Network dynamics and its relationships to topology and coupling structure in excitable complex networks*

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All dynamic complex networks have two important aspects, pattern dynamics and network topology. Discovering different types of pattern dynamics and exploring how these dynamics depend on network topologies are tasks of both great theoretical importance and broad practical significance. In this paper we study the oscillatory behaviors of excitable complex networks (ECNs) and find some interesting dynamic behaviors of ECNs in oscillatory probability, the multiplicity of oscillatory attractors, period distribution, and different types of oscillatory patterns (e.g., periodic, quasiperiodic, and chaotic). In these aspects, we further explore strikingly sharp differences among network dynamics induced by different topologies (random or scale-free topologies) and different interaction structures (symmetric or asymmetric couplings). The mechanisms behind these differences are explained physically.

Keywords: excitable complex networks, network topology, symmetric and asymmetric couplings

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1. Introduction

The problem of pattern dynamics has been a continuously hot topic in non-equilibrium statistical physics and nonlinear science for more than a century.[1,2] In the last two decades, the investigation of complex networks, including their topology and dynamics, has become one of the central points in the widely overlapping fields of physics, mathematics, biology, and even social science.[3–5] Related to these two aspects, pattern dynamics in complex networks has naturally attracted much attention in recent years.[6–8]

The brain network is one of the most important networks in nature. Different pattern dynamics in the brain network correspond to different brain functions,[9,10] and the interactions between neurons through synaptic links provide powerful computation capacities of the brain.[11,12] Complex networks consisting of excitable nodes have been extensively used to study the dynamics and topology of brain networks.[9,10,13]

In this paper, instead of studying particular brain functions and processes, we study some general problems of pattern dynamics of excitable complex networks (ECNs) and their relations to special network topologies and system structures. We investigate four characteristic features of ECNs: oscillation probability; the multiple-attractor property; period distributions of attractors, and different types of motion (periodic, quasiperiodic or chaotic). Moreover, we go further to study how different network topologies of random homogeneous networks (RHNs) and scale-free networks (SFNs) influence the above dynamic behaviors and how different kinds of couplings (symmetric and asymmetric couplings) provide distinctive dynamic features of ECNs. We find that different topologies and different coupling types can yield sharply different dynamics, and the mechanisms behind these differences can be heuristically understood.

For numerically computing dynamic ECNs we use the following coupled ordinary differential equations (ODEs)

$$\dot{\boldsymbol{X}} = \boldsymbol{f}(\boldsymbol{X}), \quad \boldsymbol{X} = (x_1, x_2, ..., x_N). \tag{1}$$

These coupled ODEs are expected to well describe the network dynamics of realistic systems if function $\boldsymbol{f}$ and the corresponding control parameters are properly chosen. In this paper we specify ECNs as

$$\begin{aligned} \frac{\mathrm{d}u_i}{\mathrm{d}t} &= -\frac{1}{\varepsilon}u_i(u_i-1)\left(u_i-\frac{v_i+b}{a}\right)+D\sum_{j\neq i}^{N}C_{ij}, \\ \frac{\mathrm{d}v_i}{\mathrm{d}t} &= f(u_i)-v_i, \quad i=1,2,...,N, \end{aligned} \tag{2}$$

where $f(u_i)=0$, $f(u_i)=1-6.75u_i(u_i-1)^2$, and $f(u_i)=1$ for $u_i \leq 1/3$, $1/3 \leq u_i \leq 1$, and $u_i \geq 1$, respectively. Here the parameters $a=0.84, b=0.07, \varepsilon=0.04$ can guarantee the excitable local dynamics.[14] $C_{ij}=M_{ij}(u_j-u_i)$ is the coupling term from node j to node i, D is the coupling intensity, $M_{ij}=1$ if j interacts i and $M_{ij}=0$ otherwise.

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Two different kinds of network structures will be studied in this paper, i.e., random homogeneous networks and scale-free networks, which are shown in Fig. 1. In a random homogeneous network each node is randomly connected to the same number of other nodes, so each node has the same connection degree and the degree distribution of network nodes is a δ function. The degree of each node is taken to be $k = 4$ in the following discussion without loss of generality. In a scale-free network most nodes have small degrees while few hubs have large ones. The degrees of all network nodes have a power law distribution $P(k) = k^{-\gamma}$, where k is the node degree and γ is a positive constant, $\gamma \approx 3.0$ in our scale-free networks.[4] In the following simulations the size of each of all the networks is chosen to be $N = 100$ for specific numerical computations. For Eq. (2) we use two types of couplings, the symmetric coupling $M_{ij} = M_{ji}$, and asymmetric couplings $M_{ij} \neq M_{ji}$, to study the pattern dynamics of ECNs.

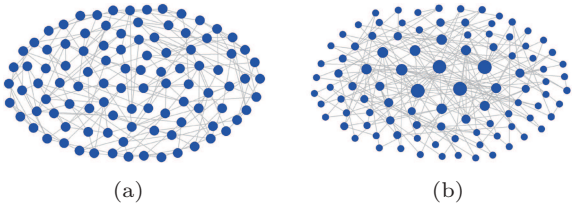


Fig. 1. (color online) (a) An example of random homogeneous networks (RHNs) with $N = 100, k = 4$. (b) An example of scale-free networks (SFNs) with $N = 100$, $\langle k \rangle = 4$. In panel (b) the nodes with large sizes are hub nodes with large connection degrees.

In the next section ODEs with symmetric couplings are discussed and essentially different types of pattern dynamics are found for different types of ECN, e.g., for random and scale-free networks. In Section 3 the pattern dynamics of random ECNs for asymmetric couplings are investigated and the essentially different behaviors for asymmetric and symmetric couplings are explored. In Section 4 the mechanisms behind the different dynamic characteristics for different network topologies and coupling natures are analyzed. In Section 5, the brief conclusions are drawn and discussed.

ECN equation (2) can show the very rich behaviors of pattern dynamics.[15,16] In this paper we focus on the following four properties: (i) the probability of self-sustained oscillation; (ii) the coexistence of multiple oscillatory attractors; (iii) the period distributions of oscillatory attractors; (iv) the different types of trajectory motions. All these properties are related to some important functions of many practical systems. However, they have not been systematically investigated, to our knowledge, in excitable complex networks to date.

2. Dynamic behaviors of ECNs with symmetrical couplings

Now we make systematical numerical simulations of ECNs Eq. (2) with different network topologies. All results are obtained by statistical computations from random initial states $u \in (0,1), v \in (0,1)$ and from different realizations of network structures. Specially, for producing each plot of the statistical computation in this section we average the results of 20 networks randomly constructed, and for each network use 5×10^4 random initial states.

In Figs. 2(a) and 2(b), we compute the values of oscillation probability P for networks with different topologies, and

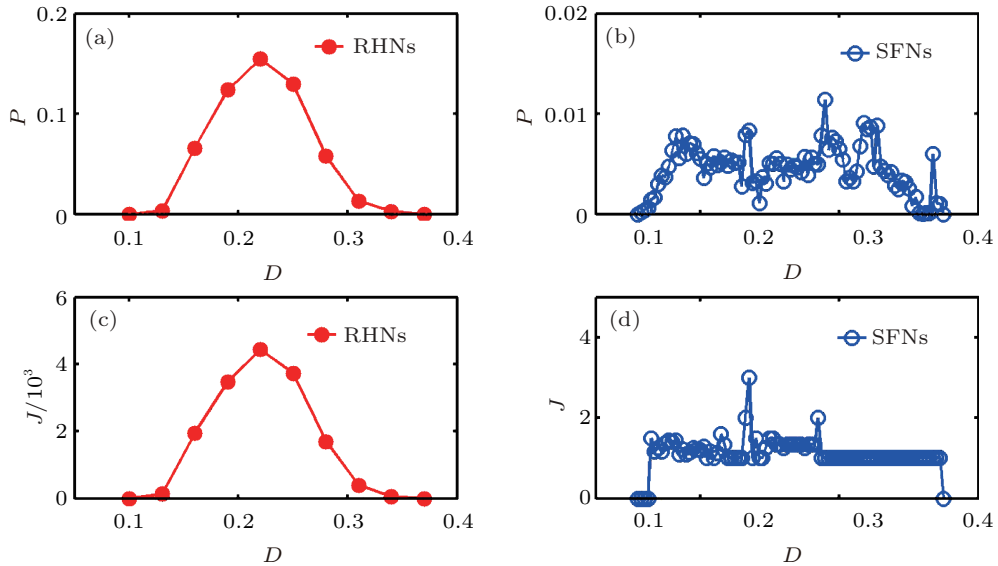


Fig. 2. (color online) Statistic dynamic features of excitable complex networks (ECNs) of ODEs Eq. (2) for symmetric couplings. Each plot is computed by the statistic averaging of 20 networks similar to Fig. 1 ($N = 100$, $k = 4$ for RHNs and $N = 100$, $\langle k \rangle = 4$ for SFNs) and 5×10^4 samples of random initial conditions for each network. Panels (a) and (b) show the oscillation probabilities P plotted versus coupling intensity D for RHNs ((a) solid disks) and SFNs ((b) empty circles), respectively. The oscillation probability of SFNs is incomparably smaller than that of RHNs. Panels (c) and (d) are the same as panels (a) and (b), respectively, with the number of oscillatory attractors J versus D. For optimal D each RHN has about 4×10^3 ((c)) distinctive attractors on average found by 5×10^4 random initial states. In sharp contrast, SFNs have one or two attractors in most of the samples of self-sustained oscillations.

plot P versus D. In the figures we find a remarkable difference between RHNs and SFNs, i.e., the oscillation probability of SFNs ((b)) is incomparably smaller than that of RHNs ((a)). The multiplicity of the attractor is also an important dynamic property of oscillatory networks. In Figs. 2(c) and 2(d), we plot the average numbers of attractors J against coupling strength D. Figure 2(c) shows that a large number of attractors coexist in RHNs (for optimal D, the number of attractors J can reach 4×10^3). However, for SFNs the J number drops off dramatically, and most of SFNs have only one or two oscillatory attractors ((d)) coexisting. This sharp contrast between RHNs and SFNs will be further discussed in Section 4.

The period of oscillation is another very important quantity for various rhythmics of complex networks. In Fig. 3 we compute the values of the average period T, for RHNs (solid circle curves) and SFNs (empty circle curves) versus D. A surprisingly interesting feature in the plots is that diffusive coupling SFNs have remarkably larger average periods than RHNs. The possibility of large oscillation periods in SFNs with electric gap junction interactions was used to explain the long-period temporal memory observed in the experiments of zebra fish.[16,17]

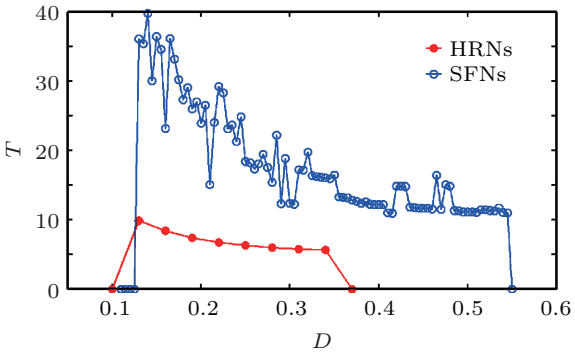


Fig. 3. (color online) The same as Fig. 2 with the statistic periodicity of networks studied, showing the variations of average period T of oscillatory attractors with D for RHNs and SFNs. SFNs have significantly longer periods than RHNs.

3. Dynamic behaviors of ECNs with asymmetrical couplings

In the last section we discussed the dynamic features of symmetrically coupled ECNs by focusing on the effects of network topologies, i.e., dynamics of RHNs and SFNs. Besides the network topology the coupling structure can also seriously influence network dynamics. In this section we focus on asymmetrically coupled RHNs and compare their dynamic behaviors with those of symmetrically coupled ECNs. Here we ignore the analysis of asymmetric SFNs since many characteristic effects of asymmetric couplings can be understood by studying the randomly interacted ECNs.

The asymmetrically coupled random homogeneous ECNs are constructed as follows. In Eq. (2) for each node we randomly choose m other nodes as the targets of outputs, and provide the randomly chosen m nodes as inputs. These operations are restricted by a strict condition that each node has m and only m inputs and m and only m outputs, i.e., as soon as a node has m inputs (m outputs) no more input (output) can be further supported in network construction.

Figure 4(a) shows the computed oscillation probability of Eq. (2) with asymmetric RHNs of $N=100$, $m=2$ ($k=2m=4$) adopted. Again for each dot of the curve 20 randomly constructed networks are considered and for each network we compute 1000 different initial states randomly chosen, and thus the probability value is the value averaged by such 2×10^4 tested samples. An obvious conclusion drawn by comparing Fig. 4(a) with Fig. 2(a) is that the former has much larger oscillation probability than the latter. In Fig. 4(b) we do the same job as in Fig. 2(c), i.e., compute the average numbers of attractors for different values of D. A striking difference between Figs. 4(b) and 2(c) is that the average numbers of attractors of asymmetric ECNs are incomparably smaller than those of symmetric ones.

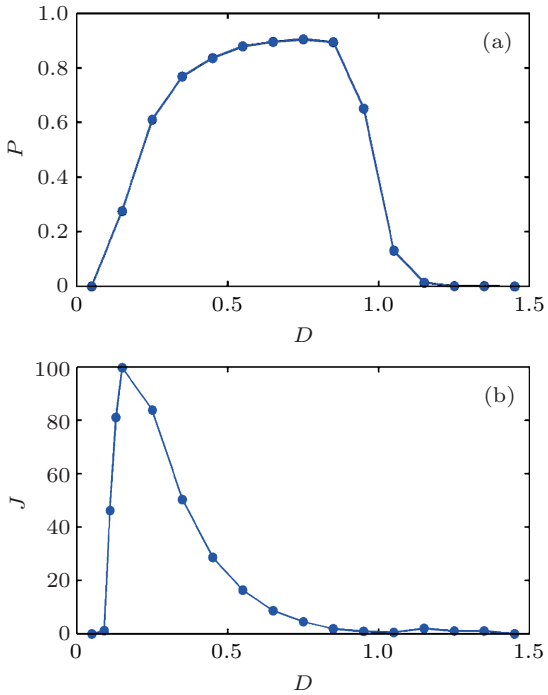


Fig. 4. (color online) (a) and (b) The same as Figs. 2(a) and 2(c), respectively, with asymmetric couplings $m_{\rm in}=m_{\rm out}=2$ considered. Here the oscillatory probability is much larger than that of symmetric networks, while numbers of attractors are incomparably smaller than that of symmetric ECNs.

By studying the dynamic patterns of oscillatory ECNs we find that the oscillatory behaviors of asymmetric ECNs are much more complicated than those of symmetric ECNs. Oscillatory symmetric ECNs often show simple and well-behaved oscillations, of which two examples are presented in

Figs. 5(a) and 5(b), corresponding to different coupling intensities respectively. In contrast, the motions of asymmetric ECNs are often rather complicated. While some simple trajectories can still be observed (with very small probability) in asymmetric ECNs (Fig. 5(c)); many trajectories are complicated even in the periodic case (Fig. 5(d)); and some trajectories are no longer periodic, they may be quasiperiodic (Fig. 5(e)); and even chaotic (Fig. 5(f)). In Figs. 6(a) and 6(b), we plot P_c versus D for both symmetrically ((a)) and asymmetrically ((b)) coupled RHNs, where P_c is defined as the probability of complicated dynamics for the given D. The complicated dynamics of a network oscillation is regarded as the fact that the corresponding orbit has some self-crossing in the plane of ($\langle u(t)\rangle$, $\langle v(t)\rangle$). It is remarkable that symmetric RHNs show simple oscillations with absolutely large probability while asymmetric RHNs exhibit complicated dynamics for most self-sustained oscillations. The reasons for the trajectory complexity of the asymmetric ECNs will be explained in Section 4.

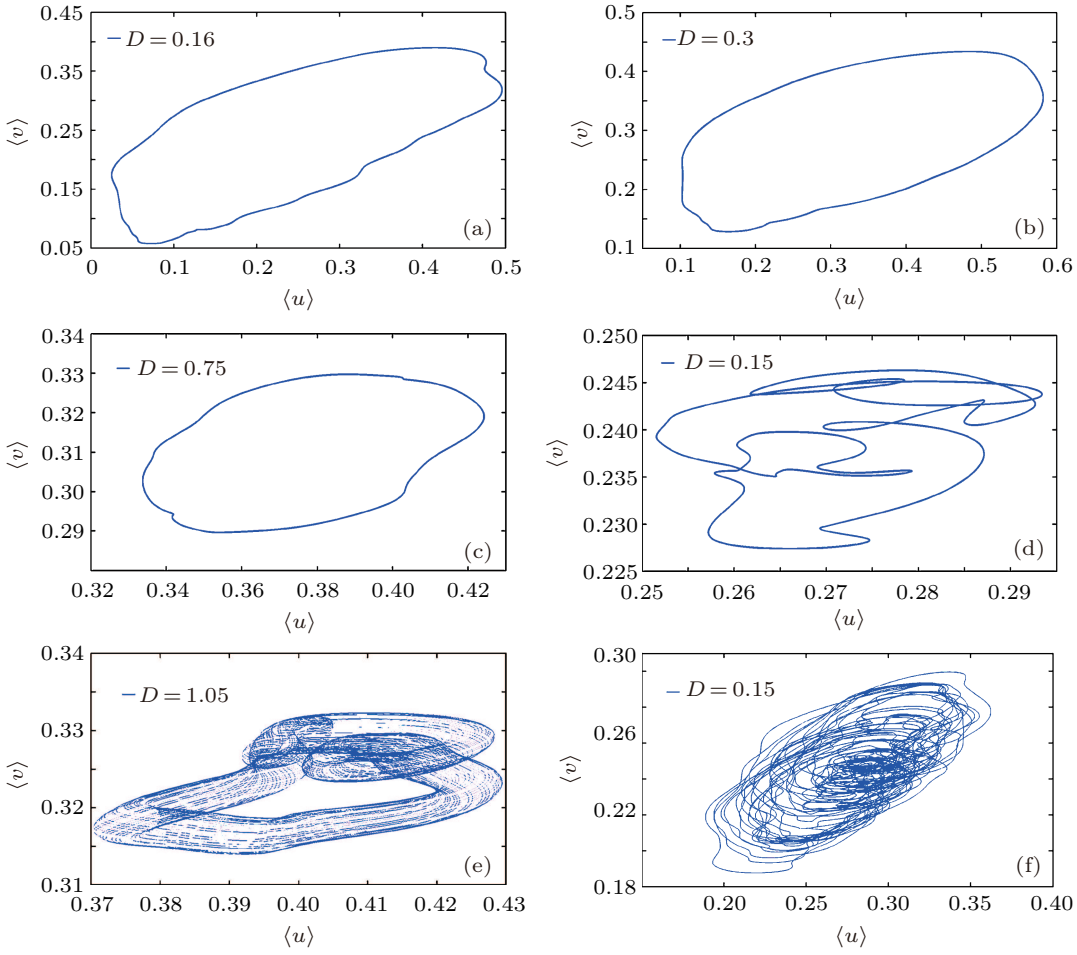


Fig. 5. (color online) Some trajectories in the ($\langle u(t)\rangle$, $\langle v(t)\rangle$) phase plane for symmetric ECNs ((a) and (b)) and asymmetric couplings ((c) and (d)). Motions are quasiperiodic in panel (e) and chaotic in panel (f).

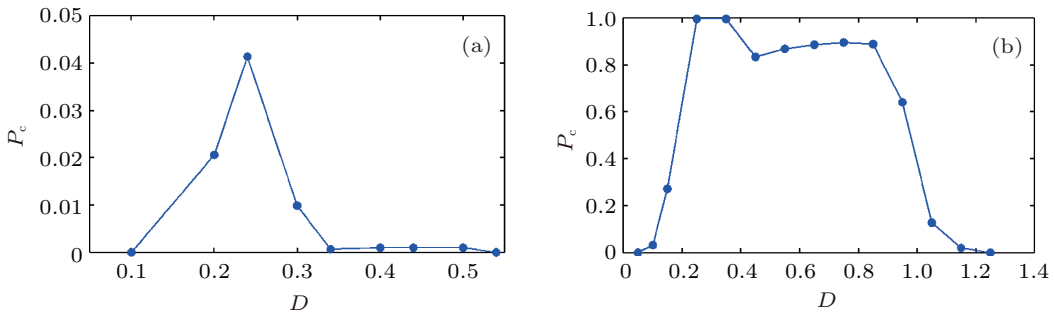


Fig. 6. (color online) The probabilities of complicated orbits in all self-sustained oscillatory samples P_c versus coupling strength D for symmetric ((a)) and asymmetric ((b)) ECNs. Here all quasiperiodic, chaotic trajectories, and periodic trajectories with self-crossings in the ($\langle u(t)\rangle$, $\langle v(t)\rangle$) phase plane are classed with complicated orbits. Obviously, the probability of complicated motions in symmetric ECNs is incomparably smaller than those in asymmetric ECNs.

Now we study the problem of oscillation period. For the complicated motions of Figs. 5(d)–5(f), the concept of period cannot be defined in a strict sense. We can however define a quantity of average period similar to that used in chaos study. Here the average period $\langle T \rangle$ is defined in such a way that for each oscillatory state we compute the average period of a network node by the average time interval for the $u(t)$ value of the given node to successively cross the threshold $u = 0.6$ from the lower part, and then average the average periods of all nodes as the period of the network for the given oscillatory sample. In Fig. 7(a) we fix $D = 0.2$ and compute the distribution of $\langle T \rangle = T$ for symmetric ECNs (in the symmetric case most of the motions are periodic and $\langle T \rangle = T$ is uniquely defined in the strict sense) by using the same samples as in Fig. 2. It is found that the period values distribute rather smoothly in the interval $6.3 \lesssim T \lesssim 8.8$. In Fig. 7(b) we do it in the same way as in Fig. 7(a) with asymmetric ECNs investigated. Here we observe a remarkable difference, i.e., in Fig. 7(b) the period distribution is concentrated in a very narrow interval around a T value which is not only incomparably smaller than that of SFNs in Fig. 3, but also much smaller than that of symmetric RHNs of Figs. 3 and 7(a).

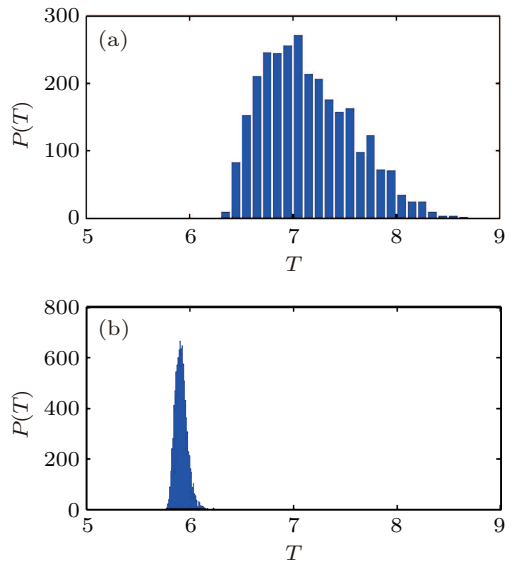


Fig. 7. (color online) Distributions of oscillation periods of ECNs. Panel (a) shows the distribution $P(T)$ with respect to T for symmetrically coupled ECNs. The samples for computations are the same as those in Figs. 2(a) and 2(c) at $D = 0.2$. Panel (b) shows the distribution $P(T)$ with respect to T with asymmetric couplings considered. The samples are the same as those used in Fig. 5 at $D = 0.2$.

4. Qualitative explanations on the relations of network dynamics to network topology and coupling structure

Network dynamics is closely related to and even determined by network topology and coupling structure. It is interesting to explore the mechanisms behind all the above-explored different characteristics of random and scale-free networks and symmetrical or asymmetrical coupled ECNs.

First, we compare the dynamic behaviors between symmetrically coupled RHNs and SFNs. It has been already understood that oscillations of all ECNs are due to self-organized source loops.[18–20] In RHNs there are an extremely large number of candidates of such loops.[21] For a random initial state one or a few of these candidates may dynamically construct successive excitation propagation in the loops which serve as the oscillation source and support the sustained oscillation of the whole network. Since each ECN with RHN topology has a large number of candidate loops, there is a large probability for a few of these candidates to become the dynamic source loop. One can therefore observe a rather large oscillation probability in Fig. 2(a) at some suitable coupling strength. Moreover, different candidate source loops produce different oscillation attractors, and lead to the huge multiplicity of attractors in Fig. 2(c). Furthermore, different source loops can produce different excitation propagation wave modes in networks, and these modes can compete with each other. It has been known that in wave competitions fast modes can always defeat slow modes.[22] Moreover, the resulting network oscillation is governed, with a large probability, by fast modes with relatively short periods, which is shown in Fig. 3 for RHNs.

The most important characteristic in the SFN topology is its strong heterogeneity in their degree distributions. In SFNs there are few hub nodes with very large link degrees and a large number of 'small' nodes with rather low degrees. The hubs with large degrees are subjected to strong dissipation. The coupling term of node i $\sum_{j\neq i}^{N} C_{ij} = \sum_{j\neq i}^{N} M_{ij}(u_j - u_i)$ provides a negative feedback term $-k_i u_i$ for the local dynamics of node i. For the electric gap junction coupling, this term is nothing but a diffusive coupling-induced dissipation.[16] Large hub nodes have large values of k_i and thus large dissipations. Therefore hub nodes are difficult to excite. For instance, in our case, rest nodes with $k_i < 6$ can be successfully excited by any single excited neighbor, but rest nodes with $k_i \geq 6$ can be excited only by two or more simultaneously excited neighbors. Therefore, hub nodes may depress a large number of excitation propagation loops and greatly reduce the number of possible candidates of source loops and thus reduce the oscillation probability (Fig. 2(b)) and reduce the number of oscillatory attractors (Fig. 2(d)). Moreover, in all actual source loops of SFNs, excitations should propagate in a long chain of low degree nodes and find a chance to excite certain hub nodes by simultaneous excitations of their neighbors which have been excited also nearly simultaneously in the previous excitation propagations. These chances must be rather rare in SFNs and the oscillation probability and number of attractors must be small. Moreover, the total process for excitations to propa-

gate among low-degree nodes and to find the chance to excite hubs by simultaneously excited neighbors usually requires a long time for excitation propagations on some long chains of nodes, and this makes the period of oscillation of SFNs much longer than that of RHNs (Fig. 3).

The mechanisms behind the different dynamic features of symmetric and asymmetric ECNs are totally different from those behind the above facts. In symmetric ECNs all embedded loops are topologically symmetric with respect to the two opposite directions of excitation propagation. Then only the loops with initial state breaking symmetry have the chance to serve as the candidates of source loops. This requirement for symmetry-breaking may reduce oscillation probability. On the other hand, also due to this symmetry-breaking requirement many embedded loops with different lengths have the chance to serve as the oscillation source loops, which leads to a large number of attractors in Fig. 2(c) and the possibility of a fairly wide period distribution spectra in Fig. 7(a). For asymmetric couplings, there are many embedded loops in ECNs already breaking symmetry topologically (i.e., these topological symmetry-breaking loops allow excitation to propagate only in one direction and not in the opposite direction for all possible initial states). Therefore for these loops the requirement for symmetry-breaking in initial states is no longer needed for the loop (nor for network) oscillation. This condition release can considerably increase the probability of sustained oscillations as we see in Fig. 4(a). In an asymmetric networks the embedded loops with topological symmetry-breaking each have a much larger probability to serve as oscillation source loops than symmetric loops. Moreover, in competitions of oscillation modes supported by different candidate source loops only the modes with fastest oscillations supported by shortest loops can survive finally. These competitions result in the fact that a much smaller number of oscillatory attractors (shown by comparing Fig. 4(b) with Fig. 2(c)) and much more concentrated and much shorter oscillation periods (shown by comparing Fig. 7(b) with Fig. 7(a)) are produced in the asymmetric networks than in the symmetric ECNs. Since in asymmetric networks, all topological symmetry-breaking loops supporting fastest oscillatory modes can produce sustained oscillations rather stably, more and different modes can coexist in the final oscillatory ECNs. The competitions among these modes can naturally generate rather complicated motions, including quasiperiodic and chaotic motions (see Figs. 5(c)–5(f), Fig. 6). In the symmetric ECNs, all embedded loops are topologically symmetric and whether they can be dynamically turned in to symmetry-breaking is sensitive to the initial states and the inputs from off-loop nodes. Finally after competition, the network oscillations are often controlled by one or two (almost no more for our observations) source loops, and they can only produce simple periodic oscillations as shown in Figs. 5(a) and 5(b).

5. Conclusion

In this paper we study the problems of the oscillatory behaviors of excitable complex networks and the relations of network dynamics to network topology and system structure. Some interesting dynamic features such as the probabilities of self-sustained oscillatory patterns, the multiplicity of oscillatory attractors, the periodicity of oscillation, and the influences of interaction topology and coupling structure on the above dynamic features and so on are studied in detail. It is found that the symmetric ECNs with random homogeneous interacted structures each have a large probability of oscillation, a large number of attractors and small average periods of oscillation, SFNs with diffusive couplings each have a much smaller probability of oscillation, incomparably smaller multiplicity of attractors, and significantly large average periods of oscillation. On the other hand, RHNs of asymmetric ECNs each have an even larger probability of oscillation, a much smaller number of attractors and much more concentrated period distributions than those of symmetric ECNs. Moreover, asymmetric ECNs have much more complicated motions than symmetric ECNs, i.e., while the latter have, in most of realizations, simple periodic oscillatory trajectories, the former often show complicated trajectories, including quasiperiodic and chaotic orbits.

The mechanisms behind the above remarkably different dynamics are intuitively understood, based on the correlations of topology and coupling structure to their dynamics. In particular, the strong heterogeneity of topology and the difficulty of hub excitation induced by their large dissipations are the key ingredients yielding the dramatically different dynamic features of symmetric SFNs and RHNs. In the cases of asymmetrically coupled ECNs, the existence and richness of directed topological loops, the stability of excitation propagation cycles in these topological loops, and the prevailing competitions between different oscillation modes induced by different source loops are crucial in understanding the different characteristics of asymmetric and symmetric ECNs.

It is interesting to apply the above understanding to practical applications. For instance, we may adjust the system topology and interaction structure to meet the realistic requirements for the oscillation probability and multiplicity of attractors according to the analysis in Section 4. Moreover, we can adjust the threshold heterogeneity of nodes and the dissipation of networks to control the periodicity of oscillations. All these may be of practical significance in manipulating the dynamics and functions of realistic ECNs.

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