Variational regularization method of solving the Cauchy problem for Laplace's equation: Innovation of the Grad–Shafranov (GS) reconstruction*

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The simplified linear model of Grad–Shafranov (GS) reconstruction can be reformulated into an inverse boundary value problem of Laplace's equation. Therefore, in this paper we focus on the method of solving the inverse boundary value problem of Laplace's equation. In the first place, the variational regularization method is used to deal with the ill-posedness of the Cauchy problem for Laplace's equation. Then, the 'L-Curve' principle is suggested to be adopted in choosing the optimal regularization parameter. Finally, a numerical experiment is implemented with a section of Neumann and Dirichlet boundary conditions with observation errors. The results well converge to the exact solution of the problem, which proves the efficiency and robustness of the proposed method. When the order of observation error δ is 10^{-1}, the order of the approximate result error can reach 10^{-3}.

Keywords: Grad–Shafranov reconstruction, variational regularization method, Cauchy problem

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1. Introduction

As shown in spacecraft data analysis, there is a novel tool called Grad–Shafranov (GS) reconstruction. We need to recover the steady two-dimensional (2D) coherent magnetic structures from over-specified observational data along the path of the spacecraft by the GS reconstruction. Firstly the Cauchy data should be produced from observations along the moving path of the spacecraft before we solve the Cauchy problem. The settlement of the Cauchy problem relies on the solving of the inverse boundary value problem of Laplace's equation. The essential idea of solving the Cauchy problem is to find an appropriate way to deal with the ill-posedness generated in the process of solving the Cauchy problem.

The Cauchy problem for Laplace's equation is ill-posed.[1–4] However, the application of the Cauchy problem to Laplace's equation is very common in real life, especially in physics and engineering, such as in corrosion non-destructive evaluation,[5] electrocardiography,[6] and geophysics.[7,8] The use of the Cauchy problem for Laplace's equation is so extensive that it has received the attention of many people.

The conception of 'inverse boundary value problems (IBVPs) for Laplace's equation" can be dated back about 110 years ago.[9] Since then, the so-called 'Cauchy problem for the Laplace equation' has been intensively studied as a typical ill-posed problem or an inverse problem. The ill-posedness of the Cauchy problem for the Laplace equation leads to the situation that any small change in the initial data may result in a dramatic change in the solution.[2–4] The theoretical analysis of Belgacem,[10] analyzed the severe ill-posedness of the Cauchy problem by using a Steklov–Poincaré approach. Calderón[11] and Engl and Leitào[12] investigated the existence and uniqueness of a weak solution for arbitrary Cauchy data. It has been proved that a solution continuously depending on the initial data can be obtained under an additional a priori boundedness condition,[13] which is called the conditional stability.

In spite of the great difficulty in achieving the exact solution due to its high ill-posedness, a large amount of literature is available in the field of Cauchy problems under the inspiration of theoretical results, e.g., the quasi-reversibility method;[14–17] the Backus Gilbert method;[18,19] the Mann iterative regularization method;[12] Li *et al.*[20] utilized the singular integral transformation and the conformal mapping method to improve the GS reconstruction; and so on. Although details for each method may be completely different from each other, they have a common characteristic, namely, an additional constraint on the discrete method, or the regularization parameters that need to satisfy the so-called stability condition.

In the 1960s, Tikhonov and Arsenin[21] proposed the now well-known idea of Tikhonov regularization method, which is closely related to the variational ideas. It is a significant problem how to choose the regularization parameter γ in the Tikhonov regularization method because the selection of γ directly affects the convergence of a regularized solution. Hence, Morozov[22] came up with the discrepancy principle.

The fundamental technique of GS reconstruction was summarized by Li *et al.*,[20] as follows: there exists a unique

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solution for the so-called elliptic Cauchy problem. According to the linearity of the elliptic operator in the GS equation, the solution can be divided into two parts. One is to be solved from a semi-linear elliptic equation with a homogeneous Dirichlet boundary condition, the other is to be solved from the inverse boundary value problem of Laplace's equation. The first part is discussed maturely in many literatures. The second part is to find a proper way to solve the inverse boundary value problems (IBVPs).

In this article, we focus on the completion of the missing data for Laplace's equation in a rectangular domain, which means recovering missing conditions in some inaccessible parts of the boundary with the help of over-specified boundary data in the remaining part. According to the idea a new solution approach is developed, that is, the variational regularization method. Papers[23–25] relevant to the variational regularization method have been published. The rest of this paper is organized as follows. In Section 2 we establish the mathematical model of the Cauchy problem for Laplace's equation, and in Section 3 we formulate the variational regularization method scheme. In Section 4, we develop a numerical discretization scheme. Then, in Section 5 the new approach is demonstrated with a numerical experiment. The optimal selection of regularization parameter γ and a summary are presented in Sections 6 and 7, respectively.

2. Mathematical model

The Cauchy problem for Laplace's equation in a rectangular domain is concerned in this paper.

Suppose that a rectangular domain is $\Omega \in R^2$, with a piecewise smooth boundary $\partial\Omega$, which is composed of four parts, Γ_i $(i=0,1,2,3)$ as shown in Fig. 1. Note Γ_4=Γ_1+Γ_2+Γ_3. The governing equations are as follows:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u|_{\Gamma_4} = u_0, \\ u_n|_{\Gamma_4} = g, \end{cases} \tag{1}$$

where $u_n = \partial u/\partial n$ and n is the normal to Γ, oriented toward the exterior of Ω.

Calculate the value of $u|_{\Gamma_0}$.

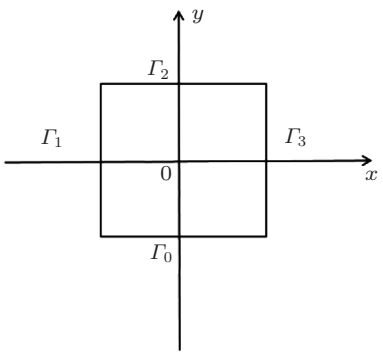


Fig. 1. Domain for the Cauchy problem.

As is well known, the Cauchy problem for the Laplace equation is ill-posed, according to theory of partial differential equations. Numerical methods usually fail to solve it. Now we use the variational regularization method to deal with the ill-posedness. First, the problem (Eq. (1)) is converted into the following problem.

Suppose $u|_{\Gamma_0} = \omega$, so the problem can be expressed in another form:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u|_{\Gamma_0} = \omega, & \text{(not given)} \\ u_n|_{\Gamma_4} = g, & \text{(given)} \end{cases} \tag{2}$$

with observational data $u|_{\Gamma_4} = u_0$, and the value of $u|_{\Gamma_0}$ is calculated.

3. Variational regularization method scheme

Step 1 Define the cost functional as follows:

$$J[\omega] = \frac{1}{2}\int_{\Gamma_4} (u-u_0)^2 \mathrm{d}s + \frac{\gamma}{2}\int_{\Omega} |\nabla u|^2 \mathrm{d}\Omega, \tag{3}$$

where $\int_\Omega |\nabla u|^2 \mathrm{d}\Omega$ is the stability functional, γ is the regularization parameter, the introduction of the regularization term $\frac{\gamma}{2}\int_\Omega |\nabla u|^2 \mathrm{d}\Omega$ can improve the convexity of functional J, and makes the numerical calculation stable.[21]

Step 2 Give ω some perturbation, then $\tilde{\omega} = \omega + \delta\omega \triangleq \omega + \hat{\omega}$. The corresponding solution of ω is $\tilde{u} = u + \delta u \triangleq u + \hat{u}$. So $\hat{u}$ satisfies the boundary value problems for Laplace's equation, i.e.,

$$\begin{cases} \Delta\hat{u} = 0, & \text{in } \Omega, \\ \hat{u}|_{\Gamma_0} = \hat{\omega}, \\ \hat{u}|_{\Gamma_4} = 0. \end{cases} \tag{4}$$

Step 3 Obtain variation of functional

$$\begin{aligned} \delta J &= \int_{\Gamma_4} (u-u_0)\hat{u}\mathrm{d}s + \gamma\int_\Omega (\nabla u\cdot\nabla\hat{u})\mathrm{d}\Omega \\ &= \int_{\Gamma_4} (u-u_0)\hat{u}\mathrm{d}s - \gamma\int_\Omega \Delta u\cdot\hat{u}\mathrm{d}\Omega \\ &\quad + \gamma\int_{\Gamma_4} \hat{u}\cdot u_n \mathrm{d}s + \gamma\int_{\Gamma_0} \hat{\omega}\cdot u_n \mathrm{d}s \\ &= \int_{\Gamma_4} [(u-u_0)+\gamma g]\hat{u}\mathrm{d}s \\ &\quad + \int_\Omega (-\gamma\Delta u)\hat{u}\mathrm{d}\Omega + \int_{\Gamma_0} \gamma u_n\hat{\omega}\mathrm{d}s. \end{aligned}$$

According to definition: $\delta J = \int_{\Gamma_0} \nabla_u J\hat{\omega}\mathrm{d}s$, we find that

$$\begin{aligned} &\int_{\Gamma_0} [\nabla_\omega J - \gamma u_n]\hat{\omega}\mathrm{d}s \\ &= \int_\Omega (-\gamma\Delta u)\hat{u}\mathrm{d}\Omega + \int_{\Gamma_4} [(u-u_0)+\gamma u_n]\hat{u}\mathrm{d}s. \end{aligned} \tag{5}$$

Step 4 Introduce the adjoint equations and adjoint boundary conditions of Eq. (4), and define the adjoint variable as P, then we will have

$$0 = (\Delta\hat{u}, P) = \int_\Omega \Delta\hat{u}\cdot P\mathrm{d}\Omega$$

$$
\begin{aligned}
&= \int_{\Omega} \Delta P \cdot \hat{u} \mathrm{d}\Omega + \int_{\partial\Omega} (P\hat{u}_n - \hat{u}P_n) \mathrm{d}s \\
&= \int_{\Omega} \Delta P \cdot \hat{u} \mathrm{d}\Omega - \int_{\Gamma_4} P_n \hat{u} \mathrm{d}s \\
&\quad + \int_{\Gamma_0} P\hat{u}_n \mathrm{d}s - \int_{\Gamma_0} P_n \hat{\omega} \mathrm{d}s,
\end{aligned}
$$

where $(f,g) = \int_{\Omega} f \cdot g \mathrm{d}\Omega$ and $P_n = \partial P / \partial n$, so that

$$
\begin{aligned}
0 = &\int_{\Omega} \Delta P \cdot \hat{u} \mathrm{d}\Omega - \int_{\Gamma_4} P_n \hat{u} \mathrm{d}s \\
&+ \int_{\Gamma_0} P\hat{u}_n \mathrm{d}s - \int_{\Gamma_0} P_n \hat{\omega} \mathrm{d}s. \qquad (6)
\end{aligned}
$$

Using Eqs. (6) and (5) we have

$$
\int_{\Omega} \Delta P \cdot \hat{u} \mathrm{d}\Omega - \int_{\Gamma_0} [\nabla_{\omega} J - \gamma u_n - P_n] \hat{\omega} \mathrm{d}s
$$

$$
+ \int_{\Gamma_4} [(u - u_0) + \gamma u_n + P_n] \hat{u} \mathrm{d}s - \int_{\Gamma_0} P\hat{u}_n \mathrm{d}s = 0,
$$

where $\hat{u}$ and $\hat{\omega}$ must remain arbitrary, $\hat{u}_n|_{\Gamma_0}$ is non-arbitrary here. We can have the adjoint equations, adjoint boundary conditions, and functional gradient as follows:

$$
\begin{cases}
\Delta P = 0, \quad \text{in } \Omega, \\
P|_{\Gamma_0} = 0, \\
P_n|_{\Gamma_4} = -(u - u_0) - \gamma u_n|_{\Gamma_4},
\end{cases} \qquad (7)
$$

$$
\nabla_{\omega} J = \gamma u_n|_{\Gamma_0} + P_n|_{\Gamma_0}. \qquad (8)
$$

Step 5 Convert the problem into solving $\omega|_{\Gamma_0}$ by using the steepest descent method to solve $\omega|_{\Gamma_0}$, the flow chart of numerical calculation is shown below in Fig. 2.

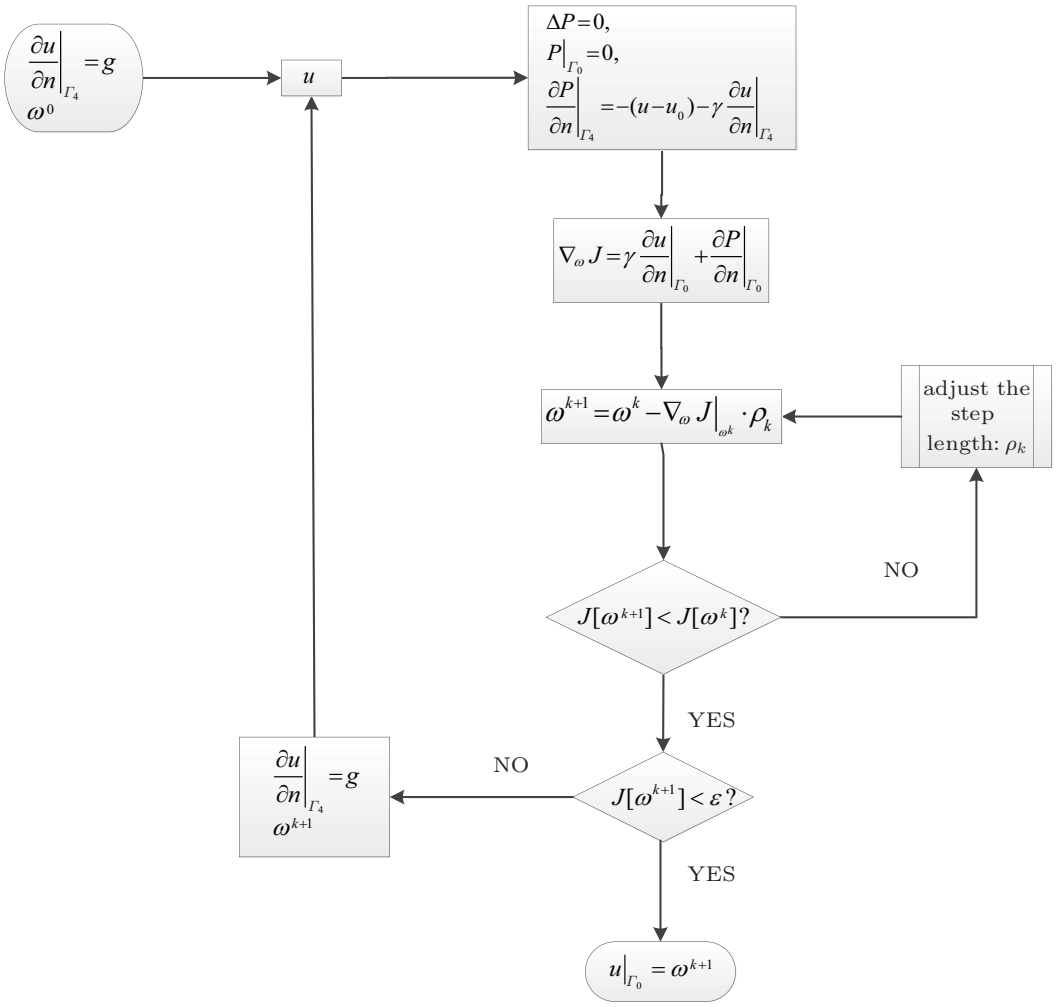


Fig. 2. Flow chart of numerical calculation.

4. Numerical discretization scheme

A square grid is chosen for the grid solutions. We select a central differential format in the internal and forward differential format or backward differential format on the boundary, when calculating the partial derivatives. We use the trapezoidal formula for integration and a cuboid formula for the double integral, when calculating the integration. Laplace's equation with mixed boundary conditions needs to be solved twice in the process of solving the above Cauchy problem nu-

merically. To solve the question, we adopt the method of solving the divergence form elliptic equation with mixed boundary condition in the book: *Difference Methods for Partial Differential Equations*.[26]

The detail of the method is as follows: first, we convert elliptic equations into the integral conservation form, and then, establish the difference scheme. Suppose $D \in \Omega$, with its boundary denoted as Γ_D. Using Green's formula, we now evaluate

$$\int_{\Gamma_D} u_n \mathrm{d}s = 0.$$

Let h be the grid spacing of x_m, such that $J \times h = 1$. We denote $u_1 = \partial u/\partial x_1$, $u_2 = \partial u/\partial x_2$. If $x \in \Omega_h$, D is a square, of which the side length is h and the center is located at x. So we have

$$A_1(\mathrm{x}) + B_1(\mathrm{x}) + C_1(\mathrm{x}) + D_1(\mathrm{x}) = 0, \tag{9}$$

where

$$\begin{aligned}
A_1(\mathrm{x}) &= -\int_{x_2-h/2}^{x_2+h/2} u_1(x_1+h/2, y_2)\,\mathrm{d}y_2,\\
B_1(\mathrm{x}) &= -\int_{x_1-h/2}^{x_1+h/2} u_2(y_1, x_2+h/2)\,\mathrm{d}y_1,\\
C_1(\mathrm{x}) &= \int_{x_2-h/2}^{x_2+h/2} u_1(x_1-h/2, y_2)\,\mathrm{d}y_2,\\
D_1(\mathrm{x}) &= \int_{x_1-h/2}^{x_1+h/2} u_2(y_1, x_2-h/2)\,\mathrm{d}y_1.
\end{aligned}$$

According to the approximate differential and integral formula, replace the above four formulas with the following four formulas:

$$\begin{aligned}
A_1(x) &\to \frac{u(x_1,x_2)-u(x_1+h,x_2)}{h}\cdot h,\\
B_1(x) &\to \frac{u(x_1,x_2)-u(x_1,x_2+h)}{h}\cdot h,\\
C_1(x) &\to \frac{u(x_1,x_2)-u(x_1-h,x_2)}{h}\cdot h,\\
D_1(x) &\to \frac{u(x_1,x_2)-u(x_1,x_2-h)}{h}\cdot h.
\end{aligned}$$

Substituting $A_1(x)$, $B_1(x)$, $C_1(x)$, $D_1(x)$ into Eq. (9), we obtain the specific formula.

When calculating the Dirichlet boundary condition, just substitute $u = u_0$ into the formula. When calculating the Neumann boundary condition, such as, $x = x_a$ on the boundary Γ_3, make D a rectangle with the side lengths being h and $h/2$ respectively, and the center of the rectangle's width on the boundary Γ_3 being x. So we have

$$A_2(\mathrm{x}) + B_2(\mathrm{x}) + C_2(\mathrm{x}) + D_2(\mathrm{x}) = 0, \tag{10}$$

where

$$\begin{aligned}
A_2(x) &= -\int_{x_2-h/2}^{x_2+h/2} u_1(x_\mathrm{a}, y_2)\,\mathrm{d}y_2\\
&= -\int_{x_2-h/2}^{x_2+h/2} g(x_\mathrm{a}, y_2)\,\mathrm{d}y_2,\\
B_2(x) &= -\int_{x_a-h/2}^{x_a} u_2(y_1, x_2+h/2)\,\mathrm{d}y_1,\\
C_2(x) &= \int_{x_2-h/2}^{x_2+h/2} u_1(x_a-h/2, y_2)\,\mathrm{d}y_2,\\
D_2(x) &= \int_{x_a-h/2}^{x_a} u_2(y_1, x_2-h/2)\,\mathrm{d}y_1,
\end{aligned}$$

and $A_2(x)$ is given. So we just need to find the other three through approximate differential and integral formula:

$$\begin{aligned}
B_2(x) &\to \frac{u(x_a,x_2)-u(x_a,x_2+h)}{h}\cdot\frac{h}{2},\\
C_2(x) &\to \frac{u(x_a,x_2)-u(x_a-h,x_2)}{h}\cdot h,\\
D_2(x) &\to \frac{u(x_a,x_2)-u(x_a,x_2-h)}{h}\cdot\frac{h}{2}.
\end{aligned}$$

Substituting $A_2(x)$, $B_2(x)$, $D_2(x)$, $D_2(x)$ into Eq. (10), we can obtain the specific formula. Similarly, we evaluate the remaining Neumann boundary condition and the corner. The Gauss–Seidel iterative method is used as an iterative scheme.

5. Numerical experiments

To verify the feasibility of the method of solving the Cauchy problem for the Laplace equation, we consider the rectangular domain:

$$\Omega = \{(x,y) \in R : 0 \le x \le \pi/2, 0 \le y \le \pi/2\}$$

with boundary

$$\begin{aligned}
\Gamma_0 &= \{(x,y) \in R : y = 0, 0 \le x \le \pi/2\},\\
\Gamma_1 &= \{(x,y) \in R : x = 0, 0 \le y \le \pi/2\},\\
\Gamma_2 &= \{(x,y) \in R : y = \pi/2, 0 \le x \le \pi/2\},
\end{aligned}$$

and $\Gamma_3 = \{(x,y) \in R : x = \pi/2, 0 \le y \le \pi/2\}$, $\Gamma_4 = \Gamma_1 + \Gamma_2 + \Gamma_3$ and $u(x,y) = \cos x \mathrm{ch} y$ in Ω, and u is called the exact solution to the problem.

So we compute artificial data $\left.\frac{\partial u}{\partial n}\right|_{\Gamma_4} = g$ and $u|_{\Gamma_4} = u_0$ from the exact solution u. In order to be closer to the actual situation we give the observation data u_0 some noise: 0.1 multiplied by uniform distribution on $(-1, 1)$. And then, solve the problem by following the flow chart in Fig. 2. The initial iterative value may be given arbitrarily, that is, the numerical results do not depend on the initial iterative value. But in order to shorten the iteration time, we use the line which connects the endpoint as the initial iterative value in this paper.

As random perturbations are added, so the results of each iteration solution are different. In order to eliminate randomness, and make sure of the feasibility of the method of solving the Cauchy problem for the Laplace equation, we decide to conduct several numerical experiments. After the procedure of the flow chart of numerical calculation the results are as follows: ($u|_{\Gamma_0}$ for the exact solution, ω for the approximate solution).

As shown in Table 1, the orders of the first errors are 10^{-3} and those of the second errors are 10^{-2}, which indicates that the variational regularization method is very effective in solving the Cauchy problem for Laplace's equation. Besides, this method also allows the observational noise and is more practical, which is a great advancement.

Table 1. Numerical errors.

	$\int_{\Gamma_0}(u\|_{\Gamma_0}-\omega)^2\,\mathrm{d}s$	$\max(\|u\|_{\Gamma_0}-\omega\|)$
First	3.48×10^{-3}	0.128
Second	2.80×10^{-3}	0.184
Third	9.53×10^{-4}	0.089
Fourth	3.52×10^{-3}	0.109
Fifth	1.22×10^{-3}	0.066
Sixth	2.47×10^{-3}	0.073
Seventh	2.66×10^{-3}	0.147
Eighth	8.18×10^{-4}	0.075
Ninth	8.52×10^{-4}	0.051
Tenth	3.29×10^{-3}	0.088
No obs. errors	5.76×10^{-4}	0.029

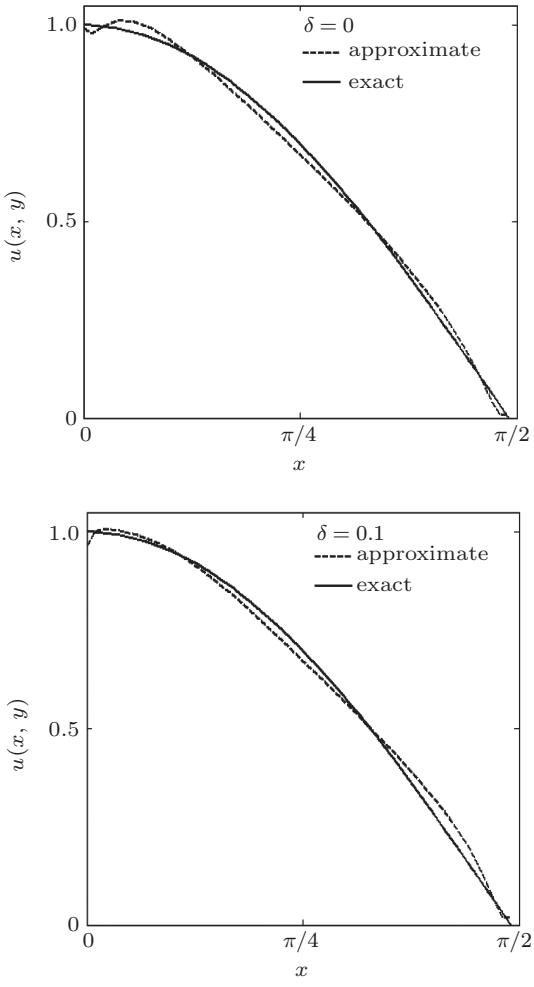


Fig. 3. Numerical results.

In order to demonstrate the inversion results more intuitively, figure 3 shows the contrast between the exact value and the inversion results. From the figure, it can be seen that the method of solving the Cauchy problem for the Laplace equation performs well and the obtained result basically accords with the true solution.

6. Optimal selection of regularization parameter γ

The essential technique of the regularization method is the selection of the regularization parameter, which contains prior selection and posteriori selection. Priori strategy usually possesses only theoretical values, and is difficult to put into practical applications. For the posteriori selection, Morozov[22] first proposed the discrepancy principle, after which people found the generalized cross-validation criteria, quasi optimal criterion, etc. But the Morozov criteria are too conservative, so researchers proposed the absorption Morozov criteria which are better than Morozov criteria. In the Morozov criteria, the Newton iteration method is used to evaluate the regularization parameter, which results in a dramatic increase in computational effort. In order to overcome the shortcomings of Newton's method that is computationally intensive, a model for using a functional method has been proposed in recent years. However, the above criteria are all based on the order of observation error δ. Engl[27] indicated that 'L-curve' criteria can be achieved by minimizing the functional. To determine the regularization parameter is to find out the regularization parameter corresponding to the L-curve 'corner point', which is the point with maximum curvature on the L-curve. The obvious advantage of this method is that it does not need to know the level of error in advance. It has been proven that L-curve criteria have a strong convergence. With the help of L-curve criteria we construct the regularization parameter image as follows: $\int_{\Gamma_0}(u-u_0)^2\,\mathrm{d}s$ and $\int_\Omega|\nabla u|^2\,\mathrm{d}\Omega$. Select the γ value of the maximum curvature point as the optimal value of γ. In this experiment, we set $\gamma=0.01$.

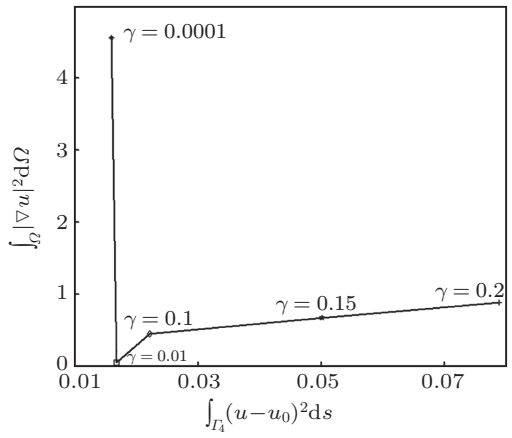


Fig. 4. L-curve.

7. Summary and conclusion

In this paper, we focus on the need for solving the inverse boundary value problem of Laplace's equation. The main technique adopted to eliminate ill-posedness is the variational regularization method. The introduction of the regularization method could deduce the effects produced by observation errors. Numerical experiments highlight the efficiency and robustness of the proposed method. When the observation data have no errors, the order of the approximate result errors can reach 10^{-4}. When the order of the observation errors is 10^{-1}, the order of the approximate result errors can reach 10^{-3}.

Compared with the previous approach to solving the Cauchy problem for Laplace's equation, this approach has some advantages. (i) The variational regularization method, which uses the L-curve method to select optimal regularization parameter, is a new way to solve the Cauchy problem. The method can not only overcome ill-posedness, but also performs steadily. Its application is not limited to the Cauchy problem for Laplace's equation, but also extends to the Cauchy problem for elliptic equations. (ii) The endpoint connected by a line is chosen as the initial iterative value, which is a universal method. So this approach would ensure the continuity of endpoints, and reduces the number of iterations. (iii) When the observation has errors (including random error), the steady solution of the inverse problem can quickly converge to the exact solution. The variational regularization method solves the instability of solutions caused by the random errors, and provides an efficient and precise method for practical application.

Next, we will focus on the study of Cauchy problems for Poisson's equation and the general elliptic partial differential equations in a triangular field. We will also make efforts to improve numerical discretization schemes to make the result more accurate and closer to practical situations.

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