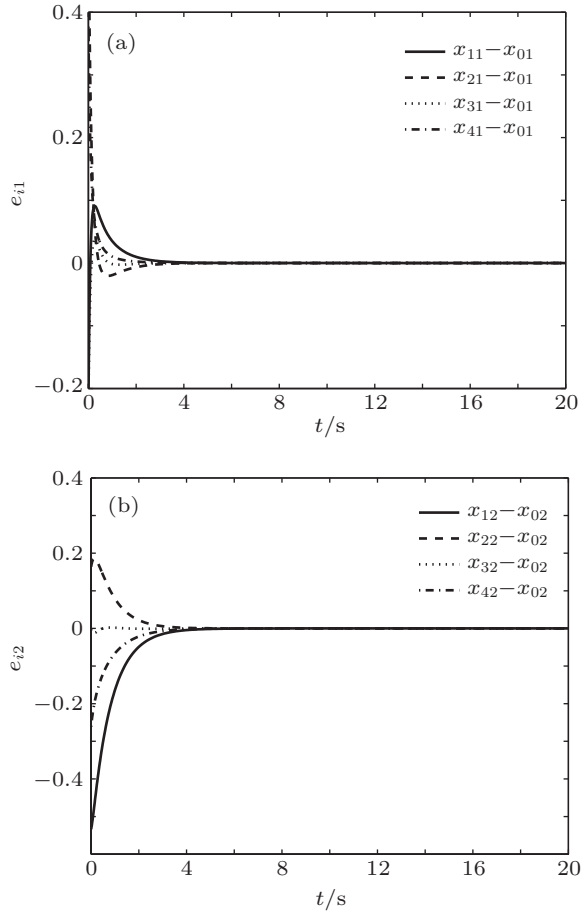


Fig. 3. Error states between the leader and each agent.

Following the procedures in Example 1, we can conclude that

$$\boldsymbol{J}_2=\boldsymbol{A}+\boldsymbol{D}\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{y}_i}+\boldsymbol{BcK}l_{ii}=\begin{pmatrix}-1&1&0\\-1&-1&1\\0&0&-1\end{pmatrix}+\begin{pmatrix}0&0&0\\0&0&0\\0&0&0.3\cos\boldsymbol{x}_{i3}\end{pmatrix}+\begin{pmatrix}0\\cl_{ii}\\0\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}^{\mathrm{T}}$$

$$=\begin{pmatrix}-1&1&0\\-1+cl_{ii}k_1&-1+cl_{ii}k_2&1+cl_{ii}k_3\\0&0&-1+0.3\cos\boldsymbol{x}_{i3}\end{pmatrix}.$$

Thus, we have

$$\boldsymbol{J}_{2\mathrm{s}}=\begin{pmatrix}-1&\frac{cl_{ii}k_1}{2}&0\\\frac{cl_{ii}k_1}{2}&-1+cl_{ii}k_2&\frac{1+cl_{ii}k_3}{2}\\0&\frac{1+cl_{ii}k_3}{2}&-1+0.3\cos\boldsymbol{x}_{i3}\end{pmatrix}.$$

In order to guarantee the matrix $J_{2\mathrm{s}}$ to be UND, some conditions need to be fulfilled.

(i) $-1<0$;

(ii) $-(-1+cl_{ii}k_2)-\dfrac{(cl_{ii}k_1)^2}{4}>0$;

(iii) $(-1+cl_{ii}k_2)(1-0.3\cos\boldsymbol{x}_{i3})+\dfrac{(1+cl_{ii}k_3)^2}{4}$

$$+(1-0.3\cos\boldsymbol{x}_{i3})\frac{(cl_{ii}k_1)^2}{4}<0.$$

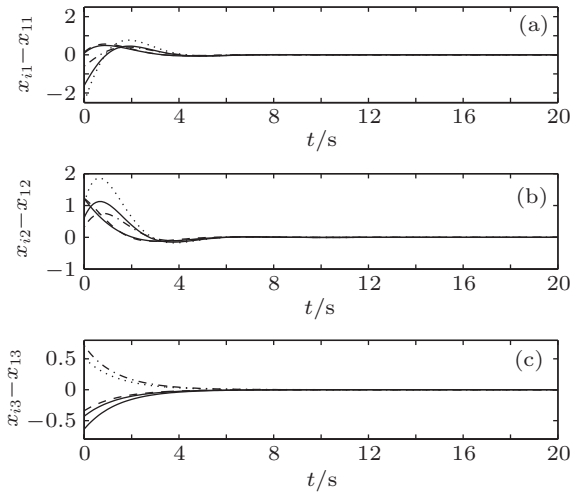


Fig. 4. Relative state trajectories of agent i ($i=2,\ldots,6$) to agent 1.

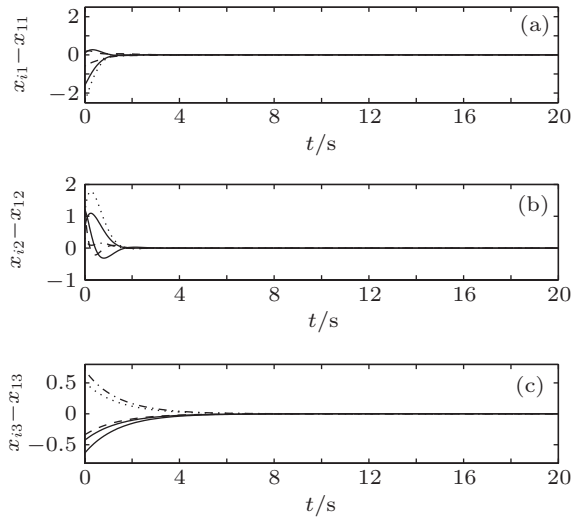


Fig. 5. Relative state trajectories of agent i ($i=2,3,\ldots,6$) to agent 1.

From (ii), (iii) and $1-0.3\cos\boldsymbol{x}_{i3}>0$, we can obtain

$$cl_{ii}k_2 < 1-\frac{(cl_{ii}k_1)^2}{4};$$
$$cl_{ii}k_2 < 1-\frac{(cl_{ii}k_1)^2}{4}-\frac{(1+cl_{ii}k_3)^2}{4(1-0.3\cos\boldsymbol{x}_{i3})}.$$

Therefore, the contracting region is

$$\mathcal{C}_1=\left\{\boldsymbol{x}_i|cl_{ii}k_2<1-\frac{(cl_{ii}k_1)^2}{4}-\frac{(1+cl_{ii}k_3)^2}{4(1-0.3\cos\boldsymbol{x}_{i3})}\right\}, \quad (37)$$

where $\boldsymbol{x}_i\in\boldsymbol{R}^3$. For convenience, we always let $1+cl_{ii}k_3=0$, then we have $cl_{ii}k_2<1-(cl_{ii}k_1)^2/4$. When $1-(cl_{ii}k_1)^2/4>0$, we take $l_{ii}=\max(\deg(i))=3$, $c=1$, and $\boldsymbol{K}=\begin{pmatrix}0 & \frac{1}{5} & -\frac{1}{3}\end{pmatrix}$. Figure 4 lists the relative state trajectories of agent i $(i=2,3,\ldots,6)$ to agent 1 of system (15). Moreover, when $1-(cl_{ii}k_1)^2/4<0$, we take $l_{ii}=\min(\deg(i))=1, c=1$ and $\boldsymbol{K}=\begin{pmatrix}-3 & -2 & -1\end{pmatrix}$. Then the error behaviors between agent i $(i=2,3,\ldots,6)$ and agent 1 of system (15) are shown in Fig. 5.

In all simulation experiments, the initial conditions are selected randomly as follows:

$$\boldsymbol{x}_1(0)=(0.5913,-0.6436,0.3803)^{\mathrm{T}},$$
$$\boldsymbol{x}_2(0)=(-1.0091,-0.0195,-0.0482)^{\mathrm{T}},$$
$$\boldsymbol{x}_3(0)=(0.0000,-0.3179,1.0950)^{\mathrm{T}},$$
$$\boldsymbol{x}_4(0)=(-1.8740,0.4282,0.8956)^{\mathrm{T}},$$
$$\boldsymbol{x}_5(0)=(0.7310,0.5779,0.0403)^{\mathrm{T}},$$
$$\boldsymbol{x}_6(0)=(0.6771,0.5689,-0.2556)^{\mathrm{T}}.$$

From these simulations, we can see that, as long as the parameters of system (15) meet the condition (37), it can reach consensus. Furthermore, if a leader is introduced in this example with $d_4=1$ and any others d_i are zero, then based on Theorem 4 we have

$$\tilde{\boldsymbol{J}}_{2\mathrm{s}}=\begin{pmatrix}-1 & \frac{mk_1}{2} & 0\\ \frac{mk_1}{2} & -1+mk_2 & \frac{1+mk_3}{2}\\ 0 & \frac{1+mk_3}{2} & -1+M\end{pmatrix},$$

where $m=cl_{ii}+cd_i, M=\partial\eta_3/\partial\tilde{\boldsymbol{y}}_{i3}$. From Assumption 3, we have $-1\le M\le 1$. To ensure the UND property of the matrix $\tilde{\boldsymbol{J}}_{2\mathrm{s}}$, the following conditions must be satisfied:

i) $-1<0$;

ii) $-(-1+mk_2)-\frac{m^2k_1^2}{4}>0$;

iii) $(-1+mk_2)(1-M)+\frac{(1+mk_3)^2}{4}+(1-M)\frac{m^2k_1^2}{4}<0.$

As discussed in the leaderless case, the convergence region is

$$\mathcal{C}_2=\left\{\boldsymbol{x}_i|mk_2<1-\frac{m^2k_1^2}{4}-\frac{(1+mk_3)^2}{4(1-M)}\right\}, \quad (38)$$

where $\boldsymbol{x}_i\in\boldsymbol{R}^3$. The selection method of the feedback matrix is similar to system (15) in the leaderless case. Let $1+mk_3=0$, if $1-m^2k_1^2/4>0$, $\boldsymbol{K}=\begin{pmatrix}0 & -1 & -\frac{1}{4}\end{pmatrix}$ and $c=1$, figure 6 lists the error behaviors of all agents to the leader of system (15). On the side, when $1-m^2k_1^2/4<0$, we take $\boldsymbol{K}=\begin{pmatrix}-4 & -4 & -1\end{pmatrix}$ and $c=1$. Then the errors between the leader and each agent of system (15) are shown in Fig. 7.

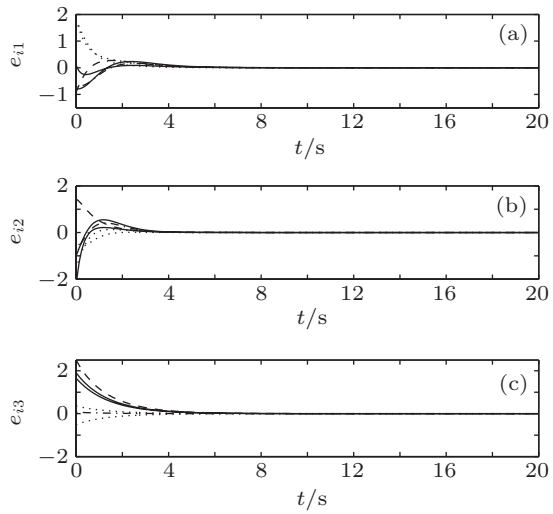


Fig. 6. Error states between the leader and each agent.

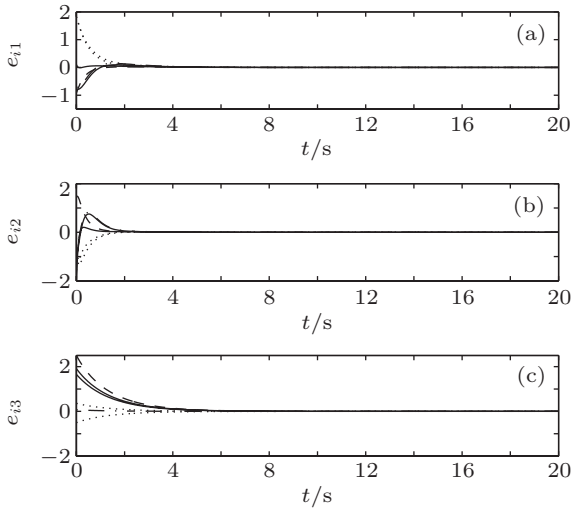


Fig. 7. Error states between the leader and each agent.

Here the initial conditions are chosen randomly as

$$\boldsymbol{x}_1(0)=(-0.0891,-2.0089,1.0839)^{\mathrm{T}},$$
$$\boldsymbol{x}_2(0)=(-0.9812,-0.6885,1.3395)^{\mathrm{T}},$$
$$\boldsymbol{x}_3(0)=(-0.9092,-0.4129,-0.5062)^{\mathrm{T}},$$
$$\boldsymbol{x}_4(0)=(1.6197,0.0809,-1.0811)^{\mathrm{T}},$$
$$\boldsymbol{x}_5(0)=(-1.1245,1.7357,1.9375)^{\mathrm{T}},$$
$$\boldsymbol{x}_6(0)=(1.6351,-1.2559,-0.2135)^{\mathrm{T}},$$
$$\boldsymbol{x}_0(0)=(-0.1989,0.3075,-0.5723)^{\mathrm{T}}.$$

Therefore, provided that the parameters of system (15) satisfy condition (38), it will reach the leader consensus.

Remark 5 In Examples 1 and 2, the selection of l_{ii} is different. In Example 1 we get $cl_{ii}k_1 < -2 - (1+cl_{ii}k_2)^2/4$, while in Example 2 we get $cl_{ii}k_2 < 1 - (cl_{ii}k_1)^2/4 - (1+cl_{ii}k_3)^2/4(1-0.3\cos x_{i3})$. Hence, the value of l_{ii} depends on the sign of the right inequality. If the right-hand number is less than zero then $l_{ii} = \min(\deg(i))$, whereas if the right-hand number is greater than zero then $l_{ii} = \max(\deg(i))$.

5. Conclusion

In this paper, we have studied the leaderless and leader-following consensus problems for multi-agent systems with general linear and nonlinear dynamics. In the case of directed fixed network topologies, we have shown that the multi-agent systems can achieve consensus by adjusting the coupling strength and feedback matrix when the network graph has a directed spanning tree. Two numerical examples have been given to verify the results. This note only considers the consensus problem of a linear multi-agent system without time delay. The consensus problem with time delay will be our future work.

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