Lie symmetry theorem of fractional nonholonomic systems*

Sun Yi(孙 毅)[a], Chen Ben-Yong(陈本永)[a], and Fu Jing-Li(傅景礼)[b]†

[a] Faculty of Mechanical-Engineering & Automation, Zhejiang Sci-Tech University, Hangzhou 310018, China
[b] Institute of Mathematical Physics, Zhejiang Sci-Tech University, Hangzhou 310018, China

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The Lie symmetry theorem of fractional nonholonomic systems in terms of combined fractional derivatives is established, and the fractional Lagrange equations are obtained by virtue of the d'Alembert–Lagrange principle with fractional derivatives. As the Lie symmetry theorem is based on the invariance of differential equations under infinitesimal transformations, by introducing the differential operator of infinitesimal generators, the determining equations are obtained. Furthermore, the limit equations, the additional restriction equations, the structural equations, and the conserved quantity of Lie symmetry are acquired. An example is presented to illustrate the application of results.

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1. Introduction

The research on the symmetries and the conserved quantities of mechanical systems has not only got important mathematical significance, but also a profound physical background. There are three kinds of symmetries: Noether symmetry, Lie symmetry, and Mei symmetry.[1–12] The symmetry theory has been approved to be a powerful tool to solve differential equations, to study constrained mechanical systems, to discuss controllable dynamical systems, to investigate mechanico-electrical systems and to establish the properties of their solution space.[13–25]

In recent years, the fractional Noether symmetry theories have been described in many of the references.[26–29] Therefore, the study on the symmetries of fractional dynamical systems has become a hot topic. Although the fractional Euler–Lagrangian equations and the fractional Hamiltonian equations have been studied and won wide development,[30–37] there are few papers discussing the fractional Lie symmetry of Lagrangian systems and Hamiltonian systems. In this paper, the fractional Lagrange equations of nonholonomic systems in terms of combined fractional derivatives are obtained. By introducing infinitesimal transformations, we study the Lie symmetry of the system, and the corresponding conserved quantity is obtained.

The present paper is organized as follows. In Section 2, some basic definitions and properties of fractional derivatives are introduced. In Section 3, the fractional Lagrange equations of nonholonomic systems in terms of combined fractional derivatives are obtained by virtue of the d'Alembert–Lagrange principle with fractional derivatives. In Section 4, the infinitesimal transformations and the determining equations of Lie symmetry are given. Section 5 gives the limit equations and the additional restriction equations. The structural equations and the conserved quantity are given in Section 6. An example is presented to illustrate the application of results in Section 7. Finally, Section 8 is dedicated to our conclusion.

2. Definitions and properties of fractional derivative

In this section, we briefly review some basic definitions and properties in terms of the left and right Riemann–Liouville fractional derivatives.

The left Riemann–Liouville fractional derivative is defined as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\theta)^{n-\alpha-1} f(\theta)\,\mathrm{d}\theta, \quad (1) $$

and the right Riemann–Liouville fractional derivative is defined as

$$ {}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (\theta-t)^{n-\alpha-1} f(\theta)\,\mathrm{d}\theta, \quad (2) $$

where α is the order of the derivative, $n-1 \leq \alpha < n$, $n \in N$, and $\alpha \neq 0$. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$ \begin{aligned} {}_aD_t^{\alpha} f(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t), \\ {}_tD_b^{\alpha} f(t) &= \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t), \quad \alpha = 1,2,\ldots \end{aligned} \quad (3) $$

Let f and g be two smooth functions defined on the interval $[a,b]$. Then, for $t \in [a,b]$, the following fractional integra-

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†Corresponding author. E-mail: sqfujingli@163.com

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tion by parts formula holds:

$$\int_a^b ({}_aD_t^\alpha f(t))g(t)\,\mathrm{d}t = \int_a^b f(t)\,({}_tD_b^\alpha g(t))\,\mathrm{d}t. \tag{4}$$

If the condition $f(a) = f(b) = 0$ for the left and right Riemann–Liouville fractional derivatives holds, respectively, then the left and right Riemann–Liouville fractional derivative equal to the left and right Caputo fractional derivatives, and the condition provides that

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_aD_t^\alpha f(t) = {}_aD_t^\alpha \frac{\mathrm{d}}{\mathrm{d}t}f(t),\quad {}_aD_t^\alpha c = 0,$$

where c is a constant.[28]

Zhang[32] gave a combined left and right Riemann–Liouville fractional derivatives operator

$$D_\gamma^{\alpha,\beta} f(t) = \gamma\,{}_aD_t^\alpha f(t) + (1-\gamma)\,(-1)^n\,{}_tD_b^\beta f(t), \tag{5}$$

where $\gamma \in [0,1]$. From Eq. (5), we have

$$\gamma = 1 : D_1^{\alpha,\beta} f(t) = {}_aD_t^\alpha f(t), \tag{6}$$

$$\gamma = 0 : D_0^{\alpha,\beta} f(t) = (-1)^n\,{}_tD_b^\beta f(t), \tag{7}$$

$$\gamma = \frac{1}{2} : D_{1/2}^{\alpha,\alpha} f(t) = \frac{1}{2}\left({}_aD_t^\alpha f(t) + (-1)^n\,{}_tD_b^\alpha f(t)\right) = {}_a^RD_b^\alpha f(t), \tag{8}$$

$$\gamma = \frac{1}{2}(1+i\mu),\ i = \sqrt{-1}, \mu \in C : D_{(1+i\mu)/2}^{\alpha,\beta} f(t) = \frac{1}{2}\left({}_aD_t^\alpha f(t) + (-1)^n\,{}_tD_b^\beta f(t)\right) + \frac{1}{2}\mathrm{i}\mu\left({}_aD_t^\alpha f(t) - (-1)^n\,{}_tD_b^\beta f(t)\right) = D_\mu^{\alpha,\beta}, \tag{9}$$

where ${}_a^RD_b^\alpha f(t)$ denotes the Riesz fractional derivative,[38] and $D_\mu^{\alpha,\beta}$ denotes the fractional derivative given by Cresson.[39]

The formula of fractional integration by part for the combined Riemann–Liouville fractional derivatives can be written as[32]

$$\int_a^b \left(D_\gamma^{\alpha,\beta} f(t)\right) g(t)\,\mathrm{d}t = (-1)^n \int_a^b f(t)\left(D_{1-\gamma}^{\beta,\alpha} g(t)\right)\mathrm{d}t, \tag{10}$$

where the function f or g and its derivatives up to order $n-1$ with respect to t are all zero at the ends $t = a$ and $t = b$. Especially, when $0 < \alpha,\ \beta < 1$, it follows Eqs. (5) and (10) that

$$D_\gamma^{\alpha,\beta} f(t) = \gamma\,{}_aD_t^\alpha f(t) - (1-\gamma)\,{}_tD_b^\beta f(t), \tag{11}$$

$$\int_a^b \left(D_\gamma^{\alpha,\beta} f(t)\right) g(t)\,\mathrm{d}t = -\int_a^b f(t)\left(D_{1-\gamma}^{\beta,\alpha} g(t)\right)\mathrm{d}t. \tag{12}$$

3. Equation of motion of fractional nonholonomic systems

Let us consider a constrained mechanical system whose configuration is determined by n generalized coordinates $q_s\,(s = 1,2,\ldots,n)$. The motion of the system is subjected to the ideal bilateral nonholonomic constraint of Appell–Chetaev type

$$f\left(t, q, D_\gamma^{\alpha,\beta} q\right) = 0, \tag{13}$$

the Appell–Chetaev condition gives

$$\frac{\partial f}{\partial D_\gamma^{\alpha,\beta} q_s}\delta q_s = 0. \tag{14}$$

The fractional Lagrange equation of holonomic conservative systems was given by[32]

$$\frac{\partial L}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha}\frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} = 0. \tag{15}$$

Assuming that the kinetic energy function of the system is $T = T(t, q_s, D_\gamma^{\alpha,\beta} q_s)$, and the Lagrangian is $L = L(t, q_s, D_\gamma^{\alpha,\beta} q_s)$, then equation (15) becomes

$$\frac{\partial T}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha}\frac{\partial T}{\partial D_\gamma^{\alpha,\beta} q_s} + Q_s' = 0, \tag{16}$$

where $Q_s' = -\partial V/\partial q_s$ is the conservative force, and $V = V(t, q_s)$ is the potential energy function.

For holonomic nonconservative systems, the equation of motion can be written as

$$\frac{\partial T}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha}\frac{\partial T}{\partial D_\gamma^{\alpha,\beta} q_s} + Q_s = 0, \tag{17}$$

where Q_s is the generalized force, which contains conservative force Q_s' and nonconservative force Q_s''.

Now we consider the d'Alembert–Lagrange principle with fractional derivatives

$$\left(\frac{\partial T}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha}\frac{\partial T}{\partial D_\gamma^{\alpha,\beta} q_s} + Q_s\right)\delta q_s = 0. \tag{18}$$

Introducing the Lagrange multiplier λ, and multiplying λ on both sides of Eq. (14), we obtain

$$\lambda\frac{\partial f}{\partial D_\gamma^{\alpha,\beta} q_s}\delta q_s = 0. \tag{19}$$

Adding Eqs. (18) and (19) yields

$$\left(\frac{\partial T}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha}\frac{\partial T}{\partial D_\gamma^{\alpha,\beta} q_s} + Q_s + \lambda\frac{\partial f}{\partial D_\gamma^{\alpha,\beta} q_s}\right)\delta q_s = 0. \tag{20}$$

Using Lagrange's method of undetermined multiplier of nonholonomic systems, we obtain

$$\frac{\partial T}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha}\frac{\partial T}{\partial D_\gamma^{\alpha,\beta} q_s} + Q_s + \lambda\frac{\partial f}{\partial D_\gamma^{\alpha,\beta} q_s} = 0, \tag{21}$$

where $Q_s = Q_s' + Q_s''$, and Q_s'' is the nonconservative force. Then we have

$$\frac{\partial T}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha}\frac{\partial T}{\partial D_\gamma^{\alpha,\beta} q_s} - \frac{\partial V}{\partial q_s} + Q_s'' + \lambda\frac{\partial f}{\partial D_\gamma^{\alpha,\beta} q_s} = 0. \tag{22}$$

Since the function V only depends on generalized coordinates, we have

$$\frac{\partial V}{\partial D_{\gamma}^{\alpha,\beta} q_s} = 0. \tag{23}$$

Using Eq. (23), we change Eq. (22) into

$$\frac{\partial (T-V)}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha} \frac{\partial (T-V)}{\partial D_{\gamma}^{\alpha,\beta} q_s} + Q_s'' + \lambda \frac{\partial f}{\partial D_{\gamma}^{\alpha,\beta} q_s} = 0. \tag{24}$$

Introducing Lagrangian $L = T - V$, we rewrite Eq. (24) as

$$\frac{\partial L}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial D_{\gamma}^{\alpha,\beta} q_s} + Q_s'' + \lambda \frac{\partial f}{\partial D_{\gamma}^{\alpha,\beta} q_s} = 0, \tag{25}$$

which is the generalized fractional Lagrange equations of nonholonomic systems in terms of the combined Riemann–Liouville fractional derivatives.

Now we come to study some important cases of Eq. (25).

Case 1 If we choose $\gamma = 1$, and the fractional Lagrangian is $L(t, q_s, {}_aD_t^{\alpha} q_s)$, then equation (25) changes into

$$\frac{\partial L}{\partial q_s} + {}_tD_b^{\alpha} \frac{\partial L}{\partial {}_aD_t^{\alpha} q_s} + Q_s'' + \lambda \frac{\partial f}{\partial {}_aD_t^{\alpha} q_s} = 0. \tag{26}$$

Case 2 If we choose $\gamma = 0$, and the fractional Lagrangian is $L(t, q_s, {}_tD_b^{\beta} q_s)$, then equation (25) becomes

$$\frac{\partial L}{\partial q_s} + {}_aD_t^{\beta} \frac{\partial L}{\partial {}_tD_b^{\beta} q_s} + Q_s'' - \lambda \frac{\partial f}{\partial {}_tD_b^{\beta} q_s} = 0. \tag{27}$$

Case 3 If we choose $\gamma = 1/2$, $\beta = \alpha$, and the fractional Lagrangian is $L\left(t, q_s, {}_a^RD_b^{\alpha} q_s\right)$, then equation (25) becomes

$$\frac{\partial L}{\partial q_s} - {}_a^RD_b^{\alpha} \frac{\partial L}{\partial {}_a^RD_b^{\alpha} q_s} + Q_s'' + \lambda \frac{\partial f}{\partial {}_a^RD_b^{\alpha} q_s} = 0. \tag{28}$$

Case 4 If we choose $\gamma = (1 + \mathrm{i}\mu)/2$, $\mathrm{i} = \sqrt{-1}, \mu \in \mathrm{C}$, and the fractional Lagrangian is $L(t, q_s, D_{\mu}^{\alpha,\beta} q_s)$, then equation (25) becomes

$$\frac{\partial L}{\partial q_s} - D_{-\mu}^{\beta,\alpha} \frac{\partial L}{\partial D_{\mu}^{\alpha,\beta} q_s} + Q_s'' + \lambda \frac{\partial f}{\partial D_{\mu}^{\alpha,\beta} q_s} = 0. \tag{29}$$

Case 5 In these cases mentioned above, if we choose $\alpha, \beta \to 1$, then equation (25) gives the classical Lagrange equations of nonholonomic systems

$$\frac{\partial L}{\partial q_s} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_s} + Q_s'' + \lambda \frac{\partial f}{\partial \dot{q}_s} = 0. \tag{30}$$

Assuming that the system (25) is a nonsingular system, that is

$$\Delta = \det \left(\frac{\partial^2 L}{\partial D_{\gamma}^{\alpha,\beta} q_s \partial D_{\gamma}^{\alpha,\beta} q_k} \right) \neq 0, \quad s,k = 1,2,\ldots,n. \tag{31}$$

Before integrating the equations of motion, we can determine the Lagrange multiplier λ, equation (25) can be written as

$$D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial D_{\gamma}^{\alpha,\beta} q_s} - \frac{\partial L}{\partial q_s} = Q_s'' + \Lambda_s, \tag{32}$$

where $\Lambda_s = \Lambda_s \left(t, q_s, D_{\gamma}^{\alpha,\beta} q_s\right) = \lambda \partial f / \partial D_{\gamma}^{\alpha,\beta} q_s$.

In order to simplify our calculations, we consider the Lagrangian

$$L = L\left(t, q_s, D_{\gamma}^{\alpha,\beta} q_s\right) = \frac{1}{2} \left(D_{\gamma}^{\alpha,\beta} q_s\right)^2 - V(t, q_s), \tag{33}$$

and the fractional nonholonomic constraint

$$f = a_s D_{\gamma}^{\alpha,\beta} q_s + \chi(t, q_s) = 0, \tag{34}$$

where a_s are constants.

From Eqs. (32)–(34), we obtain

$$D_{1-\gamma}^{\beta,\alpha} \left(D_{\gamma}^{\alpha,\beta} q_s\right) = Q_s'' + \lambda a_s - \frac{\partial V}{\partial q_s} = h_s(t, q_s, D_{\gamma}^{\alpha,\beta} q_s). \tag{35}$$

Equation (35) is called the equation of motion of holonomic systems corresponding to the nonholonomic systems (13) and (25). If the initial conditions satisfy Eq. (13), the solution of Eq. (35) gives the motion of the nonholonomic systems.

From Eqs. (34) and (35), we can obtain the Lagrange multiplier λ as

$$\lambda = \frac{1}{\sum_{s=1}^{n} a_s^2} \left\{ \sum_{s=1}^{n} \left(\frac{\partial V}{\partial q_s} - Q_s'' \right) a_s - D_{1-\gamma}^{\beta,\alpha} (\chi(t, q_s)) \right\}. \tag{36}$$

4. Infinitesimal transformations and the determining equations of the Lie symmetry

Introducing the infinitesimal transformations in terms of time and generalized coordinates

$$t^* = t + \Delta t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s, \tag{37}$$

and their expansion formulae are

$$\begin{aligned} t^* &= t + \varepsilon \xi(t, q_s, D_{\gamma}^{\alpha,\beta} q_s), \\ q_s^*(t^*) &= q_s(t) + \varepsilon \eta_s(t, q_s, D_{\gamma}^{\alpha,\beta} q_s), \end{aligned} \tag{38}$$

where ε is a small parameter, and ξ, η_s are generating functions of the infinitesimal transformations.

Introducing the differential operator of infinitesimal generator

$$X^{(0)} = \xi \frac{\partial}{\partial t} + \eta_s \frac{\partial}{\partial q_s}, \tag{39}$$

we have the fractional first-order extended $\tilde{X}^{(1)}$ [40]

$$\tilde{X}^{(1)} = \xi \frac{\partial}{\partial t} + \eta_s \frac{\partial}{\partial q_s} + \tilde{\eta}_s^{(1)} \frac{\partial}{\partial D_{\gamma}^{\alpha,\beta} q_s}, \tag{40}$$

where

$$\tilde{\eta}_s^{(1)} = D_\gamma^{\alpha,\beta}\eta_s + D_\gamma^{\alpha,\beta}\left(\dot{\xi}q_s\right) - \frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}\left(\xi q_s\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi. \tag{41}$$

Assuming that the condition $f(a) = f(b) = 0$ for the left and right Riemann–Liouville fractional derivatives holds, then equation (41) becomes

$$\begin{aligned}\tilde{\eta}_s^{(1)} &= D_\gamma^{\alpha,\beta}\eta_s + D_\gamma^{\alpha,\beta}\left(\dot{\xi}q_s\right) - \frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}\left(\xi q_s\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi \\ &= D_\gamma^{\alpha,\beta}\left(\eta_s - \frac{\mathrm{d}}{\mathrm{d}t}\left(\xi q_s\right) + \dot{\xi}q_s\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi \\ &= D_\gamma^{\alpha,\beta}\left(\eta_s - \dot{q}_s\xi\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi. \end{aligned} \tag{42}$$

Therefore, the fractional first-order extended $\tilde{X}^{(1)}$ has the form of

$$\tilde{X}^{(1)} = \xi\frac{\partial}{\partial t} + \eta_s\frac{\partial}{\partial q_s} + \left[D_\gamma^{\alpha,\beta}\left(\eta_s - \dot{q}_s\xi\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi\right] \times \frac{\partial}{\partial D_\gamma^{\alpha,\beta}q_s}. \tag{43}$$

If the fractional second-order extended operator $\tilde{\eta}_s^{(2)}$ is given by the prolongation formula defined in the usual sense

$$\begin{aligned}\tilde{\eta}_s^{(2)} &= D_{1-\gamma}^{\beta,\alpha}\eta_s^{(\alpha,\beta)} - D_{1-\gamma}^{\beta,\alpha}\xi\cdot D_{1-\gamma}^{\beta,\alpha}D_\gamma^{\alpha,\beta}q_s \\ &= D_{1-\gamma}^{\beta,\alpha}[D_\gamma^{\alpha,\beta}\left(\eta_s - \dot{q}_s\xi\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi] \\ &\quad - D_{1-\gamma}^{\beta,\alpha}\xi\cdot D_{1-\gamma}^{\beta,\alpha}D_\gamma^{\alpha,\beta}q_s, \end{aligned} \tag{44}$$

then the fractional second-order extended $\tilde{X}^{(2)}$ has

$$\begin{aligned}\tilde{X}^{(2)} &= \tilde{X}^{(1)} + \tilde{\eta}_s^{(2)}\frac{\partial}{\partial D_{1-\gamma}^{\beta,\alpha}D_\gamma^{\alpha,\beta}q_s} \\ &= \xi\frac{\partial}{\partial t} + \eta_s\frac{\partial}{\partial q_s} + \left[D_\gamma^{\alpha,\beta}\left(\eta_s - \dot{q}_s\xi\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi\right] \\ &\quad \times \frac{\partial}{\partial D_\gamma^{\alpha,\beta}q_s} + \left\{D_{1-\gamma}^{\beta,\alpha}\left[D_\gamma^{\alpha,\beta}\left(\eta_s - \dot{q}_s\xi\right)\right.\right. \\ &\quad \left.\left. + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi\right] - D_{1-\gamma}^{\beta,\alpha}\xi\cdot D_{1-\gamma}^{\beta,\alpha}D_\gamma^{\alpha,\beta}q_s\right\} \\ &\quad \times \frac{\partial}{\partial D_{1-\gamma}^{\beta,\alpha}D_\gamma^{\alpha,\beta}q_s}. \end{aligned} \tag{45}$$

It should be emphasized that the operators $\tilde{\eta}_s^{(1)}$, $\tilde{\eta}_s^{(2)}$, $\tilde{X}^{(1)}$, and $\tilde{X}^{(2)}$ differing from the operator are defined in the usual sense.

The basic idea of the Lie symmetry is to keep the differential equation (35) invariant under the infinitesimal transformations (38), if and only if

$$\tilde{X}^{(2)}\left[D_{1-\gamma}^{\beta,\alpha}\left(D_\gamma^{\alpha,\beta}q_s\right) - h_s(t,q_s,D_\gamma^{\alpha,\beta}q_s)\right] = 0, \tag{46}$$

the differential equation (35) has the infinitesimal transformations.

Therefore, we have

$$\begin{aligned}&\tilde{X}^{(2)}\left[D_{1-\gamma}^{\beta,\alpha}\left(D_\gamma^{\alpha,\beta}q_s\right) - h_s(t,q_s,D_\gamma^{\alpha,\beta}q_s)\right] \\ &= D_{1-\gamma}^{\beta,\alpha}\left[D_\gamma^{\alpha,\beta}\left(\eta_s - \dot{q}_s\xi\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi\right] \\ &\quad - D_{1-\gamma}^{\beta,\alpha}\xi\cdot D_{1-\gamma}^{\beta,\alpha}D_\gamma^{\alpha,\beta}q_s - \tilde{X}^{(1)}\left(h_s\right) \\ &= 0, \end{aligned} \tag{47}$$

namely,

$$\begin{aligned}&D_{1-\gamma}^{\beta,\alpha}\left[D_\gamma^{\alpha,\beta}\left(\eta_s - \dot{q}_s\xi\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}D_\gamma^{\alpha,\beta}q_s\right)\xi\right] \\ &- D_{1-\gamma}^{\beta,\alpha}\xi\cdot D_{1-\gamma}^{\beta,\alpha}D_\gamma^{\alpha,\beta}q_s = \tilde{X}^{(1)}\left(h_s\right). \end{aligned} \tag{48}$$

Equation (48) is called the determining equation that the generating functions ξ and η_s should satisfy.

Definition 1 If the generating functions ξ and η_s satisfy the determining equation (48), then the symmetry is called Lie symmetry of the fractional holonomic Lagrangian systems corresponding to the fractional nonholonomic Lagrangian systems.

On the basis of the determining equations, we can study two classes of problems. The first class: using the differential equation (35), we can obtain the generating functions ξ and η_s of Lie symmetry. The second class: if the generating functions ξ and η_s of Lie symmetry are given, we can obtain the function h_s.

5. The limit equation and the additional restriction equation

The invariance of the nonholonomic constraint equation (13) under the infinitesimal transformation (38) boils down to the following limit equation:

$$\tilde{X}^{(1)}\left[f\left(t,q,D_\gamma^{\alpha,\beta}q\right)\right] = 0. \tag{49}$$

Definition 2 If the generating functions ξ and η_s satisfy the determining equation (48) and the limit equation (49), then the corresponding symmetry is called weak Lie symmetry of the fractional nonholonomic systems.

If we only consider the invariance of the differential equation (35) under the infinitesimal transformation (38), then the weak Lie symmetry is the usual Lie symmetry of the fractional nonholonomic systems.

If we also consider the derivation process of the differential equation and the additional restrictions on the generating functions ξ and η_s, then we need to define additional Lie

symmetry. In the derivation process of Eq. (25), we used the Appell–Chetaev condition (14), and the condition constraints δq_s.

Noting that

$$\delta q_s = \Delta q_s - \dot{q}_s \Delta t = \varepsilon(\eta_s - \xi \dot{q}_s), \tag{50}$$

and substituting Eq. (50) into Eq. (14), we have

$$\frac{\partial f}{\partial D_\gamma^{\alpha,\beta} q_s}(\eta_s - \xi \dot{q}_s) = 0, \tag{51}$$

which is called the additional restriction equations.

Definition 3 If the generating functions ξ and η_s satisfy the determining equation (48), the limit equation (49), and the additional restriction equation (51), then the corresponding symmetry is called strong Lie symmetry of the fractional nonholonomic systems.

6. Structural equations and conserved quantity

For the fractional nonholonomic systems, the Lie symmetry does not necessarily lead to the conserved quantity. The condition and the form of conserved quantity are given by the following theorem.

Theorem 1 If the generating functions ξ and η_s satisfy the determining equation (48), and there exist gauge functions $G = G\left(t, q_s, D_\gamma^{\alpha,\beta} q_s\right)$, which satisfy the following structural equations:

$$L\dot{\xi} + \tilde{X}^{(1)}(L) + \left(Q_s'' + \Lambda_s\right)(\eta_s - \xi \dot{q}_s) + \dot{G} = 0. \tag{52}$$

Then the Lie symmetry conserved quantity of the fractional holonomic system (35) corresponding to the fractional nonholonomic systems has the following form:

$$\begin{aligned} I &= L\xi + \int_a^t \left[D_\gamma^{\alpha,\beta}(\eta_s - \dot{q}_s\xi) \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} \right. \\ &\quad \left. + (\eta_s - \dot{q}_s\xi) D_{1-\gamma}^{\beta,\alpha}\left(\frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s}\right)\right] \mathrm{d}\gamma + G \\ &= \text{const.} \end{aligned} \tag{53}$$

Proof We want to insert Eq. (32) into the structural equation (52). Hence, we will write the latter in a suitable form

$$\begin{aligned} &L\dot{\xi} + \tilde{X}^{(1)}(L) + \left(Q_s'' + \Lambda_s\right)(\eta_s - \xi\dot{q}_s) + \dot{G} \pm \xi \dot{q}_s \frac{\partial L}{\partial q_s} \\ &= L\dot{\xi} + \xi \frac{\partial L}{\partial t} + \eta_s \frac{\partial L}{\partial q_s} + \left[D_\gamma^{\alpha,\beta}(\eta_s - \dot{q}_s\xi) \right. \\ &\quad \left. + \left(\frac{\mathrm{d}}{\mathrm{d}t} D_\gamma^{\alpha,\beta} q_s\right)\xi \right] \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} + \left(Q_s'' + \Lambda_s\right)(\eta_s - \xi\dot{q}_s) \\ &\quad + \dot{G} \pm \xi \dot{q}_s \frac{\partial L}{\partial q_s} \\ &= \xi \left[\frac{\partial L}{\partial t} + \dot{q}_s \frac{\partial L}{\partial q_s} + \left(\frac{\mathrm{d}}{\mathrm{d}t} D_\gamma^{\alpha,\beta} q_s\right) \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} \right] \\ &\quad + (\eta_s - \dot{q}_s\xi) \frac{\partial L}{\partial q_s} + D_\gamma^{\alpha,\beta}(\eta_s - \dot{q}_s\xi) \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} \\ &\quad + L\dot{\xi} + \left(Q_s'' + \Lambda_s\right)(\eta_s - \xi\dot{q}_s) + \dot{G} \\ &= \xi \frac{\mathrm{d}}{\mathrm{d}t} L + (\eta_s - \dot{q}_s\xi) \frac{\partial L}{\partial q_s} + D_\gamma^{\alpha,\beta}(\eta_s - \dot{q}_s\xi) \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} \\ &\quad + L\dot{\xi} + \left(Q_s'' + \Lambda_s\right)(\eta_s - \xi\dot{q}_s) \\ &\quad + \dot{G} \pm (\eta_s - \dot{q}_s\xi) D_{1-\gamma}^{\beta,\alpha}\left(\frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(L\xi) + (\eta_s - \dot{q}_s\xi)\left[\frac{\partial L}{\partial q_s} - D_{1-\gamma}^{\beta,\alpha}\left(\frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s}\right)\right. \\ &\quad \left. + Q_s'' + \Lambda_s\right] + \dot{G} + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \left[D_\gamma^{\alpha,\beta}(\eta_s - \dot{q}_s\xi) \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} \right. \\ &\quad \left. + (\eta_s - \dot{q}_s\xi) D_{1-\gamma}^{\beta,\alpha}\left(\frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s}\right)\right] \mathrm{d}\gamma \\ &= 0. \end{aligned} \tag{54}$$

The second term in the last equation is the generalized fractional Lagrange equations of nonholonomic systems multiplied by $(\eta_s - \dot{q}_s\xi)$. Therefore, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{ L\xi + G + \int_a^t \left[D_\gamma^{\alpha,\beta}(\eta_s - \dot{q}_s\xi) \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} \right. \right. \\ &\quad \left. \left. + (\eta_s - \dot{q}_s\xi) D_{1-\gamma}^{\beta,\alpha}\left(\frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s}\right)\right] \mathrm{d}\gamma \right\} = 0, \end{aligned} \tag{55}$$

that is,

$$\begin{aligned} &L\xi + \int_a^t \left[D_\gamma^{\alpha,\beta}(\eta_s - \dot{q}_s\xi) \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s} \right. \\ &\quad \left. + (\eta_s - \dot{q}_s\xi) D_{1-\gamma}^{\beta,\alpha}\left(\frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q_s}\right)\right] \mathrm{d}\gamma + G = \text{const.} \end{aligned} \tag{56}$$

Theorem 2 If the generating functions ξ and η_s satisfy the determining equation (48) and the limit equation (49), and there exist gauge functions $G = G\left(t, q_s, D_\gamma^{\alpha,\beta} q_s\right)$, which satisfy the structural equation (52), then the fractional nonholonomic systems have the weak Lie symmetry conserved quantity (53).

Theorem 3 If the generating functions ξ and η_s satisfy the determining equation (48), the limit equation (49), and the additional restriction equation (51), and there exist gauge functions $G = G\left(t, q_s, D_\gamma^{\alpha,\beta} q_s\right)$, which satisfy the structural equation (52). Then the fractional nonholonomic systems have the strong Lie symmetry conserved quantity (53).

For the three kinds of Lie symmetry, we can obtain the conserved quantities only when the gauge function $G = G(t, q_s, D_\gamma^{\alpha,\beta} q_s)$ is given by the structural equation (52).

7. An example

The Lagrangian of a fractional mechanical system is

$$L=\frac{1}{2}\left[\left(D_{\gamma}^{\alpha,\beta}q_1\right)^2+\left(D_{\gamma}^{\alpha,\beta}q_2\right)^2\right]-q_1,\ Q_1''=Q_2''=0, \tag{57}$$

and the nonholonomic constraint is

$$f=D_{\gamma}^{\alpha,\beta}q_1+D_{\gamma}^{\alpha,\beta}q_2-q_2=0. \tag{58}$$

The equations of motion are given by Eq. (25)

$$\begin{cases} D_{1-\gamma}^{\beta,\alpha}\left(D_{\gamma}^{\alpha,\beta}q_1\right)=\lambda-1, \\ D_{1-\gamma}^{\beta,\alpha}\left(D_{\gamma}^{\alpha,\beta}q_2\right)=\lambda. \end{cases} \tag{59}$$

Using Eqs. (58) and (59), we have

$$\lambda=\frac{1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}. \tag{60}$$

Therefore, the equation of motion (59) can be written as

$$\begin{cases} D_{1-\gamma}^{\beta,\alpha}\left(D_{\gamma}^{\alpha,\beta}q_1\right)=\dfrac{-1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}, \\ D_{1-\gamma}^{\beta,\alpha}\left(D_{\gamma}^{\alpha,\beta}q_2\right)=\dfrac{1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}. \end{cases} \tag{61}$$

From Eqs. (32), (35), and (61), we obtain

$$\begin{cases} \Lambda_1=\dfrac{1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}, & \Lambda_2=\dfrac{1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}, \\ h_1=\dfrac{-1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}, & h_2=\dfrac{1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}. \end{cases} \tag{62}$$

The determining equation (48) becomes

$$\begin{aligned} &D_{1-\gamma}^{\beta,\alpha}\left[D_{\gamma}^{\alpha,\beta}\left(\eta_1-\dot{q}_1\xi\right)+\left(\frac{\mathrm{d}}{\mathrm{d}t}D_{\gamma}^{\alpha,\beta}q_1\right)\xi\right] \\ &-\frac{-1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}D_{1-\gamma}^{\beta,\alpha}\xi=0, \end{aligned} \tag{63}$$

$$\begin{aligned} &D_{1-\gamma}^{\beta,\alpha}\left[D_{\gamma}^{\alpha,\beta}\left(\eta_2-\dot{q}_2\xi\right)+\left(\frac{\mathrm{d}}{\mathrm{d}t}D_{\gamma}^{\alpha,\beta}q_2\right)\xi\right] \\ &-\frac{1+D_{1-\gamma}^{\beta,\alpha}q_2}{2}D_{1-\gamma}^{\beta,\alpha}\xi=0. \end{aligned} \tag{64}$$

Assuming that the condition $f(a)=f(b)=0$ for the left and right Riemann–Liouville fractional derivatives holds, and choosing $\gamma=1$, then we obtain the following solutions:

$$\xi=0,\ \ \eta_1=1,\ \ \eta_2=-1. \tag{65}$$

Substituting the generating functions into the limit equation (49) and the additional restriction equation (51), we have

$$\tilde{X}^{(1)}\left[f\left(t,q_s,{}_aD_t^{\alpha}q_s\right)\right]=1\neq 0, \tag{66}$$

and

$$\frac{\partial f}{\partial\,{}_aD_t^{\alpha}q_s}\left(\eta_s-\xi\dot{q}_s\right)={}_aD_t^{\alpha}q_1-{}_aD_t^{\alpha}q_2\neq 0. \tag{67}$$

Therefore, the symmetry is only the fractional Lie symmetry of Eq. (61).

Inserting the generating functions into the structural equation (52), we obtain

$$\begin{aligned} 0&=L\dot{\xi}+\tilde{X}^{(1)}(L)+\left(Q_s''+\Lambda_s\right)\left(\eta_s-\xi\dot{q}_s\right)+\dot{G} \\ &=-1+\dot{G}. \end{aligned} \tag{68}$$

Therefore, the gauge function has the form of

$$G=t. \tag{69}$$

Substituting Eqs. (65) and (69) into Eq. (53), we obtain the conserved quantity

$$\begin{aligned} I&=L\xi+\int_a^t\left[{}_aD_x^{\alpha}\left(\eta_s-\dot{q}_s\xi\right)\frac{\partial L}{\partial\,{}_aD_x^{\alpha}q_s}\right. \\ &\quad\left.-\left(\eta_s-\dot{q}_s\xi\right){}_xD_b^{\alpha}\left(\frac{\partial L}{\partial\,{}_aD_x^{\alpha}q_s}\right)\right]\mathrm{d}x+G \\ &=\int_a^t\left(-{}_xD_b^{\alpha}q_2+2{}_xD_b^{\alpha}\,{}_aD_x^{\alpha}q_2\right)\mathrm{d}x+t=\text{const}. \end{aligned} \tag{70}$$

8. Conclusions

In this paper, we study the Lie symmetry of the fractional nonholonomic systems in terms of combined fractional derivatives. Especially, we obtain the determining equations that the infinitesimal transformation groups of the Lie symmetry have to satisfy. The infinitesimal transformation groups are used for the limit equations and the additional restriction equations which generate the weak Lie symmetry and strong Lie symmetry of the fractional nonholonomic systems, respectively. Furthermore, the infinitesimal transformation groups also satisfy the structural equations, which generate the conserved quantity of Lie symmetry for the fractional nonholonomic systems.

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