Third-order nonlinear differential operators preserving invariant subspaces of maximal dimension*

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In this paper, third-order nonlinear differential operators are studied. It is shown that they are quadratic forms when they preserve invariant subspaces of maximal dimension. A complete description of third-order quadratic operators with constant coefficients is obtained. One example is given to derive special solutions for evolution equations with third-order quadratic operators.

Keywords: nonlinear evolution equation, quadratic operator, invariant subspace method, blow-up solution

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1. Introduction

It is well known that seeking explicit solutions of nonlinear partial differential equations (PDEs) is one of the most important topics in the study of mathematical physics problems. There are various techniques to construct exact solutions of PDEs, which involve the Lie point symmetry,[1] the conditional symmetry,[2] the direct method,[3] the generalized conditional symmetry,[4–6] the nonlocal symmetry,[7] the sign-invariant method,[8–10] the invariant subspace method,[11–22] and so on.[23–27]

The invariant subspace method, introduced in Refs. [11] and [12], plays an important role in finding exact solutions of nonlinear PDEs. In fact, various invariant subspaces to nonlinear evolution equations arising from fields of mathematics, mechanics, and physics have been derived. In Ref. [16], Galaktionov and Svirshchevskii offered a systematic and advanced approach to investigate invariant subspaces of nonlinear equations and obtained many applications and interesting results.

Consider the general evolution equation

$$u_t = F[u], \tag{1}$$

where $F[u]$ is a kth-order nonlinear differential operator, and $F[u] \equiv F(x,u,u_x,u_{xx},\ldots)$ is a sufficiently smooth function of its arguments. The n-dimensional linear subspace

$$W_n = \mathscr{L}\{f_1(x),f_2(x),\ldots,f_n(x)\} \equiv \left\{\sum_{i=1}^{n} C_i f_i(x) | C_i \in R\right\}, \tag{2}$$

(a linear hull of linearly independent functions $\{f_1(x),f_2(x),\ldots,f_n(x)\}$) is invariant with respect to the operator F if $F[W_n] \subseteq W_n$, then F is said to preserve or admit W_n. This means

$$F\left[\sum_{i=1}^{n} C_i f_i(x)\right] = \sum_{i=1}^{n} \Psi_i(C_1,C_2,\ldots,C_n) f_i(x)$$

for any $C \equiv (C_1,C_2,\ldots,C_n) \in \mathbb{R}^n$, where $\{\Psi_i\}$ are the expansion coefficients of $F[u] \in W_n$ in the basis $\{f_i\}$. Equation (1) has solutions of the form

$$u(x,t) = \sum_{i=1}^{n} C_i(t) f_i(x), \tag{3}$$

where the coefficients $\{C_i(t)\}$ fulfill the n-dimensional dynamical system

$$C_i'(t) = \Psi_i(C_1(t),C_2(t),\ldots,C_n(t)), \quad i = 1,2,\ldots,n.$$

Suppose that the invariant subspace W_n is defined as the space of solutions to a linear nth-order ordinary differential equation (ODE)

$$L[y] \equiv y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0. \tag{4}$$

Then the invariant condition with respect to F takes the form

$$L[F[u]]|_{[H]} = 0, \tag{5}$$

where $[H]$ denotes the equation $L[u] = 0$ and its differential consequences with respect to x.

The invariant condition (5) means that the invariant subspace method is closely related to the conditional Lie–Bäcklund symmetry. On the other hand, using the invariant condition (5), one can obtain the maximal dimension of the invariant subspaces preserved by an arbitrary kth-order nonlinear differential operator $F[u]$, which is described by the following theorem.

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Theorem 1 If a linear subspace W_n is invariant under a nonlinear ordinary differential operator F of order k, then

$$n \le 2k+1.$$

In Ref. [13], the author studied the nonlinear differential operators of first and second order possessing invariant subspaces of maximal dimension, and showed that they are quadratic in y. For the third-order nonlinear differential operators, according to Theorem 1, we know that the dimension of invariant subspaces cannot exceed 7. When the dimension of invariant subspaces preserved by the third-order nonlinear differential operators (6) is less than 7, the classification results corresponding to Eq. (1) include KdV-type equations. Does this hold in the case of invariant subspaces of the maximal dimension 7?

The purpose of the paper is to characterize the nonlinear differential operators of third order possessing invariant subspaces with the maximal dimension 7. In Section 2, we claim that these operators are quadratic in u_{xxx}, u_{xx}, u_x and u. A complete description of the quadratic operators with constant coefficients is given in Section 3. The last section gives the concluding remarks.

2. Third-order operators possessing invariant subspaces of maximal dimension

In this section, we discuss the invariant subspaces of the maximal dimension 7 for the third-order nonlinear operator

$$F[u] \equiv F(x,u,u_x,u_{xx},u_{xxx}), \tag{6}$$

which is defined by the following ODE:

$$\begin{aligned} L[y] \equiv\ & y^{(7)} + a_6(x)y^{(6)} + a_5(x)y^{(5)} + a_4(x)y^{(4)} + a_3(x)y''' \\ & + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \end{aligned} \tag{7}$$

In terms of Eq. (5), the corresponding invariant condition with respect to $F[u]$ takes the form

$$\begin{aligned} & D^7F + a_6(x)D^6F + a_5(x)D^5F + a_4(x)D^4F + a_3(x)D^3F \\ & + a_2(x)D^2F + a_1(x)DF + a_0(x)F|_{L[u]} = 0, \end{aligned} \tag{8}$$

where

$$\begin{aligned} D =\ & \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + u_4\frac{\partial}{\partial u_3} + u_5\frac{\partial}{\partial u_4} \\ & + u_6\frac{\partial}{\partial u_5} - [a_6(x)u_6 + a_5(x)u_5 + a_4(x)u_4 + a_3(x)u_3 \\ & + a_2(x)u_2 + a_1(x)u_1 + a_0(x)u]\frac{\partial}{\partial u_6}. \end{aligned}$$

For simplicity, we use the notation $u_i = (\partial^i u)/(\partial x^i)$ $(i = 1,2,\ldots)$. From the invariant condition (8), we have the following theorem.

Theorem 2 The operator (6) preserving a seven-dimensional invariant subspace can be expressed in the form of

$$F[u] = F_2[u] + F_1[u] + F_0[u],$$

where

$$\begin{aligned} F_2[u] =\ & F_3(x)u_{xxx}^2 + F_{22}(x)u_{xxx}u_{xx} + F_{212}(x)u_{xxx}u_x \\ & + F_{2112}(x)u_{xxx}u + F_{13}(x)u_{xx}^2 + F_{122}(x)u_{xx}u_x \\ & + F_{1212}(x)u_{xx}u + F_{113}(x)u_x^2 + F_{1122}(x)u_xu \\ & + F_{1113}(x)u^2, \\ F_1[u] =\ & F_{2111}(x)u_{xxx} + F_{1211}(x)u_{xx} + F_{1121}(x)u_x \\ & + F_{1112}(x)u, \\ F_0[u] =\ & F_{1111}(x). \end{aligned}$$

Proof The left-hand side of Eq. (8) is a polynomial in the variables u_4, u_5, and u_6, whose coefficients should be zero.

Observing the coefficient of $u_6u_5^2$, we find that

$$F_{u_3u_3u_3} = 0. \tag{9}$$

Then the coefficient of u_5^3 becomes

$$F_{u_2u_3u_3} = 0. \tag{10}$$

By virtue of Eqs. (9) and (10), the coefficient of $u_5^2u_4$ yields

$$3F_{u_1u_3u_3} + 4F_{u_2u_2u_3} = 0.$$

Meanwhile, the coefficient of u_6^2, on differentiation in u_2, leads to

$$2F_{u_1u_3u_3} + F_{u_2u_2u_3} = 0.$$

The above two equalities imply that

$$F_{u_1u_3u_3} = F_{u_2u_2u_3} = 0. \tag{11}$$

Thus, F has the form

$$\begin{aligned} F =\ & F_3(x,u)u_3^2 + [F_{22}(x,u,u_1)u_2 \\ & + F_{21}(x,u,u_1)]u_3 + F_1(x,u,u_1,u_2). \end{aligned} \tag{12}$$

Substituting Eq. (12) into Eq. (8) and differentiating the coefficient of u_6^2 in u_1 and the coefficient of u_6u_5 in u_2, we have

$$140\frac{\partial F_3}{\partial u} + 35\frac{\partial F_{22}}{\partial u_1} = 0, \quad 420\frac{\partial F_3}{\partial u} + 301\frac{\partial F_{22}}{\partial u_1} + 35\frac{\partial^3 F_1}{\partial u_2^3} = 0,$$

and the coefficient of $u_6u_4u_3$ implies that

$$364\frac{\partial F_3}{\partial u} + 553\frac{\partial F_{22}}{\partial u_1} + 105\frac{\partial^3 F_1}{\partial u_2^3} = 0.$$

From the above equalities, we obtain $\partial F_3/\partial u = 0$, $\partial F_{22}/\partial u_1 = 0$, $\partial^3 F_1/\partial u_2^3 = 0$. Then we can see that the coefficient of u_6^2 turns into

$$35F_{22}(x,u) - 50a_6(x)F_3(x) + 14F_3'(x) = 0,$$

which, on differentiation with respect to u, leads to $\partial F_{22}/\partial u = 0$. Moreover, collecting the coefficients of u_4^3 and $u_5u_4u_3$ respectively yields

$$70\frac{\partial^2 F_{21}}{\partial u_1^2} + 105\frac{\partial^3 F_1}{\partial u_1 \partial u_2^2} = 0,$$
$$350\frac{\partial^2 F_{21}}{\partial u_1^2} + 455\frac{\partial^3 F_1}{\partial u_1 \partial u_2^2} = 0,$$

which means that $\partial^2 F_{21}/\partial u_1^2 = 0$, $\partial^3 F_1/\partial u_1 \partial u_2^2 = 0$. Thus

$$\begin{aligned} F &= F_3(x)u_3^2 + [F_{22}(x)u_2 + F_{212}(x,u)u_1 + F_{211}(x,u)]u_3 \\ &\quad + F_{13}(x,u)u_2^2 + F_{12}(x,u,u_1)u_2 + F_{11}(x,u,u_1). \end{aligned} \tag{13}$$

Inserting Eq. (13) into Eq. (8), we find that the left-hand side of Eq. (8) is a polynomial in the variables u_2, u_3, u_4, u_5, u_6, and collecting the coefficients of $u_5u_4u_2$ and $u_5u_3^2$ respectively leads to

$$532\frac{\partial F_{13}}{\partial u} + 280\frac{\partial^2 F_{12}}{\partial u_1^2} + 462\frac{\partial F_{212}}{\partial u} = 0,$$
$$322\frac{\partial F_{13}}{\partial u} + 210\frac{\partial^2 F_{12}}{\partial u_1^2} + 266\frac{\partial F_{212}}{\partial u} = 0.$$

Differentiating the coefficient of u_5^2 with respect to u_1, we obtain

$$140\frac{\partial F_{13}}{\partial u} + 35\frac{\partial^2 F_{12}}{\partial u_1^2} + 126\frac{\partial F_{212}}{\partial u} = 0,$$

which leads to $\partial F_{13}/\partial u = 0$, $\partial^2 F_{12}/\partial u_1^2 = 0$, and $\partial F_{212}/\partial u = 0$. In view of the above results, considering the coefficients of $u_4^2u_2$, $u_4u_3^2$, and vanishing them, we have

$$105\frac{\partial^2 F_{211}}{\partial u^2} + 350\frac{\partial^2 F_{12}}{\partial u \partial u_1} + 70\frac{\partial^3 F_{11}}{\partial u_1^3} = 0,$$
$$105\frac{\partial^2 F_{211}}{\partial u^2} + 455\frac{\partial^2 F_{12}}{\partial u \partial u_1} + 105\frac{\partial^3 F_{11}}{\partial u_1^3} = 0,$$

and the coefficient of u_5^2, on differentiation in u, implies that

$$21\frac{\partial^2 F_{211}}{\partial u^2} + 35\frac{\partial^2 F_{12}}{\partial u \partial u_1} = 0,$$

then we have $\partial^2 F_{211}/\partial u^2 = 0$, $\partial^2 F_{12}/\partial u \partial u_1 = 0$, $\partial^3 F_{11}/\partial u_1^3 = 0$. Therefore, F can be expressed as follows:

$$\begin{aligned} F &= F_3(x)u_3^2 + [F_{22}(x)u_2 + F_{212}(x)u_1 + F_{2112}(x)u \\ &\quad + F_{2111}(x)]u_3 + F_{13}(x)u_2^2 + [F_{122}(x)u_1 + F_{121}(x,u)]u_2 \\ &\quad + F_{113}(x,u)u_1^2 + F_{112}(x,u)u_1 + F_{111}(x,u). \end{aligned} \tag{14}$$

Substituting Eq. (14) into Eq. (8) and equating the coefficients of $u_4u_3u_2$ and u_3^3 to zero, we have

$$350\frac{\partial^2 F_{121}}{\partial u^2} + 910\frac{\partial F_{113}}{\partial u} = 0,$$
$$70\frac{\partial^2 F_{121}}{\partial u^2} + 210\frac{\partial F_{113}}{\partial u} = 0,$$

which means that $F_{121}(x,u) = F_{1212}(x)u + F_{1211}(x)$, $F_{113}(x,u) = F_{113}(x)$. Finally, noting that with the coefficients of $u_3^2u_2$ and $u_5u_1^2$, we have $\partial^2 F_{112}/\partial u^2 = 0$, $\partial^3 F_{111}/\partial u^3 = 0$, then $F_{112}(x,u) = F_{1122}(x)u + F_{1121}(x)$, $F_{111}(x,u) = F_{1113}(x)u^2 + F_{1112}(x)u + F_{1111}(x)$. This completes the proof.

3. Third-order quadratic operators preserving maximal-dimensional invariant subspaces

Now we discuss the third-order quadratic operators with constant coefficients

$$\begin{aligned} F[u] &= b_1u_{xxx}^2 + b_2u_{xx}u_{xxx} + b_3u_xu_{xxx} + b_4uu_{xxx} + b_5u_{xx}^2 \\ &\quad + b_6u_xu_{xx} + b_7uu_{xx} + b_8u_x^2 + b_9uu_x + b_{10}u^2, \end{aligned} \tag{15}$$

preserving seven-dimensional invariant subspaces, given by Eq. (7), where $b_i = \text{const}$, $i = 1, 2, \ldots, 10$. Further analysis of the invariant condition (8) leads to the following statement.

Lemma 1 If the subspace defined by Eq. (7) is invariant with respect to operator (15), then the coefficients $a_i(x)$ $(i = 0, 1, \ldots, 6)$ of Eq. (7) are independent of x.

This lemma can be verified similarly as done in Refs. [19] and [20].

The next theorem gives a complete description of operator (15) possessing the invariant subspaces defined by Eq. (7).

Theorem 3 The operators (15) with constant coefficients preserving seven-dimensional invariant subspaces given by Eq. (7) belong to the following list:

(I)

$$\begin{aligned} F[u] &= F_1[u] + b_4F_2[u], \\ F_1[u] &= u_{xxx}^2 + 10u_{xxx}u_{xx} + 8u_{xxx}u_x + \frac{77}{5}u_{xx}^2 - \frac{232}{5}u_{xx}u_x \\ &\quad - \frac{192}{5}u_{xx}u - \frac{772}{5}u_x^2 + \frac{96}{5}u_xu + \frac{1728}{5}u^2, \\ F_2[u] &= \frac{1}{4}u_{xxx}u_x + u_{xxx}u - \frac{1}{5}u_{xx}^2 - \frac{11}{20}u_{xx}u_x \\ &\quad + \frac{21}{5}u_{xx}u - \frac{33}{10}u_x^2 - \frac{8}{5}u_xu - \frac{24}{5}u^2, \\ W_7 &= \mathscr{L}\{1, \mathrm{e}^{-4x}, \mathrm{e}^{-3x}, \mathrm{e}^{-2x}, \mathrm{e}^{-x}, \mathrm{e}^{x}, \mathrm{e}^{2x}\}, \\ y^{(7)} &= -7y^{(6)} - 7y^{(5)} + 35y^{(4)} + 56y''' - 28y'' - 48y', \end{aligned}$$

where b_4 is an arbitrary constant;

(II)

$$\begin{aligned} F[u] &= \left(u_{xxx} + 6u_{xx} + \frac{47}{4}u_x + \frac{15}{2}u\right) \\ &\quad \times \left(u_{xxx} + 4u_{xx} + \frac{11}{4}u_x - \frac{5}{2}u\right), \\ W_7 &= \mathscr{L}\{1, \mathrm{e}^{-5x/2}, \mathrm{e}^{-2x}, \mathrm{e}^{-3x/2}, \mathrm{e}^{-x}, \mathrm{e}^{-x/2}, \mathrm{e}^{x/2}\}, \\ y^{(7)} &= -7y^{(6)} - \frac{35}{2}y^{(5)} - \frac{35}{2}y^{(4)} - \frac{49}{16}y''' + \frac{77}{16}y'' + \frac{15}{8}y'; \end{aligned}$$

(III)

$$\begin{aligned} F[u] &= (u_{xxx}+17u_{xx}-518u_x-5880u)\\ &\quad\times(u_{xxx}-7u_{xx}-686u_x+8232u),\\ W_7 &= \mathscr{L}\{1,\mathrm{e}^{-28x},\mathrm{e}^{-14x},\mathrm{e}^{-7x},\mathrm{e}^{7x},\mathrm{e}^{14x},\mathrm{e}^{21x}\},\\ y^{(7)} &= -7y^{(6)}+833y^{(5)}+1715y^{(4)}-153664y'''\\ &\quad-67228y''+5647152y'; \end{aligned}$$

(IV)

$$\begin{aligned} F[u] &= \left(u_{xxx}+5u_{xx}+\frac{74}{9}u_x+\frac{40}{9}u\right)^2,\\ W_7 &= \mathscr{L}\{1,\mathrm{e}^{-2x},\mathrm{e}^{-5x/3},\mathrm{e}^{-4x/3},\mathrm{e}^{-x},\mathrm{e}^{-2x/3},\mathrm{e}^{-x/3}\},\\ y^{(7)} &= -7y^{(6)}-\frac{175}{9}y^{(5)}-\frac{245}{9}y^{(4)}\\ &\quad-\frac{1624}{81}y'''-\frac{196}{27}y''-\frac{80}{81}y'; \end{aligned}$$

(V)

$$\begin{aligned} F[u] &= u_{xxx}^2+10u_{xxx}u_{xx}+\frac{119}{32}u_{xxx}u_x-\frac{245}{32}u_{xxx}u+\frac{315}{16}u_{xx}^2\\ &\quad-\frac{245}{8}u_{xx}u_x-\frac{25725}{512}u_{xx}u-\frac{409199}{4096}u_x^2\\ &\quad+\frac{12005}{512}u_xu+\frac{13865775}{65536}u^2,\\ W_7 &= \mathscr{L}\{1,\mathrm{e}^{-35x/8},\mathrm{e}^{-21x/8},\mathrm{e}^{-7x/4},\mathrm{e}^{-7x/8},\mathrm{e}^{7x/8},\mathrm{e}^{7x/4}\},\\ y^{(7)} &= -7y^{(6)}-\frac{245}{32}y^{(5)}+\frac{1715}{64}y^{(4)}+\frac{170471}{4096}y'''\\ &\quad-\frac{16807}{1024}y''-\frac{1764735}{65536}y'; \end{aligned}$$

(VI)

$$\begin{aligned} F[u] &= u_{xxx}^2+10u_{xxx}u_{xx}+\frac{196}{81}u_{xxx}u_x-\frac{784}{243}u_{xxx}u+21u_{xx}^2\\ &\quad-\frac{2156}{81}u_{xx}u_x-\frac{5488}{243}u_{xx}u-\frac{570752}{6561}u_x^2\\ &\quad+\frac{268912}{19683}uu_x+\frac{6722800}{59049}u^2,\\ W_7 &= \mathscr{L}\{1,\mathrm{e}^{-14x/3},\mathrm{e}^{-7x/3},\mathrm{e}^{-14x/9},\mathrm{e}^{-7x/9},\mathrm{e}^{7x/9},\mathrm{e}^{14x/9}\},\\ y^{(7)} &= -7y^{(6)}-\frac{637}{81}y^{(5)}+\frac{1715}{81}y^{(4)}+\frac{206486}{6561}y'''\\ &\quad-\frac{67228}{6561}y''-\frac{941192}{59049}y'; \end{aligned}$$

(VII)

$$\begin{aligned} F[u] &= \left(u_{xxx}+\frac{21}{5}u_{xx}+\frac{686}{225}u_x-\frac{2744}{1125}u\right)\\ &\quad\times\left(u_{xxx}+\frac{29}{5}u_{xx}+\frac{2366}{225}u_x+\frac{6664}{1125}u\right),\\ W_7 &= \mathscr{L}\{1,\mathrm{e}^{-14x/5},\mathrm{e}^{-28x/15},\mathrm{e}^{-7x/5},\mathrm{e}^{-14x/15},\\ &\quad\mathrm{e}^{-7x/15},\mathrm{e}^{7x/15}\},\\ y^{(7)} &= -7y^{(6)}-\frac{3871}{225}y^{(5)}-\frac{3773}{225}y^{(4)}-\frac{153664}{50625}y'''\\ &\quad+\frac{67228}{16875}y''+\frac{1882384}{1265625}y'; \end{aligned}$$

(VIII)

$$\begin{aligned} F[u] &= u_{xxx}^2+b_3\left(u_{xxx}u_x-\frac{4}{5}u_{xx}^2\right),\\ W_7 &= \mathscr{L}\{1,x,x^2,x^3,x^4,x^5,x^6\},\\ y^{(7)} &= 0, \end{aligned}$$

where b_3 is an arbitrary constant;

(IX)

$$\begin{aligned} F[u] &= F_1[u]+b_3F_2[u],\\ F_1[u] &= u_{xxx}^2+\frac{46}{35}u_{xx}^2+\frac{9}{245}u_{xx}u+\frac{99}{196}u_x^2+\frac{243}{3430}u^2,\\ F_2[u] &= u_{xxx}u_x-\frac{4}{5}u_{xx}^2+\frac{9}{35}u_{xx}u-\frac{1}{14}u_x^2+\frac{9}{245}u^2,\\ W_7 &= \mathscr{L}\left\{1,\sin\frac{\sqrt{14}x}{14},\cos\frac{\sqrt{14}x}{14},\sin\frac{\sqrt{14}x}{7},\cos\frac{\sqrt{14}x}{7},\right.\\ &\quad\left.\sin\frac{3\sqrt{14}x}{14},\cos\frac{3\sqrt{14}x}{14}\right\},\\ y^{(7)} &= -y^{(5)}+\frac{1}{4}y'''+\frac{9}{686}y', \end{aligned}$$

where b_3 is an arbitrary constant;

(X)

$$\begin{aligned} F[u] &= u_{xxx}^2+\frac{32}{21}u_{xxx}u_x+\frac{2}{21}u_{xx}^2+\frac{64}{441}u_{xx}u\\ &\quad+\frac{256}{441}u_x^2+\frac{512}{9261}u^2,\\ W_7 &= \mathscr{L}\left\{1,\sin\frac{\sqrt{21}x}{21},\cos\frac{\sqrt{21}x}{21},\sin\frac{2\sqrt{21}x}{21},\right.\\ &\quad\left.\cos\frac{2\sqrt{21}x}{21},\sin\frac{4\sqrt{21}x}{21},\cos\frac{4\sqrt{21}x}{21}\right\},\\ y^{(7)} &= -y^{(5)}+\frac{4}{21}y'''+\frac{64}{9261}y'; \end{aligned}$$

(XI)

$$\begin{aligned} F[u] &= u_{xxx}u_x-\frac{4}{5}u_{xx}^2+\frac{9}{35}u_{xx}u-\frac{1}{14}u_x^2+\frac{9}{245}u^2,\\ W_7 &= \mathscr{L}\left\{1,\sin\frac{\sqrt{14}x}{14},\cos\frac{\sqrt{14}x}{14},\sin\frac{\sqrt{14}x}{7},\right.\\ &\quad\left.\cos\frac{\sqrt{14}x}{7},\sin\frac{3\sqrt{14}x}{14},\cos\frac{3\sqrt{14}x}{14}\right\},\\ y^{(7)} &= -y^{(5)}+\frac{1}{4}y'''+\frac{9}{686}y''+\frac{48}{117649}y'; \end{aligned}$$

(XII)

$$\begin{aligned} F[u] &= u_{xxx}u_x+\frac{4}{7}u_{xxx}u-\frac{4}{5}u_{xx}^2-\frac{11}{35}u_{xx}u_x\\ &\quad+\frac{12}{35}u_{xx}u-\frac{66}{245}u_x^2-\frac{32}{1715}u_xu-\frac{96}{12005}u^2,\\ W_7 &= \mathscr{L}\{1,\mathrm{e}^{-4x/7},\mathrm{e}^{-3x/7},\mathrm{e}^{-2x/7},\mathrm{e}^{-x/7},\mathrm{e}^{x/7},\mathrm{e}^{2x/7}\},\\ y^{(7)} &= -y^{(6)}+\frac{1}{7}y^{(5)}-\frac{5}{49}y^{(4)}-\frac{8}{343}y'''+\frac{4}{2401}y''\\ &\quad+\frac{48}{117649}y'. \end{aligned}$$

Proof According to Lemma 1, we investigate operators (15) preserving seven-dimensional invariant subspaces defined by the following ODE:

$$\begin{aligned} L[y] \equiv\ & y^{(7)} + a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y''' \\ & + a_2 y'' + a_1 y' + a_0 y \\ =\ & 0, \end{aligned} \tag{16}$$

where a_i $(i = 0, 1, \ldots, 6)$ are constants. Then the corresponding invariant condition is

$$\begin{aligned} & D^7 F + a_6 D^6 F + a_5 D^5 F + a_4 D^4 F + a_3 D^3 F \\ & + a_2 D^2 F + a_1 DF + a_0 F|_{L[u]} = 0. \end{aligned} \tag{17}$$

The left-hand side of the invariant condition (17) turns into a polynomial in variables u and u_i $(i = 1, 2, \ldots, 6)$, namely,

$$\begin{aligned} & (35b_2 - 50a_6b_1)u_6^2 + (12b_1a_6^2 + 56b_3 + 70b_5 - 92a_5b_1 \\ & - 21b_2a_6)u_6u_5 + (-12a_6b_6 - 13a_5b_2 + 56b_6 - 16a_6b_3 \\ & + 6a_6b_1a_5 + 7b_2a_6^2 - 76a_4b_1 + 42b_4 - 2b_1a_6^3)u_6u_4 \\ & + (42b_7 - 7a_6b_6 - 4a_5b_5 + 42b_8 - b_2a_6^3 + 3b_2a_6a_5 \\ & - 70b_1a_3 + 2b_5a_6^2 - 11b_3a_5 - 3b_2a_4 + 6b_3a_6^2 - 15a_6b_4)u_6u_3 \\ & + (6b_4a_6^2 - 3a_4b_3 - 11a_5b_4 - 2a_6b_8 + 3b_3a_6a_5 - 70b_1a_2 \\ & - b_3a_6^3 + 28b_9 - 6a_6b_7 + b_6a_6^2 - 2a_5b_6)u_6u_2 + (b_7a_6^2 \\ & - 3b_4a_4 + 3a_6b_4a_5 - a_6b_9 - 2a_5b_7 - b_4a_6^3 + 14b_{10} \\ & - 70a_1b_1)u_6u_1 - 70b_1a_0u_6u + (21b_4 - 46a_5b_2 + 12a_6b_1a_5 \\ & - 36a_4b_1 + 15a_6b_3 + 20a_6b_5 + 35b_6)u_5^2 + (21a_6b_4 \\ & - 27b_3a_5 - 74b_2a_4 + 35a_6b_6 + 4b_1a_5^2 - 22a_5b_5 + 56b_7 \\ & + 7b_2a_6a_5 - 2b_1a_6^2a_5 + 14a_6b_1a_4 + 70b_8 - 50b_1a_3)u_5u_4 \\ & + (-13a_5b_6 - 6a_4b_5 - 42b_1a_2 + a_6b_2a_4 + 2a_6b_5a_5 \\ & + 12b_1a_6a_3 + 2b_2a_5^2 + 56b_9 + 30a_6b_8 + 26a_6b_7 - 15a_4b_3 \\ & + 6b_3a_6a_5 - 25a_5b_4 - b_2a_6^2a_5 - 60b_2a_3)u_5u_3 + (-b_3a_6^2a_5 \\ & + 6a_6b_4a_5 + b_3a_6a_4 - 4b_3a_3 - 15b_4a_4 + 12b_1a_6a_2 \\ & + 21a_6b_9 - 56b_2a_2 + b_6a_6a_5 - 11a_5b_7 - 4a_5b_8 - 3b_6a_4 \\ & + 42b_{10} + 2b_3a_5^2 - 42a_1b_1)u_5u_2 + (a_6b_4a_4 - 3a_4b_7 \\ & - 56a_1b_2 + 12a_6b_{10} + 2b_4a_5^2 + a_6b_7a_5 - 42a_0b_1 - 2a_5b_9 \\ & - 4b_4a_3 - b_4a_6^2a_5 + 12b_1a_6a_1)u_5u_1 + (12b_1a_6a_0 \\ & - 56b_2a_0)u_5u + (-12b_1a_2 + 10a_5b_6 + 15a_6b_7 + 35b_9 \\ & + 4a_5b_1a_4 - 2b_1a_6^2a_4 - 36a_4b_5 - 38a_4b_3 + 7a_6b_2a_4 \\ & + 20a_6b_8 + 2b_1a_6a_3 + 5a_5b_4 - 25b_2a_3)u_4^2 + (15a_5b_7 \\ & + 20a_5b_8 + 70b_{10} + 2b_1a_6a_2 + 35a_6b_9 + 2b_2a_5a_4 - 50b_5a_3 \\ & + 6b_3a_6a_4 - 14a_1b_1 + 2a_6b_5a_4 - 2b_1a_6^2a_3 - b_2a_6^2a_4 \\ & - 31b_4a_4 + 4a_5b_1a_3 + 8b_2a_6a_3 - 18b_6a_4 - 33b_2a_2 \\ & - 60b_3a_3)u_4u_3 + (-b_3a_6^2a_4 + 30a_6b_{10} + 4a_5b_1a_2 \\ & - 2b_1a_6^2a_2 + b_6a_6a_4 - 4b_6a_3 - 18b_4a_3 + 15a_5b_9 \\ & + 6a_6b_4a_4 + a_6b_3a_3 - 15a_4b_7 + 7a_6b_2a_2 - 42b_5a_2 \\ & - 47b_3a_2 - 6a_4b_8 + 2a_5b_3a_4 - 14a_0b_1 - 28a_1b_2 \\ & + 2b_1a_6a_1)u_4u_2 + (a_6b_7a_4 + a_6b_4a_3 + 2b_1a_6a_0 - 28b_2a_0 \\ & + 7b_2a_6a_1 + 2a_5b_4a_4 - 42a_1b_5 + 4b_1a_5a_1 + 10a_5b_{10} \\ & - 42a_1b_3 - 2b_1a_6^2a_1 - 4b_7a_3 - 5b_4a_2 - b_4a_6^2a_4 \\ & - 3a_4b_9)u_4u_1 + (-2b_1a_6^2a_0 + 7b_2a_6a_0 - 42a_0b_5 \\ & - 42b_3a_0 + 4b_1a_5a_0)u_4u + (6a_4b_8 - 12b_5a_2 + 10a_5b_9 \\ & + 2b_5a_6a_3 + 2a_5b_2a_3 + 20a_6b_{10} - a_0b_1 + 4a_4b_7 - b_2a_6^2a_3 \\ & + a_6b_2a_2 + 6a_6b_3a_3 - 21b_3a_2 - 35b_4a_3 - 7a_1b_2 \\ & - 25b_6a_3)u_3^2 + (-b_2a_6^2a_2 - 56b_4a_2 + 20a_5b_{10} + 2a_5b_2a_2 \\ & + b_6a_6a_3 - 14a_1b_5 - 8b_8a_3 + 2b_5a_6a_2 - 28a_1b_3 - 18b_7a_3 \\ & - b_3a_6^2a_3 + 2a_5b_3a_3 - 33b_6a_2 + 10a_4b_9 + 6a_6b_4a_3 - 8b_2a_0 \\ & + 7a_6b_3a_2 + b_2a_6a_1)u_3u_2 + (6b_3a_6a_1 + 2b_5a_6a_1 + b_2a_6a_0 \\ & + 2b_2a_5a_1 - 22b_3a_0 - 14a_0b_5 + a_6b_4a_2 + a_6b_7a_3 - 4b_9a_3 \\ & - 42b_4a_1 - 28a_1b_6 - b_4a_6^2a_3 + 2a_5b_4a_3 - 5b_7a_2 - b_2a_6^2a_1 \\ & + 8a_4b_{10})u_3u_1 + (2b_2a_5a_0 - b_2a_6^2a_0 + 2b_5a_6a_0 - 28b_6a_0 \\ & - 36b_4a_0 + 6b_3a_6a_0)u_3u + (-7a_1b_6 + 6a_4b_{10} + 2a_5b_3a_2 \\ & + 3b_9a_3 - 12b_8a_2 - a_0b_5 - 7b_3a_0 + b_6a_6a_2 + 6a_6b_4a_2 \\ & - b_3a_6^2a_2 - 21b_4a_1 + b_3a_6a_1 - 21b_7a_2)u_2^2 + (-5b_9a_2 \\ & + 7b_4a_6a_1 - b_4a_6^2a_2 + 2a_5b_4a_2 - 28a_1b_7 + a_6b_7a_2 - 14a_1b_8 \\ & - 8b_6a_0 - 28b_4a_0 + b_6a_6a_1 - b_3a_6^2a_1 + 6b_{10}a_3 + b_3a_6a_0 \\ & + 2b_3a_5a_1)u_2u_1 + (-14a_0b_8 + 6b_4a_6a_0 - b_3a_6^2a_0 + 2b_3a_5a_0 \\ & + b_6a_6a_0 - 22b_7a_0)u_2u + (b_4a_6a_0 - 7a_1b_9 + 2b_4a_5a_1 \\ & - b_4a_6^2a_1 - b_7a_6a_1 - 7b_7a_0 - a_0b_8 + 2b_{10}a_2)u_1^2 \\ & + (2b_4a_5a_0 + b_7a_6a_0 - 8a_0b_9 - b_4a_6^2a_0)u_1u - a_0b_{10}u^2. \end{aligned} \tag{18}$$

Equating all the coefficients to zero yields an algebraic system for a_i, b_j $(i = 0, \ldots, 6, j = 1, \ldots, 10)$. To solve the system, first, we deduce from the coefficients of u_6^2 that $b_2 = 10a_6b_1/7$, so we need to discuss the following two cases: $b_1 \neq 0$ and $b_1 = 0$.

Case (i) $b_1 \neq 0$. For convenience, let $b_1 = 1$, then $a_6 = 7b_2/10$. Here we distinguish two subcases: $b_2 \neq 0$ and $b_2 = 0$.

Subcase (i.1) $b_2 \neq 0$. Using the scaling in x means $b_2 = 10$. Then solving the system, we derive the following seven solutions:

$$\begin{aligned} & a_0 = 0,\ a_1 = 48,\ a_2 = 28,\ a_3 = -56,\ a_4 = -35, \\ & a_5 = 7,\ a_6 = 7,\ b_1 = 1,\ b_2 = 10,\ b_3 = 8 + \frac{1}{4}b_4, \\ & b_5 = \frac{77}{5} - \frac{1}{5}b_4,\ b_6 = -\frac{11}{20}b_4 - \frac{232}{5}, \\ & b_7 = -\frac{192}{5} + \frac{21}{5}b_4,\ b_8 = -\frac{772}{5} - \frac{33}{10}b_4, \\ & b_9 = -\frac{8}{5}b_4 + \frac{96}{5},\ b_{10} = \frac{1728}{5} - \frac{24}{5}b_4, \end{aligned}$$

with an arbitrary constant b_4,

$$a_0 = 0,\ a_1 = -\frac{15}{8},\ a_2 = -\frac{77}{16},\ a_3 = \frac{49}{16},$$

$$a_4=\frac{35}{2},\ a_5=\frac{35}{2},\ a_6=7,\ b_1=1,\ b_2=10,$$
$$b_3=\frac{29}{2},\ b_4=5,\ b_5=24,\ b_6=\frac{127}{2},\ b_7=15,$$
$$b_8=\frac{517}{16},\ b_9=-\frac{35}{4},\ b_{10}=-\frac{75}{4};$$
$$a_0=0,\ a_1=-5647152,\ a_2=67228,\ a_3=153664,$$
$$a_4=-1715,\ a_5=-833,\ a_6=7,\ b_1=1,\ b_2=10,$$
$$b_3=-1204,\ b_4=2352,\ b_5=-119,\ b_6=-8036,$$
$$b_7=181104,\ b_8=355348,\ b_9=-230496,$$
$$b_{10}=-48404160;$$
$$a_0=0,\ a_1=\frac{80}{81},\ a_2=\frac{196}{27},\ a_3=\frac{1624}{81},\ a_4=\frac{245}{9},$$
$$a_5=\frac{175}{9},\ a_6=7,\ b_1=1,\ b_2=10,\ b_3=\frac{148}{9},$$
$$b_4=\frac{80}{9},\ b_5=25,\ b_6=\frac{740}{9},\ b_7=\frac{400}{9},$$
$$b_8=\frac{5476}{81},\ b_9=\frac{5929}{81},\ b_{10}=\frac{1600}{81};$$
$$a_0=0,\ a_1=\frac{1764735}{65536},\ a_2=\frac{16807}{1024},\ a_3=-\frac{170471}{4096},$$
$$a_4=-\frac{1715}{64},\ a_5=\frac{245}{32},\ a_6=7,\ b_1=1,\ b_2=10,$$
$$b_3=\frac{119}{32},\ b_4=-\frac{245}{32},\ b_5=\frac{315}{16},\ b_6=-\frac{245}{8},$$
$$b_7=-\frac{25725}{512},\ b_8=-\frac{409199}{4096},\ b_9=\frac{12005}{512},$$
$$b_{10}=\frac{13865775}{65536};$$
$$a_0=0,\ a_1=\frac{941192}{59049},\ a_2=\frac{67228}{6561},\ a_3=-\frac{206486}{6561},$$
$$a_4=-\frac{1715}{81},\ a_5=\frac{637}{81},\ a_6=7,\ b_1=1,\ b_2=10,$$
$$b_3=\frac{196}{81},\ b_4=-\frac{784}{243},\ b_5=21,\ b_6=-\frac{2156}{81},$$
$$b_7=-\frac{5488}{243},\ b_8=-\frac{570752}{6561},\ b_9=\frac{268912}{19683},$$
$$b_{10}=\frac{6722800}{59049};$$
$$a_0=0,\ a_1=-\frac{1882384}{1265625},\ a_2=-\frac{67228}{16875},\ a_3=\frac{153664}{50625},$$
$$a_4=\frac{3773}{225},\ a_5=\frac{3871}{225},\ a_6=7,\ b_1=1,\ b_2=10,$$
$$b_3=\frac{3052}{225},\ b_4=\frac{784}{225},\ b_5=\frac{609}{25},\ b_6=\frac{13916}{225},$$
$$b_7=\frac{60368}{5625},\ b_8=\frac{1623076}{50625},\ b_9=-\frac{76832}{10125},$$
$$b_{10}=-\frac{18286016}{1265625}.$$

Plugging these solutions into Eqs. (15) and (7), we obtain results (1)–(7) in the theorem.

Subcase (i.2) $b_2=0$. In this subcase, solving the system yields

$$a_0=0,\ a_1=\frac{9}{686}a_5^3,\ a_2=0,\ a_3=\frac{1}{4}a_5^2,\ a_4=0,$$
$$a_6=0,\ b_1=1,\ b_2=0,\ b_4=0,\ b_5=-\frac{4}{5}b_3+\frac{46}{35}a_5,$$
$$b_6=0,\ b_7=\frac{9}{35}b_3a_5+\frac{9}{245}a_5^2,\ b_8=-\frac{1}{14}b_3a_5+\frac{99}{196}a_5^2,$$
$$b_9=0,\ b_{10}=\frac{9}{245}b_3a_5^2+\frac{243}{3430}a_5^3, \tag{19}$$

with arbitrary constants a_5 and b_3;

$$a_0=0,\ a_1=\frac{64}{9261}a_5^3,\ a_2=0,\ a_3=\frac{4}{21}a_5^2,\ a_4=0,$$
$$a_6=0,\ b_1=1,\ b_2=0,\ b_3=\frac{32}{21}a_5,\ b_4=0,\ b_5=\frac{2}{21}a_5,$$
$$b_6=0,\ b_7=\frac{64}{441}a_5^2,\ b_8=\frac{256}{441}a_5^2,$$
$$b_9=0,\ b_{10}=\frac{512}{9261}a_5^3, \tag{20}$$

with an arbitrary constant a_5.

Inserting the solution (19) into Eqs. (15) and (7), we obtain

$$\begin{aligned}F[u]\ =\ & u_{xxx}^2+\frac{9}{245}a_5^2uu_{xx}+\frac{46}{35}a_5u_{xx}^2+\frac{99}{196}a_5^2u_x^2+\frac{243}{3430}a_5^3u^2\\ &+b_3\Big(u_{xxx}u_x-\frac{4}{5}u_{xx}^2+\frac{9}{35}a_5uu_{xx}\\ &-\frac{1}{14}a_5u_x^2+\frac{9}{245}a_5^2u^2\Big),\\ L[y]\ \equiv\ & y^{(7)}+a_5y^{(5)}+\frac{1}{4}a_5^2y'''+\frac{9}{686}a_5^3y'=0.\end{aligned}$$

If $a_5=0$, we have case (8) of the theorem. If $a_5\neq 0$, further analysis of Eqs. (19) and (20) leads to results (9) and (10) in the theorem.

Case (ii) $b_1=0$. Then the solutions of the system for a_i and b_j are

$$a_0=0,\ a_1=\frac{9}{686}a_5^3,\ a_2=0,\ a_3=\frac{1}{4}a_5^2,\ a_4=0,$$
$$a_6=0,\ b_1=0,\ b_2=0,\ b_4=0,\ b_5=-\frac{4}{5}b_3,$$
$$b_6=0,\ b_7=\frac{9}{35}b_3a_5,\ b_8=-\frac{1}{14}b_3a_5,$$
$$b_9=0,\ b_{10}=\frac{9}{245}b_3a_5^2,$$

with arbitrary constants a_5 and b_3;

$$a_0=0,\ a_1=\frac{48}{117649}a_6^6,\ a_2=\frac{4}{2401}a_6^5,\ a_3=-\frac{8}{343}a_6^4,$$
$$a_4=-\frac{5}{49}a_6^3,\ a_5=\frac{1}{7}a_6^2,\ b_1=0,\ b_2=0,\ b_4=\frac{4}{7}a_6b_3,$$
$$b_5=-\frac{4}{5}b_3,\ b_6=-\frac{11}{35}a_6b_3,\ b_7=\frac{12}{35}b_3a_6^2,$$
$$b_8=-\frac{66}{245}b_3a_6^2,\ b_9=-\frac{32}{1715}b_3a_6^3,\ b_{10}=-\frac{96}{12005}a_6^4b_3,$$

with arbitrary constants a_6 and b_3.

Inserting these two solutions into Eqs. (15) and (7), we obtain cases (11) and (12) in the theorem. This completes the proof of Theorem 3.

Example 1 Consider the equation

$$u_t=u_{xxx}^2, \tag{21}$$

with the simple third-order quadratic operator in Theorem 3; it has solutions on the invariant subspace of maximal dimension 7

$$\begin{aligned} W_7 &= \mathscr{L}\{1,x,x^2,x^3,x^4,x^5,x^6\}, \\ u(x,t) &= C_1(t)+C_2(t)x+C_3(t)x^2+C_4(t)x^3 \\ &\quad + C_5(t)x^4+C_6(t)x^5+C_7(t)x^6. \end{aligned} \tag{22}$$

Substituting Eq. (22) into Eq. (21) gives the following system of ODEs:

$$\begin{aligned} &C_1' = 36C_4^2,\ \ C_2' = 288C_4C_5,\ \ C_3' = 720C_4C_6+576C_5^2, \\ &C_4' = 1440C_4C_7+2880C_5C_6,\ \ C_5' = 5760C_5C_7+3600C_6^2, \\ &C_6' = 14400C_6C_7,\ \ C_7' = 14400C_7^2. \end{aligned}$$

We find equation (21) has the special solution

$$u(x,t) = \frac{(C-6x)(C^5-30C^4x-72C^3x^2-4752C^2x^3-7072Cx^4-7776x^5)}{46656(T-14400t)},$$

where C is an arbitrary positive constant, and $T>0$ is the finite blow-up time. This solution blows up as the time $t = T/14400$.

4. Concluding remarks

In this paper, we have investigated the nonlinear differential operators of third order possessing invariant subspaces with the maximal dimension 7. Under the invariant condition, we have shown that these operators are quadratic in u_{xxx}, u_{xx}, u_x, and u. Moreover, we have completely characterized the third-order quadratic operators with constant coefficients.

There are still some interesting topics to be considered. For instance, what is the form of the third-order nonlinear differential operators possessing invariant subspaces with lower dimensions? How can we verify the general assertion that any nonlinear differential operator of kth order preserving invariant subspaces with the maximal dimension is quadratic?

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