Bäcklund transformations for the Burgers equation via localization of residual symmetries*

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We obtain the non-local residual symmetry related to truncated Painlevé expansion of Burgers equation. In order to localize the residual symmetry, we introduce new variables to prolong the original Burgers equation into a new system. By using Lie's first theorem, we obtain the finite transformation for the localized residual symmetry. More importantly, we also localize the linear superposition of multiple residual symmetries to find the corresponding finite transformations. It is interesting to find that the n-th Bäcklund transformation for Burgers equation can be expressed by determinants in a compact way.

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1. Introduction

The integrable equation has wide applications in the field of nonlinear science, like nonlinear optics, the theory of deep water waves, plasma physics, etc. It is well known that there are many ways to obtain explicit solutions of integrable equations, such as the inverse Scattering transformation,[1] the Hirota technique,[2–5] symmetry reduction,[6–8] Bäcklund transformation (BT) and Darboux transformation (DT),[9–12] the algebra-geometric method, etc. Among these methods, the BT and DT are the most direct and efficient approach for the construction of exact solutions (see, e.g., Ref. [13] and [14]). It is interesting that, through iterations, compact representations in terms of special determinants such as the Wronskian or Grammian for N-soliton solutions can be obtained. The essential idea of BT or DT is to construct the solution of integrable equations by the seed solutions, which means that there exist underlying symmetries related to these transformations for the equations under consideration.

One of the most powerful methods to prove the integrability of a model is the Painlevé analysis developed by Weiss, Tabor, and Canvela (WTC). For many Painlevé integrable equations, one can obtain the corresponding truncated Painlevé expansion, which serves as a BT for the original equation. It is known that to search for non-local symmetries is of considerable interest. Recent studies[15,16] have found that the symmetries related to the truncated Painlevé expansion is just the residue of the expansion with respect to singular manifold, so it is called residual symmetry. Like other symmetries related to BT, the residual symmetry is nonlocal and great effort has been made to construct new solutions by using these symmetries. One of the direct ways is to prolong the original equation to localize the nonlocal symmetries and then obtain new symmetry reduction solutions by the standard Lie symmetry approach.[17] In this paper, we will derive the BT for the Burgers equation by localizing the residual symmetry and then generalize the procedure to find the n-th BT.

The Burgers equation

$$u_t + uu_x = \nu u_{xx}, \tag{1}$$

with viscosity coefficient ν, is one of the most simplest integrable equations. It serves as a fundamental equation in the field of fluid mechanics and also occurs in various areas of applied physics such as modeling of gas dynamics and traffic flow. The exact solutions of the Burgers equation have been studied extensively.[18–21] In Ref. [22], the infinite many symmetries and exact solutions of the Burgers equation are obtained by the repeated symmetry reduction approach.

The paper is organized as follows. In Section 2, we obtain the non-local residual symmetry of the Burgers equation and then localize it in a new prolonged system. Consequently, the corresponding finite transformation group is obtained by using Lie's first theorem. In Section 3, we first localize the linear superposition of multiple residual symmetries by introducing various new variables, then derive the n-th BT of the prolonged Burgers system formulated in terms of determinants.

2. Localization of residual symmetry and related Bäcklund transformation

As the Burgers equation (1) is Painlevé integrable, we search for the truncated Painlevé expansion in the general form

$$u = \sum_{i=0}^{\alpha} u_i \phi^{i-\alpha}, \tag{2}$$

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where u_α is an arbitrary solution of the discussed equation while $u_{\alpha-1}, u_{\alpha-2}, \ldots, u_0$ are all related to derivatives of ϕ. After substituting $u \sim u_0\phi^\alpha$ into Eq. (1) and balancing the nonlinear and dispersion terms, we obtain the truncated Painlevé expansion as

$$u = \frac{u_0}{\phi} + u_1, \tag{3}$$

where u_0 and u_1 are to be determined.

Now substituting Eq. (3) into Eq. (1), we have

$$\begin{aligned} &u_{1t} + u_1u_{1x} - \nu u_{1xx} + \phi^{-1}(u_0u_{1x} + u_1u_{0x} + u_{0t} - \nu u_{0xx}) \\ &+ \phi^{-2}(-u_0\phi_t + 2\nu u_{0x}\phi_x + \nu u_0\phi_{xx} + u_0u_{0x} - u_1u_0\phi_x) \\ &- \phi^{-3}\phi_x u_0(2\nu\phi_x + u_0) = 0, \end{aligned} \tag{4}$$

Vanishing the coefficients of ϕ^{-3} and ϕ^{-2}, we have

$$u_0 = -2\nu\phi_x, \tag{5}$$

and

$$u_1 = \frac{-\phi_t + \nu\phi_{xx}}{\phi_x}. \tag{6}$$

While from the coefficients of ϕ^0 and ϕ^{-1}, we find that u_0 is a symmetry of the Burgers equation (1) with a solution of u_1.

Substituting Eq. (6) into the coefficient of ϕ^0 in Eq. (4), we obtain the Schwarz form of the Burgers equation as

$$K_t - KK_x - 2\nu K_{xx} + \nu^2\{\phi;x\}_x = 0, \tag{7}$$

where the Schwarzian derivative $\{\phi;x\} = \phi_{xxx}/\phi_x - 3\phi_{xx}^2/(2\phi_x^2)$ and $K = \phi_t/\phi_x$ are all invariant under the Mobious transformation, i.e.,

$$\phi \to \frac{a_1\phi + b_1}{a_2\phi + b_2}, \quad a_1a_2 \neq b_1b_2. \tag{8}$$

From the properties of any Schwarzian equation,[23] equation (7) possesses three symmetries $\sigma^\phi = d_1$, $\sigma^\phi = d_2\phi$, and

$$\sigma^\phi = d_3\phi^2$$

with arbitrary constants d_1, d_2, and d_3.

Since u_0 is a non-local symmetry related to the BT (3), one can naturally believe that it can be localized to a Lie point symmetry such that we can use the Lie's first theorem to recover the original BT. To this end, we prolong the Burgers equation into the following system:

$$u_t + uu_x - \nu u_{xx} = 0, \tag{9a}$$

$$u = \frac{-\phi_t + \nu\phi_{xx}}{\phi_x}, \tag{9b}$$

$$g = \phi_x, \tag{9c}$$

and obtain the localized Lie point symmetry

$$\sigma^u = g, \tag{10a}$$

$$\sigma^g = \frac{\phi g}{\nu}, \tag{10b}$$

$$\sigma^\phi = \frac{\phi^2}{2\nu}, \tag{10c}$$

where d_3 is fixed as $1/2\nu$. It can be easily verified that equation (10) indeed satisfies the linearized form of Eq. (9)

$$\sigma_x^\phi - \sigma^g = 0, \tag{11a}$$

$$\sigma_t^u + \sigma^u u_x + u\sigma_x^u - \nu\sigma_{xx}^u = 0, \tag{11b}$$

$$\sigma_x^\phi\phi_t - \sigma^u\phi_x^2 + (\nu\sigma_{xx}^\phi - \sigma_t^\phi)\phi_x - \nu\sigma_x^\phi\phi_{xx} = 0. \tag{11c}$$

The result (10) indicates that the residual symmetry (5) is localized in the properly prolonged system (9) with the Lie point symmetry vector in the form of

$$V = g\partial_u + \frac{\phi g}{\nu}\partial_g + \frac{\phi^2}{2\nu}\partial_\phi. \tag{12}$$

Now we can obtain the finite transformation of Lie point symmetry (12) straightforwardly, which can be stated in the following theorem.

Theorem 1 If $\{u, g, \phi\}$ is a solution of the prolonged system (9), then so is $\{\hat{u}, \hat{g}, \hat{\phi}\}$ with

$$\hat{u} = u - \frac{2\epsilon\nu g}{\epsilon\phi - 2\nu}, \tag{13a}$$

$$\hat{g} = \frac{4\nu^2 g}{(\epsilon\phi - 2\nu)^2}, \tag{13b}$$

$$\hat{\phi} = -\frac{2\nu\phi}{\epsilon\phi - 2\nu}, \tag{13c}$$

where ϵ is an arbitrary group parameter.

Proof Using Lie's first theorem on vector (12) with the corresponding initial condition as follows:

$$\frac{\mathrm{d}\hat{u}(\epsilon)}{\mathrm{d}\epsilon} = \hat{g}(\epsilon), \quad \hat{u}(0) = u, \tag{14}$$

$$\frac{\mathrm{d}\hat{g}(\epsilon)}{\mathrm{d}\epsilon} = \frac{\hat{\phi}(\epsilon)\hat{g}(\epsilon)}{\nu}, \quad \hat{g}(0) = g, \tag{15}$$

$$\frac{\mathrm{d}\hat{\phi}(\epsilon)}{\mathrm{d}\epsilon} = \frac{\hat{\phi}^2(\epsilon)}{2\nu}, \quad \hat{\phi}(0) = \phi, \tag{16}$$

one can easily obtain the solutions of the above equations in Theorem 1. Thus, the theorem is proved.

3. Bäcklund transformation related to multiple residual symmetries

From the expression (5), we know that the residual symmetry depends only on the solution of the original equation in Schwarz form, i.e., Eq. (7), for the Burgers equation. Hence, there exist an infinite number of residual symmetries expressed by different solutions of the Schwarzian equation and any linear superposition of these symmetries takes the form

$$\sigma_n^u = \sum_{i=1}^{n} c_i\phi_{i,x}, \tag{17}$$

for arbitrary positive integer n, where ϕ_i $(i = 1, 2, \ldots, n)$ are different solutions of Eq. (7). It should be pointed out that such residues are not symmetries of the original equation since it satisfies the linearized equation only for a single solution.[24]

We can also localize the symmetry (17) by prolonging the Burgers system (9) further, which can be stated in the following theorem.

Theorem 2 The symmetry (17) is localized to a Lie point

symmetry

$$\sigma_n^u = \sum_{i=1}^{n} c_i g_i, \tag{18a}$$

$$\sigma^{\phi_i} = \frac{c_i}{2\nu}\phi_i^2 + \frac{1}{2\nu}\sum_{j\neq i}^{n} c_j\phi_j\phi_i, \tag{18b}$$

$$\sigma^{g_i} = \frac{c_i}{\nu} g_i\phi_i + \frac{1}{2\nu}\sum_{j\neq i}^{n} c_j(g_i\phi_j + g_j\phi_i), \tag{18c}$$

where $\{u, \phi_i, g_i\}$ is a solution of the enlarged system

$$u_t + uu_x - \nu u_{xx} = 0, \tag{19a}$$

$$u = \frac{-\phi_{i,t} + \nu\phi_{i,xx}}{\phi_{i,x}}, \tag{19b}$$

$$g_i = \phi_{i,x}. \tag{19c}$$

Proof We write down the linearized form of the enlarged system (19)

$$\sigma_t^u + \sigma^u u_x + u\sigma_x^u - \nu\sigma_{xx}^u = 0, \tag{20a}$$

$$\sigma^u = \frac{\nu\sigma_{xx}^{\phi_i} - \sigma_t^{\phi_i}}{\phi_{i,x}} - \frac{(\nu\phi_{i,xx} - \phi_{i,t})\sigma_x^{\phi_i}}{\phi_{i,x}^2}, \tag{20b}$$

$$\sigma^{g_i} = \sigma_x^{\phi_i}, \quad i = 1,2,\ldots,n, \tag{20c}$$

and the explicit form of the linearized equation for g_i can be obtained based on Eqs. (7) and (20c).

To prove this theorem, we first consider the special case, i.e., for any fixed k, $c_k \neq 0$ while $c_j = 0$ $(j \neq k)$ in Eq. (17). In this case, we obtain the localized symmetry for u, ϕ_k, and g_k from Eq. (10)

$$\sigma^u = c_k g_k, \tag{21a}$$

$$\sigma^{\phi_k} = \frac{c_k\phi_k^2}{2\nu}, \tag{21b}$$

$$\sigma^{g_k} = \frac{c_k\phi_k g_k}{\nu}. \tag{21c}$$

For $j \neq k$, we eliminate u through Eq. (19b) by taking $i = k$ and $i = j$, respectively. Then we have

$$\phi_{j,xx} = -\frac{\phi_{j,x}\phi_{k,t}}{\nu\phi_{k,x}} + \frac{\phi_{j,x}\phi_{k,xx}}{\phi_{k,x}} + \frac{\phi_{j,t}}{\nu}. \tag{22}$$

Substituting Eq. (21a) into Eq. (20b) with $i = j$ and vanishing $\phi_{j,xx}$ through Eq. (22), we have

$$c_k\phi_{k,x}^2\phi_{j,x} = \nu\phi_{k,x}\sigma_{xx}^{\phi_j} - \phi_{k,x}\sigma_t^{\phi_j} + \sigma_x^{\phi_j}\phi_{k,t} - \nu\sigma_x^{\phi_j}\phi_{k,xx}. \tag{23}$$

It can be easily verified that equation (23) has the solution

$$\sigma^{\phi_j} = \frac{1}{2\nu}c_k\phi_k\phi_j. \tag{24}$$

By using Eq. (7), the symmetry for g_j can be easily obtained from Eq. (20c) with $i = j$

$$\sigma^{g_j} = \frac{1}{2\nu}c_k(g_j\phi_k + g_k\phi_j). \tag{25}$$

From the linear superposition property of symmetry, we derive the desired result (18) by taking the linear combination of the above results for $k = 1,2,\ldots,n$, the theorem is thus proved.

Now, we can obtain the finite transformations according to the localized symmetry (18) by using the Lie's first theorem and the corresponding initial value problem takes the form

$$\frac{\mathrm{d}U(\epsilon)}{\mathrm{d}\epsilon} = \sum_{j=1}^{n} c_j\Phi_{j,x}(\epsilon), \tag{26a}$$

$$\frac{\mathrm{d}\Phi_i(\epsilon)}{\mathrm{d}\epsilon} = \frac{c_i}{2\nu}\Phi_i^2(\epsilon) + \frac{1}{2\nu}\sum_{j\neq i}^{n} c_j\Phi_j(\epsilon)\Phi_i(\epsilon), \tag{26b}$$

$$\frac{\mathrm{d}G_i(\epsilon)}{\mathrm{d}\epsilon} = \frac{c_i}{\nu}G_i(\epsilon)\Phi_i(\epsilon) + \frac{1}{2\nu}\sum_{j\neq i}^{n} c_j(G_i(\epsilon)\Phi_j(\epsilon) + G_j(\epsilon)\Phi_i(\epsilon)), \tag{26c}$$

$$U(0) = u, \quad \Phi_i(0) = \phi_i, \quad G_i(0) = g_i, \quad i = 1,2,\ldots,n. \tag{26d}$$

By solving Eq. (26), we obtain the following n-th Bäcklund transformation theorem.

Theorem 3 If $\{u, \phi_i, g_i, i = 1,2,\ldots,n\}$ is a solution of the prolonged Burgers system (19), so is $\{U(\epsilon), \Phi_i(\epsilon), G_i(\epsilon), i = 1,2,\ldots,n\}$ where

$$U(\epsilon) = u - 2\nu(\ln\Delta)_x, \tag{27a}$$

$$\Phi_i(\epsilon) = -2\nu\frac{\Delta_i}{\Delta}, \tag{27b}$$

$$G_i(\epsilon) = \Phi_{i,x}(\epsilon), \tag{27c}$$

with Δ and Δ_i being the determinants of

$$\begin{pmatrix} \frac{c_1}{2\nu}\epsilon\phi_1 - 1 & c_1\epsilon\mu_{1,2} & \cdots & c_1\epsilon\mu_{1,j} & \cdots & c_1\epsilon\mu_{1,n} \\ c_2\epsilon\mu_{1,2} & \frac{c_2}{2\nu}\epsilon\phi_2 - 1 & \cdots & c_2\epsilon\mu_{2,j} & \cdots & c_2\epsilon\mu_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_j\epsilon\mu_{1,j} & c_j\epsilon\mu_{2,j} & \cdots & \frac{c_j}{2\nu}\epsilon\phi_j - 1 & \cdots & c_j\epsilon\mu_{j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon\mu_{1,n} & c_n\epsilon\mu_{2,n} & \cdots & c_n\epsilon\mu_{j,n} & \cdots & \frac{c_n}{2\nu}\epsilon\phi_n - 1 \end{pmatrix},$$

$$\mu_{i,j} = \frac{1}{2\nu}\sqrt{\phi_i\phi_j}, \tag{28}$$

and

$$\begin{pmatrix} \frac{c_1}{2\nu}\epsilon\phi_1 - 1 & c_1\epsilon\mu_{1,2} & \cdots & c_1\epsilon\mu_{1,i-1} & c_1\epsilon\mu_{1,i} & c_1\epsilon\mu_{1,i+1} & \cdots & c_1\epsilon\mu_{1,n} \\ c_2\epsilon\mu_{1,2} & \frac{c_2}{2\nu}\epsilon\phi_2 - 1 & \cdots & c_2\epsilon\mu_{2,i-1} & c_2\epsilon\mu_{2,i} & c_2\epsilon\mu_{2,i+1} & \cdots & c_2\epsilon\mu_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i-1}\epsilon\mu_{1,i-1} & c_{i-1}\epsilon\mu_{2,i-1} & \cdots & \frac{c_{i-1}}{2\nu}\epsilon\phi_{i-1} - 1 & c_{i-1}\epsilon\mu_{i-1,i} & c_{i-1}\epsilon\mu_{i-1,i+1} & \cdots & c_{i-1}\epsilon\mu_{i-1,n} \\ \mu_{1,i} & \mu_{2,i} & \cdots & \mu_{i-1,i} & \frac{1}{2\nu}\phi_i & \mu_{i,i+1} & \cdots & \mu_{i,n} \\ c_{i+1}\epsilon\mu_{1,i+1} & c_{i+1}\epsilon\mu_{2,i+1} & \cdots & c_{i+1}\epsilon\mu_{i-1,i+1} & c_{i+1}\epsilon\mu_{i,i+1} & \frac{c_{i+1}}{2\nu}\epsilon\phi_{i+1} - 1 & \cdots & c_{i+1}\epsilon\mu_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ c_n\epsilon\mu_{1,n} & c_n\epsilon\mu_{2,n} & \cdots & c_n\epsilon\mu_{i-1,n} & c_n\epsilon\mu_{i,n} & c_n\epsilon\mu_{i+1,n} & \cdots & \frac{c_n}{2\nu}\epsilon\phi_n - 1 \end{pmatrix}. \tag{29}$$

From Theorem 3, we can obtain an infinite number of new solutions from any seed solutions of the Burgers equation and its Schwarz form. Especially, one can obtain new soliton solutions of the Burgers equation from the known ones. For example, taking the following kink soliton as the seed solution for the Burgers equation,[25] i.e.,

$$u = -2kv[1+\tanh(kx+2vk^2t)], \tag{30}$$

with an arbitrary constant k, we obtain the solution of the Schwarzian Burgers equation (7) by solving Eq. (19b), which takes the form

$$\phi_i = A_i \tanh(kx+2vk^2t), \tag{31}$$

with arbitrary constants A_i, and then

$$\mu_{i,j} = \frac{\sqrt{A_iA_j}}{2v}\tanh(kx+2vk^2t). \tag{32}$$

Substituting Eqs. (30), (31), and (32) into Theorem 3, we obtain a new kink soliton solution for Burgers equation

$$u = -\frac{2kv[(c_1\epsilon A_1+c_2\epsilon A_2-2v)\tanh(kx+2vk^2t)+c_1\epsilon A_1+c_2\epsilon A_2-2v]}{\epsilon(c_1A_1+c_2A_2)\tanh(kx+2vk^2t)-2v}, \tag{33}$$

for $n=2$ case.

4. Conclusion and discussion

To sum up, the non-local residual symmetry of the Burgers equation related to a truncated Painlevé expansion can be localized if we introduce a new variable g to eliminate the derivative of ϕ. Using the Lie's first theorem to the Lie point symmetry of the prolonged system $\{u,\phi,g\}$, one can obtain the generalized BT for the Burgers system, which includes the truncated Painlevé expansion as its special one.

Furthermore, we also obtain the finite transformations for linear superposition of multiple residual symmetries, which is equivalent to the n-th BT for Burgers. From Theorem 3, it is interesting to find that the n-th BT can be constructed in terms of determinants in a compact way and various solutions, including soliton solutions, can be obtained through it if the seed solutions are given. It should be noted that the commutable property for BT or DT becomes trivial in the framework of symmetry, because any symmetry group allows the commutativity law in the addition algorithm.

In this paper, we discuss the only procedure for the localization of residual symmetry for constructing a new BT for the Burgers equation. It is obvious that the same procedure can be executed for other nonlocal symmetries, such as those obtained from Bäcklund transformation, the bilinear forms and negative hierarchies, the nonlinearizations[26,27] and self-consistent sources,[28] etc. We believe that the idea is rather general and could be applied to other physically interesting models as well.

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