Consensus of heterogeneous multi-agent systems based on sampled data with a small sampling delay*

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In this paper, consensus problems of heterogeneous multi-agent systems based on sampled data with a small sampling delay are considered. First, a consensus protocol based on sampled data with a small sampling delay for heterogeneous multi-agent systems is proposed. Then, the algebra graph theory, the matrix method, the stability theory of linear systems, and some other techniques are employed to derive the necessary and sufficient conditions guaranteeing heterogeneous multi-agent systems to asymptotically achieve the stationary consensus. Finally, simulations are performed to demonstrate the correctness of the theoretical results.

Keywords: heterogeneous multi-agent systems, consensus, sampled-data, small sampling delay

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1. Introduction

In recent years, distributed coordination control of multi-agent systems has attracted considerable attention from various research communities due to its broad applications including in swarming, flocking,[1] formation, synchronization,[2] automated highway systems, distributed sensor networks, cooperative control of unmanned air vehicles, congestion control in communication networks, etc. As one of the most important issues in the distributed coordination control of multi-agent systems, consensus requires that the outputs of several spatially distributed agents reach a common value without a central controller or global communication.[3] Most of the relevant results primarily focus on the agents with first-order integrator dynamics.[4–6] With the help of graph theory and matrix theory, in Refs. [7] and [8] there was proved that the agents over arbitrary switching topologies converge to a common steady state if the undirected interaction topology is jointly connected or the directed topology is jointly rooted. In Refs. [9] and [10] it was further demonstrated that if each agent compares the delayed states of its neighbors with its own current state, the consensus is independent of time-varying communication delays. By employing the frequency-domain analysis, in Ref. [11] it was shown that the first-order consensus condition depends on input delay rather than communication delay.

Compared with the case of first-order integrators, the consensus problem of second-order integrator agents is more complicated. Dynamical second-order consensus problems have been studied in Refs. [12]–[21]. In Ref. [12], Tian and Liu studied the effects of constant input delays on the convergence of second-order consensus protocols and provided delay-dependent consensus conditions. By employing the frequency-domain analysis, a simplified sufficient condition was given to guarantee the consensus of the second-order multi-agent systems with nonuniform time delays in Ref. [13]. Park *et al.*[14] investigated new delay-dependent consensus criteria for multi-agent systems with time-varying delays and switching interconnection topologies by constructing a suitable Lyapunov–Krasovskii functional and using the reciprocally convex approach. In Ref. [15], using the graph theory and the common Lyapunov function method, Gao *et al.* obtained some sufficient conditions for second-order multi-agent systems with undirected and balanced switching interconnection topology. There are also some results investigating the consensus problems of multi-agent systems with higher order dynamics.[22–25]

Most of the above results are on multi-agent systems with the same-order integrators, that is to say, all agents have the same dynamic behaviors. However, in the practical systems, the dynamic behaviors of the agents coupled with each other might be different because of various restrictions or the common goals that the mixed agents have. Therefore, it is necessary to study the consensus of multi-agent systems with different dynamic behaviors. Recently, there were some results on the consensus problems of heterogeneous multi-agent systems.[26–32] Liu C L and Liu F[26] studied the stationary consensus of discrete-time heterogeneous multi-agent systems with bounded communication delays, and obtained the sufficient consensus criteria by using the properties of non-negative matrices. Zheng *et al.*[27] gave some sufficient conditions

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for the consensus of heterogeneous multi-agent systems with undirected, connected, and leader-following network topologies. Lee and Spong[28] studied the consensus of continuous-time heterogeneous multi-agent systems with non-uniform communication delays, and obtained delay-independent consensus conditions by employing the frequency-domain analysis and spectral radius theory. In Ref. [29], by using graph theory and Lyapunov theory, Zheng and Wang gave some consensus protocols to resolve the consensus problem for heterogeneous multi-agent systems without velocity measurements. The finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements was investigated in Ref. [30]. Employing frequency-domain analysis, Tian[31] obtained a necessary and sufficient condition for the existence of a high-order consensus solution to heterogeneous multi-agent systems with unknown communication delays. According to Mason's rule and the Routh stability criterion, Yan *et al*.[32] provided the convergence conditions for heterogeneous multi-agent systems, which ensure that agents can catch up with the target in finite time.

In Refs. [26]–[32], concerning the consensus of heterogeneous multi-agent systems, continuous-time consensus protocols were used for continuous-time multi-agent systems, and discrete-time consensus protocols were used for discrete-time multi-agent systems. However, due to the application of digital sensors and controllers, in many cases, though the system itself is a continuous process, only sampled-data at discrete sampling instants are available for the synthesis of control laws. Compared with full-state information transmission resulting in high communication traffic, the transmission of the state information at the sampling instants can effectively save the bandwidth of networks and communication cost. Considering the broad applications of digital communication and control in distributed systems, it is of great significance to study the consensus of heterogeneous multi-agent systems based on sampled-data. To the best of our knowledge, there has been no report on the consensus of heterogeneous multi-agent systems based on sampled-data control.

Moreover, in real applications, the sampling delay induced by the signal sampling process indeed exists and might cause multi-agent systems based on sampled-data to oscillate or diverge. Thus, the effects of the sampling delay on consensus of multi-agent systems should be considered. Recently, in Refs. [33] and [34], Wu *et al*. investigated the stochastic bounded consensus tracking problems of first-order and second-order multi-agent systems with measurement noises based on sampled-data with a small sampling delay. However, to date, there have been no results on the consensus of heterogeneous multi-agent systems based on sampled data with a small sampling delay.

According to the above considerations, in this paper we consider the consensus problems of heterogeneous multi-agent systems based on sampled data with a small sampling delay. First, the consensus protocol based on sampled data with a small sampling delay for heterogeneous multi-agent systems is proposed. Second, by employing the algebra graph theory, the matrix method, the stability theory of linear systems, and some other techniques, we obtain the necessary and sufficient conditions guaranteeing that heterogeneous multi-agent systems asymptotically achieve the stationary consensus. Finally, simulations are performed to demonstrate the correctness of the theoretical results.

The remainder of this paper is organized as follows. In Section 2, some preliminaries are provided and the problem is formulated. Convergence analysis is investigated in Section 3. Numerical simulations are provided in Section 4. Finally, concluding remarks are stated in Section 5.

2. Preliminaries and problem statement

2.1. Algebra graph theory and some notations

Let $G=(V,E,A)$ be a weighted undirected graph with a set of nodes $V=\{v_1,v_2,\ldots,v_N\}$, a set of edges $E\subseteq V\times V$, and the weighted adjacency matrix $A=[a_{ij}]$ with nonnegative adjacency elements a_{ij}. The node indexes of G belong to a finite index set $I=\{1,2,\ldots,N\}$. An edge of G is denoted by $e_{ij}=(v_i,v_j)$. The adjacency elements associated with the edges are positive, i.e., $e_{ij}\in E\Leftrightarrow a_{ij}>0$. Moreover, we assume $a_{ii}=0$ for all $i\in I$. For the undirected graph G, the adjacency matrix A is symmetric, i.e., $a_{ij}=a_{ji}$. The set of neighbors of node v_i is denoted by $N_i=\{v_j\in V: e_{ij}\in E\}$. The degree of node v_i is defined as $d_i=\sum\limits_{j\in N_i}a_{ij}$. The Laplacian matrix of G is defined as $L=D-A$, where $D=\mathrm{Diag}(d_1,d_2,\ldots,d_N)$ is the degree matrix of G with diagonal elements d_i and zero off-diagonal elements. An important fact of L is that all row sums are zero and thus L has a right eigenvector 1_N associated with the zero eigenvalue, i.e., $L1_N=0_N$, where 1_N denotes the N-dimensional column vector being unity. A path between two distinct nodes v_i and v_j mean a sequence of distinct edges of the form $(v_i,v_{k_1}),(v_{k_1},v_{k_2}),\ldots,(v_{k_l},v_j)$. A graph is called 'connected' if there is a path between any two distinct nodes of the graph. An important property of the Laplacian matrix, which is instrumental in the convergence analysis of consensus protocols, is that the graph G is connected if and only if $\mathrm{rank}(L)=N-1$.[35] Thus, for a connected graph, L has one and only one zero eigenvalue and the remaining eigenvalues of L are all positive. Without loss of generality, for a connected graph G, all the eigenvalues of L can be ordered sequentially in an ascending order as $0=\lambda_1(L)<\lambda_2(L)\leq\cdots\leq\lambda_N(L)$.

For heterogeneous multi-agent systems, N-order weighted Laplacian matrix of the connected graph G is given

by

$$L=\begin{bmatrix} L_s+D_{sf} & -A_{sf} \\ -A_{fs} & L_f+D_{fs} \end{bmatrix},$$

where L_s and L_f are the Laplacian matrices of the interaction topology composed of the M second-order agents and the $N-M$ first-order agents, respectively, and $A_{sf}\in R^{M\times(N-M)}$ and $A_{fs}\in R^{(N-M)\times M}$ are the adjacency matrix between the first-order agents and the second-order agents, respectively. Denote

$$D_{sf}=\mathrm{diag}\left\{\sum_{j=M+1}^{N}a_{ij}, i\in I_M\right\}$$

and

$$D_{fs}=\mathrm{diag}\left\{\sum_{j=1}^{M}a_{ij}, i\in\{M+1,\ldots,N\}\right\}.$$

The following notations will be used throughout this paper. 0_N denotes the N-dimensional column vectors with all zeros. I_N denotes the N-dimensional identity matrix. Let R denote the set of real numbers. For a given vector or matrix P, P^{T} and $\det(P)$ denote its transpose and determinant, respectively. λ_i and u_j are the i-th, j-th eigenvalues of L_s+D_{sf} and L_f+D_{fs}, respectively, where $i=1,2,\ldots,M$; $j=1,2,\ldots,N-M$. The symbols $\otimes$ and $\oplus$ denote the Kronecker product and the Kronecker sum, respectively.

2.2. Problem statement

Suppose that heterogeneous multi-agent systems consist of M second-order agents and $N-M$ first-order agents. Without loss of generality, the first M $(M<N)$ agents are assumed to be second-order ones, while the remaining $N-M$ agents are first-order ones. The system of dynamic agents is given as follows:

$$\begin{cases} \dot{x}_i(t)=y_i(t),\ i\in I_M, \\ \dot{y}_i(t)=u_i(t),\ i\in I_M, \\ \dot{x}_i(t)=u_i(t),\ i\in\{M+1,\ldots,N\}, \end{cases} \tag{1}$$

where $x_i(t)\in R$, $y_i(t)\in R$, and $u_i(t)\in R$ are regarded as the position, velocity, and control input of agent i, respectively.

Definition 1 The heterogeneous multi-agent system (1) is called to asymptotically achieve the stationary consensus, if

$$\begin{cases} \lim_{t\to\infty}x_i(t)=c, \text{ for } i\in I_N, \\ \lim_{t\to\infty}y_i(t)=0, \text{ for } i\in I_M, \end{cases}$$

where c is a constant.

Zheng *et al.*[27] proposed the following linear consensus protocol for the heterogeneous multi-agent systems (1):

$$u_i(t)=\begin{cases} -\gamma y_i(t)+\sum_{j=1}^{N}a_{ij}[x_j(t)-x_i(t)],\ i\in I_M, \\ \sum_{j=1}^{N}a_{ij}[x_j(t)-x_i(t)],\ i\in\{M+1,\ldots,N\}, \end{cases} \tag{2}$$

where $\gamma>0$ is the control parameter.

Denote the sampling period and the sampling delay by h and τ, respectively, where $h>0$ and the sampling delay is fixed and less than one sampling period, i.e., $\tau\in(0,h)$. Using the periodic sampling technology and zero-order hold circuit, the following consensus protocol based on sampled data with a small sampling delay is induced from the continuous-time consensus protocol (2)

$$u_i(t)=\begin{cases} u_i(kh-h)=-\gamma y_i(kh-h) \\ \qquad\qquad +\sum_{j=1}^{N}a_{ij}[x_j(kh-h)-x_i(kh-h)], \\ t\in[kh,kh+\tau),\ k=0,1,2,\ldots,\ i\in I_M, \\ u_i(kh)=-\gamma y_i(kh)+\sum_{j=1}^{N}a_{ij}[x_j(kh)-x_i(kh)], \\ t\in[kh+\tau,kh+h),\ k=0,1,2,\ldots,\ i\in I_M, \\ u_i(kh-h)=\sum_{j=1}^{N}a_{ij}[x_j(kh-h)-x_i(kh-h)], \\ t\in[kh,kh+\tau), \\ k=0,1,2,\ldots,\ i\in\{M+1,\ldots,N\}, \\ u_i(kh)=\sum_{j=1}^{N}a_{ij}[x_j(kh)-x_i(kh)], \\ t\in[kh+\tau,kh+h), \\ k=0,1,2,\ldots,\ i\in\{M+1,\ldots,N\}. \end{cases} \tag{3}$$

In the following, we will find the necessary and sufficient conditions guaranteeing the systems (1) using the consensus protocol (3) will achieve the stationary consensus.

Before doing so, we need to provide the following lemmas.

Lemma 1 (Schur's Formula[36]) Let A, B, C, $D\in R^{n\times n}$, where $n\in N^+$ denotes the dimension of a matrix. If A, B, C, and D commute pairwise, then

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)=\det(AD-BC).$$

Lemma 2 (Horn and Jornson.[37]) For matrices A, B, C, and D with appropriate dimensions, the Kronecker product $\otimes$ has the following properties:

(i) $(A+B)\otimes C=A\otimes C+B\otimes C$; (ii) $k(A\otimes B)=kA\otimes B=A\otimes kB$, $k\in c$; (iii) $I_M\otimes I_N=I_N\otimes I_M=I_{MN}$.

Lemma 3 (Horn and Jornson[37]) Suppose that A is an $N\times N$ matrix and B is an $M\times M$ matrix, then

i) the NM eigenvalues of $A\otimes B$ will be $\lambda_i u_j$, $i=1,2,\ldots,N$; $j=1,2,\ldots,M$, where $\lambda_1,\lambda_2,\ldots,\lambda_N$ are the N eigenvalues of A, and $u_1,u_2,\ldots,u_M$ are the M eigenvalues of B.

ii) Define the Kronecker sum $\oplus$ as $A\oplus B=A\otimes I_M+I_N\otimes B$, then the NM eigenvalues of $A\oplus B$ will be λ_i+u_j, $i=1,2,\ldots,N$; $j=1,2,\ldots,M$.

Lemma 4 (Ren and Cao[38]) All roots of the equation $s^2+as+b=0$, where a, $b\in R$, are within the unit circle, if and only if all roots of the equation $(1+a+b)t^2+2(1-b)t+1-a+b=0$ are in the open left half plane.

3. Convergence analysis

In this section, by employing the algebra graph theory, the matrix method, the stability theory of linear systems, and some other techniques, we derive the necessary and sufficient conditions guaranteeing that the heterogeneous multi-agent systems (1) using the consensus protocol (3) achieve the stationary consensus. The main results are as follows.

Theorem 1 The heterogeneous multi-agent system (1) with a fixed, undirected and connected network topology using the consensus protocol (3), achieves the stationary consensus if and only if all roots of the equation

$$\begin{aligned}&s^4+[hr-\tau r-2+(h-\tau)u_j+(h-\tau)^2\lambda_i/2]s^3\\&+[1-hr+2\tau r+((h-\tau)^2r+2\tau-h)u_j\\&+(h\tau-\tau^2+h^2/2)\lambda_i]s^2+[(2\tau rh-2\tau^2r-\tau)u_j\\&+\tau^2\lambda_i/2-\tau r]s+\tau^2ru_j=0,\ i\in I_M,\ j\in I_{N-M}\end{aligned}\tag{4}$$

are within the unit circle.

Proof When $t\in[kh,kh+\tau)$, from Eqs. (1) and (3) we obtain

$$\begin{cases}\begin{pmatrix}x_i(t)\\y_i(t)\end{pmatrix}=\exp\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}(t-kh)\right\}\begin{pmatrix}x_i(kh)\\y_i(kh)\end{pmatrix}+\displaystyle\int_{kh}^{t}\exp\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}(t-t')\right\}\begin{pmatrix}0\\1\end{pmatrix}u_i(kh-h)\mathrm{d}t'\\\qquad\quad=\begin{bmatrix}1&t-kh\\0&1\end{bmatrix}\begin{pmatrix}x_i(kh)\\y_i(kh)\end{pmatrix}+\begin{pmatrix}(t-kh)^2/2\\t-kh\end{pmatrix}u_i(kh-h),\quad k=0,1,2,\ldots,\ i\in I_M,\\x_i(t)=x_i(kh)+(t-kh)u_i(kh-h),\quad k=0,1,2,\ldots,\quad i\in\{M+1,\ldots,N\},\end{cases}\tag{5}$$

which further leads to

$$\begin{cases}\begin{pmatrix}x_i(kh+\tau)\\y_i(kh+\tau)\end{pmatrix}=\begin{bmatrix}1&\tau\\0&1\end{bmatrix}\begin{pmatrix}x_i(kh)\\y_i(kh)\end{pmatrix}+\begin{pmatrix}\tau^2/2\\\tau\end{pmatrix}u_i(kh-h),\quad k=0,1,2,\ldots,\ i\in I_M,\\x_i(kh+\tau)=x_i(kh)+\tau u_i(kh-h),\quad k=0,1,2,\ldots,\quad i\in\{M+1,\ldots,N\}.\end{cases}\tag{6}$$

Similarly, when $t\in[kh+\tau,kh+h)$, from Eqs. (1) and (3) we obtain

$$\begin{cases}\begin{pmatrix}x_i(t)\\y_i(t)\end{pmatrix}=\exp\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}(t-kh-\tau)\right\}\begin{pmatrix}x_i(kh+\tau)\\y_i(kh+\tau)\end{pmatrix}+\displaystyle\int_{kh+\tau}^{t}\exp\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}(t-t')\right\}\begin{pmatrix}0\\1\end{pmatrix}u_i(kh)\mathrm{d}t'\\\qquad\quad=\begin{bmatrix}1&t-kh-\tau\\0&1\end{bmatrix}\begin{pmatrix}x_i(kh+\tau)\\y_i(kh+\tau)\end{pmatrix}+\begin{pmatrix}(t-kh-\tau)^2/2\\t-kh-\tau\end{pmatrix}u_i(kh),\quad k=0,1,2,\ldots,\ i\in I_M,\\x_i(t)=x_i(kh+\tau)+(t-kh-\tau)u_i(kh),\ k=0,1,2,\ldots,\ i\in\{M+1,\ldots,N\},\end{cases}\tag{7}$$

which further results in

$$\begin{cases}\begin{pmatrix}x_i(kh+h)\\y_i(kh+h)\end{pmatrix}=\begin{bmatrix}1&h-\tau\\0&1\end{bmatrix}\begin{pmatrix}x_i(kh+\tau)\\y_i(kh+\tau)\end{pmatrix}+\begin{pmatrix}(h-\tau)^2/2\\h-\tau\end{pmatrix}u_i(kh),\quad k=0,1,2,\ldots,\ i\in I_M,\\x_i(kh+h)=x_i(kh+\tau)+(h-\tau)u_i(kh),\quad k=0,1,2,\ldots,\ i\in\{M+1,\ldots,N\}.\end{cases}\tag{8}$$

Combining Eqs. (6) and (8) yields the dynamic equation at the sampling instants

$$\begin{cases}x_i(kh+h)=x_i(kh)+hy_i(kh)+(h\tau-\tau^2/2)u_i(kh-h)+[(h-\tau)^2/2]u_i(kh),\quad k=0,1,2,\ldots,\ i\in I_M,\\y_i(kh+h)=y_i(kh)+\tau u_i(kh-h)+(h-\tau)u_i(kh),\quad k=0,1,2,\ldots,\ i\in I_M,\\x_i(kh+h)=x_i(kh)+(h-\tau)u_i(kh)+\tau u_i(kh-h),\quad k=0,1,2,\ldots,\ i\in\{M+1,\ldots,N\}.\end{cases}\tag{9}$$

Notice $L1_N=0_N$, then the above dynamics equation (9) can be rewritten in a compact form

$$\begin{cases}x_s(kh+h)=[I_M-(h-\tau)^2(L_s+D_{sf})/2]x_s(kh)+[(h-\tau)^2A_{sf}/2]x_f(kh)\\\qquad\qquad-(h\tau-\tau^2/2)ry_s(kh-h),\quad k=0,1,2,\ldots,\quad i\in I_M,\\y_s(kh+h)=(\tau-h)(L_s+D_{sf})x_s(kh)+(h-\tau)A_{sf}x_f(kh)-\tau(L_s+D_{sf})x_s(kh-h)\\\qquad\qquad+\tau A_{sf}x_f(kh-h)+(1-hr+\tau r)y_s(kh)-\tau ry_s(kh-h),\quad k=0,1,2,\ldots,\quad i\in I_M,\\x_f(kh+h)=[I_{N-M}-(h-\tau)(L_f+D_{fs})]x_f(kh)+(h-\tau)A_{fs}x_s(kh)+\tau A_{fs}x_s(kh-h)\\\qquad\qquad-\tau(L_f+D_{fs})x_f(kh-h),\quad k=0,1,2,\ldots,\quad i\in\{M+1,\ldots,N\},\end{cases}\tag{10}$$

where $x_s(kh)=[x_1(kh),\ x_2(kh),\ \ldots,\ x_M(kh)]^{\mathrm{T}}$, $y_s(kh)=[y_1(kh),\ y_2(kh),\ \ldots,\ y_M(kh)]^{\mathrm{T}}$ and

$$x_f(kh)=[x_{M+1}(kh),\ x_{M+2}(kh),\ \ldots,\ x_N(kh)]^{\mathrm{T}}.$$

Define $\xi(kh)=\left[x_s^{\mathrm{T}}(kh),\ y_s^{\mathrm{T}}(kh),\ x_f^{\mathrm{T}}(kh),\ x_s^{\mathrm{T}}(kh-h),\ y_s^{\mathrm{T}}(kh-h),\ x_f^{\mathrm{T}}(kh-h)\right]^{\mathrm{T}}$, then we will rewrite the systems (10) as

$$\xi(kh+h)=\begin{bmatrix} A & B \\ I_{N+M} & 0_{N+M}\end{bmatrix}\xi(kh),\quad k=0,1,2,\ldots, \tag{11}$$

where

$$A=\begin{bmatrix} I_M-[(h-\tau)^2(L_s+D_{sf})/2] & [h-(h-\tau)^2r/2]I_M & (h-\tau)^2A_{sf}/2 \\ (\tau-h)(L_s+D_{sf}) & (1-hr+\tau r)I_M & (h-\tau)A_{sf} \\ (h-\tau)A_{fs} & 0_{N-M} & I_{N-M}-(h-\tau)(L_f+D_{fs})\end{bmatrix}_{(N+M)\times(N+M)}$$

and

$$B=\begin{bmatrix} -(h\tau-\tau^2/2)(L_s+D_{sf}) & -(h\tau-\tau^2/2)rI_M & (h\tau-\tau^2/2)A_{sf} \\ -\tau(L_s+D_{sf}) & -\tau rI_M & \tau A_{sf} \\ \tau A_{fs} & 0_{N-M} & -\tau(L_f+D_{fs})\end{bmatrix}_{(N+M)\times(N+M)}.$$

From Eq. (11), we have

$$\xi(nh)=\begin{bmatrix} A & B \\ I_{N+M} & 0_{N+M}\end{bmatrix}^n\xi(0). \tag{12}$$

Denote the characteristic polynomial of $\begin{bmatrix} A & B \\ I_{N+M} & 0_{N+M}\end{bmatrix}$ by $F(s)$. Employing Lemma 1, Lemma 2, and Lemma 3 yields

$$\begin{aligned}
F(s) &= \det\left(sI_{2(N+M)\times 2(N+M)}-\begin{bmatrix} A & B \\ I_{N+M} & 0_{N+M}\end{bmatrix}\right)\\
&= \det(s^2I_{N+M}-sA-B)\\
&= \det\left(\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33}\end{bmatrix}\right)\\
&= \det([s^6-(3-hr+\tau r)s^5+(3-2hr+3\tau r)s^4\\
&\quad -(1-hr+3\tau r)s^3+\tau rs^2]I_{(N-M)M}\\
&\quad +[(h-\tau)^2s^5/2+(4h\tau-3\tau^2)s^4/2\\
&\quad -(h^2+2h\tau-3\tau^2)s^3/2-\tau^2s^2/2]\\
&\quad \times((L_s+D_{sf})\otimes I_{N-M})\\
&\quad +[(h-\tau)s^5+((h-\tau)^2r-2h+3\tau)s^4\\
&\quad +(h-h^2r-3\tau-3\tau^2r+4\tau rh)s^3\\
&\quad +(\tau-2\tau hr+3\tau^2r)s^2-\tau^2rs]\\
&\quad \times(I_M\otimes(L_f+D_{fs})))\\
&= (s^2-s)\prod_{i=1,2,\ldots,M;j=1,2,\ldots,N-M}\\
&\quad \times\{s^4+[hr-\tau r-2+(h-\tau)u_j+(h-\tau)^2\lambda_i/2]s^3\\
&\quad +[1-hr+2\tau r+((h-\tau)^2r+2\tau-h)u_j\\
&\quad +(h\tau-\tau^2+h^2/2)\lambda_i]s^2+[(2\tau rh-2\tau^2r-\tau)u_j\\
&\quad +\tau^2\lambda_i/2-\tau r]s+\tau^2ru_j\},
\end{aligned}\tag{13}$$

where c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, and c_{33} in Eqs. (13) satisfy

$$\begin{aligned}
c_{11} &= (s^2-s)I_M+[s(h-\tau)^2/2+(h\tau-\tau^2/2)](l_s+D_{sf}),\\
c_{12} &= [(h-\tau)^2rs/2-hs+(h\tau-\tau^2/2)r]I_M,\\
c_{13} &= [-s(h-\tau)^2/2+(\tau^2/2-h\tau)]A_{sf},\\
c_{21} &= [s(h-\tau)+\tau](L_s+D_{sf}),\\
c_{22} &= [s^2-s(1-hr+\tau r)+\tau r]I_M,\\
c_{23} &= [s(\tau-h)-\tau]A_{sf},\\
c_{31} &= [s(\tau-h)-\tau]A_{fs},\quad c_{32}=0_{N-M},
\end{aligned}$$

and

$$c_{33}=(s^2-s)I_{N-M}+[s(h-\tau)+\tau](L_f+D_{fs}).$$

It is obvious that equation (13) has one and only one root equal to 1 and one and only one root equal to 0. Therefore, system (13) is asymptotically stable if and only if all roots of Eq. (4) are within a unit circle.

Now it is assumed that all roots of Eq. (4) are within a unit circle. Denote the static state that the state $x_i(t)$ of the systems (10) converges to, as x_i^*, i.e., $\lim_{k\to\infty}x_i(kh)=x_i^*$, $\forall i\in I_N$, then

$$\lim_{k\to\infty}y_i(kh)=0,\ \ \forall i\in I_M. \tag{14}$$

From Eq. (10) we have $L(x_1^*,\ x_2^*,\ \ldots,\ x_N^*)^{\mathrm{T}}=0$. Notice $L1_N=0_N$, thus $(x_1^*,\ x_2^*,\ \ldots,\ x_N^*)^{\mathrm{T}}=c1_N$, i.e.,

$$\lim_{k\to\infty}x_i(kh)=c,\quad \forall i\in I_N. \tag{15}$$

It follows from Eq. (5) that when $t\in[kh,kh+\tau)$,

$$\begin{cases}\begin{pmatrix} x_s(t)\\ y_s(t)\end{pmatrix}-\begin{bmatrix} 1 & t-kh\\ 0 & 1\end{bmatrix}\begin{pmatrix} x_s(kh)\\ y_s(kh)\end{pmatrix}=\begin{pmatrix}(t-kh)^2/2\\ t-kh\end{pmatrix}[-ry_s(kh-h)-(L_s+D_{sf})x_s(kh-h)+A_{sf}x_f(kh-h)],\\ x_f(t)-x_f(kh)=-(t-kh)(L_f+D_{fs})x_f(kh-h)+(t-kh)A_{fs}x_s(kh-h).\end{cases} \tag{16}$$

Expanding Eq. (16) yields

$$\begin{aligned}\begin{pmatrix}\begin{pmatrix}x_s(t)-x_s(kh)\\ x_f(t)-x_f(kh)\end{pmatrix}\\ y_s(t)-y_s(kh)\end{pmatrix} &= \begin{pmatrix}\begin{pmatrix}(t-kh)y_s(kh)\\ 0_{N-M}\end{pmatrix}\\ 0_M\end{pmatrix}+\begin{pmatrix}\begin{pmatrix}-(t-kh)^2 r y_s(kh-h)/2\\ 0_{N-M}\end{pmatrix}\\ (kh-t)r y_s(kh-h)\end{pmatrix}\\ &+\begin{pmatrix}\begin{pmatrix}-(t-kh)^2(L_s+D_{sf})/2 & (t-kh)^2 A_{sf}/2\\ (t-kh)A_{fs} & (kh-t)(L_f+D_{fs})\end{pmatrix}\begin{pmatrix}x_s(kh-h)\\ x_f(kh-h)\end{pmatrix}\\ (t-kh)\begin{pmatrix}-(L_s+D_{sf}) & A_{sf}\end{pmatrix}\begin{pmatrix}x_s(kh-h)\\ x_f(kh-h)\end{pmatrix}\end{pmatrix}.\end{aligned} \tag{17}$$

From Eq. (17) we obtain that when $t\in[kh,kh+\tau)$,

$$\begin{cases}\lim_{t\to\infty}(x_s(t)-x_s(kh))=0,\\ \lim_{t\to\infty}(y_s(t)-y_s(kh))=0,\\ \lim_{t\to\infty}(x_f(t)-x_f(kh))=0.\end{cases} \tag{18}$$

Similarly, from Eqs. (6) and (7) we obtain that when $t\in[kh+\tau,kh+h)$,

$$\begin{cases}\begin{pmatrix}x_s(t)\\ y_s(t)\end{pmatrix}-\begin{bmatrix}1 & t-kh\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_s(kh)\\ y_s(kh)\end{pmatrix}=\begin{pmatrix}\tau t-\tau kh-\tau^2/2\\ \tau\end{pmatrix}[-\gamma y_s(kh-h)-(L_s+D_{sf})x_s(kh-h)+A_{sf}x_f(kh-h)]\\ \qquad\qquad+\begin{pmatrix}(t-kh-\tau)^2/2\\ t-kh-\tau\end{pmatrix}[-\gamma y_s(kh)-(L_s+D_{sf})x_s(kh)+A_{sf}x_f(kh)],\\ x_f(t)-x_f(kh)=-\tau(L_f+D_{fs})x_f(kh-h)+\tau A_{fs}x_s(kh-h)-(t-kh-\tau)(L_f+D_{fs})x_f(kh)+(t-kh-\tau)A_{fs}x_s(kh).\end{cases} \tag{19}$$

Expanding Eq. (19) results in

$$\begin{aligned}\begin{pmatrix}\begin{pmatrix}x_s(t)-x_s(kh)\\ x_f(t)-x_f(kh)\end{pmatrix}\\ y_s(t)-y_s(kh)\end{pmatrix} &= \begin{pmatrix}\begin{pmatrix}(t-kh-(t-kh-\tau)^2 r/2)y_s(kh)\\ 0_{N-M}\end{pmatrix}\\ (kh+\tau-t)r y_s(kh)\end{pmatrix}+\begin{pmatrix}\begin{pmatrix}(\tau kh-\tau t+\tau^2/2)r y_s(kh-h)\\ 0_{N-M}\end{pmatrix}\\ -\tau r y_s(kh-h)\end{pmatrix}\\ &+\begin{pmatrix}\begin{pmatrix}-(t-kh-\tau)^2(L_s+D_{sf})/2 & (t-kh-\tau)^2 A_{sf}/2\\ (t-kh-\tau)A_{fs} & (\tau+kh-t)(L_f+D_{fs})\end{pmatrix}\begin{pmatrix}x_s(kh)\\ x_f(kh)\end{pmatrix}\\ (t-kh-\tau)\begin{pmatrix}-(L_s+D_{sf}) & A_{sf}\end{pmatrix}\begin{pmatrix}x_s(kh)\\ x_f(kh)\end{pmatrix}\end{pmatrix}\\ &+\begin{pmatrix}\begin{pmatrix}(\tau kh-\tau t+\tau^2/2)(L_s+D_{sf})2 & (\tau t-\tau kh-\tau^2/2)A_{sf}/2\\ \tau A_{fs} & -\tau(L_f+D_{fs})\end{pmatrix}\begin{pmatrix}x_s(kh-h)\\ x_f(kh-h)\end{pmatrix}\\ \tau\begin{pmatrix}-(L_s+D_{sf}) & A_{sf}\end{pmatrix}\begin{pmatrix}x_s(kh-h)\\ x_f(kh-h)\end{pmatrix}\end{pmatrix}.\end{aligned} \tag{20}$$

From Eq. (20) we obtain that when $t\in[kh+\tau,kh+h)$,

$$\begin{cases}\lim_{t\to\infty}(x_s(t)-x_s(kh))=0,\\ \lim_{t\to\infty}(y_s(t)-y_s(kh))=0,\\ \lim_{t\to\infty}(x_f(t)-x_f(kh))=0.\end{cases} \tag{21}$$

Combining Eqs. (18) and (21) leads to

$$\begin{cases}\lim_{t\to\infty}(x_s(t)-x_s(kh))=0,\\ \lim_{t\to\infty}(y_s(t)-y_s(kh))=0,\\ \lim_{t\to\infty}(x_f(t)-x_f(kh))=0,\end{cases}\quad t\in[kh,kh+h). \tag{22}$$

Combining Eqs. (14), (15), and (22) results in

$$\begin{cases}\lim_{t\to\infty}x_i(t)=c, & \forall i\in I_N,\\ \lim_{t\to\infty}y_i(t)=0, & \forall i\in I_M.\end{cases} \tag{23}$$

Based on the above analysis, the heterogeneous multi-agent systems (1) using the consensus protocol (3), achieves the stationary consensus if and only if all roots of Eq. (4) are within a unit circle. The proof is completed.

Remark 1 While implementing the consensus protocol (3), we need to initialize $(x_i(-h),y_i(-h))$, $i\in I_M$; $x_i(-h)$, $i\in M+1,\ldots,N$. In the above analysis, it is assumed that $(x_i(-h),y_i(-h))=(x_i(0),y_i(0)), i\in I_M$; $x_i(-h)=x_i(0)$, $i\in M+1,\ldots,N$, which does not affect the results obtained in Theorem 1.

Remark 2 Although the condition that all roots of Eq. (4) are within a unit circle is necessary and sufficient, for the small sampling delay, it is hard to derive from this condition the explicit allowable scope of the sampling period that guarantees that the heterogeneous multi-agent systems (1) using the consensus protocol (3) achieve the stationary consensus. Therefore, the results of Theorem 1 are mainly used for testing the

effectiveness of the sampling period guaranteeing the achievement of stationary consensus.

Remark 3 When $\tau = 0$, i.e., there is no sampling delay, the consensus protocol (3) with the small sampling delay degenerates into the consensus protocol without the sampling delay

$$u_i(t) = \begin{cases} -\gamma y_i(kh) + \sum_{j=1}^{N} a_{ij}[x_j(kh) - x_i(kh)], \\ t \in [kh, kh+h), \quad k = 0,1,2,\ldots, \quad i \in I_M, \\ \sum_{j=1}^{N} a_{ij}[x_j(kh) - x_i(kh)], \quad t \in [kh, kh+h), \\ k = 0,1,2,\ldots, \quad i \in \{M+1,\ldots,N\}, \end{cases} \tag{24}$$

and we have the following corollary.

Corollary 1 A heterogeneous multi-agent system (1) with a fixed, undirected, and connected network topology using the consensus protocol (24), achieves the stationary consensus if and only if

$$0 < h < \min_{i \in I_M, j \in I_{N-M}} \left\{ \frac{r+u_j}{ru_j + \lambda_i/2}, \frac{r+u_j - |r-u_j|}{ru_j} \right\}. \tag{25}$$

Proof When $\tau = 0$, equation (4) degenerates into

$$s^2 + (hr - 2 + hu_j + h^2\lambda_i/2)s + 1 - hr + (h^2 r - h)u_j + h^2\lambda_i/2 = 0, \quad i \in I_M, j \in I_{N-M}. \tag{26}$$

From Lemma 4, all roots of Eq. (26) are within a unit circle if and only if all roots of the equation

$$h^2(\lambda_i + ru_j)t^2 + 2[h(r+u_j) - h^2(ru_j + \lambda_i/2)]t + h^2 ru_j - 2h(r+u_j) + 4 = 0, \quad i \in I_M, \ j \in I_{N-M} \tag{27}$$

are in the open left half plane. Therefore, to prove Corollary 1, we only need to prove that all roots of Eq. (27) are in the open left half plane if and only if inequality (25) holds.

It is easy to prove that two roots of Eq. (27) are always in the open left half plane if and only if the following inequalities:

$$2[h(r+u_j) - h^2(ru_j + \lambda_i/2)] > 0, \quad i \in I_M, j \in I_{N-M} \tag{28}$$

and

$$h^2 ru_j - 2h(r+u_j) + 4 > 0, \quad j \in I_{N-M} \tag{29}$$

hold simultaneously. By further computation, we obtain that inequalities (28) and (29) simultaneously hold if and only if

$$0 < h < \min \left\{ \frac{r+u_j}{ru_j + \lambda_i/2}, \frac{r+u_j - |r-u_j|}{ru_j} \right\},$$

$i \in I_M, j \in I_{N-M}$, which is equivalent to Eq. (25). The proof is completed.

4. Simulations

In this section we provide numerical simulations to illustrate the correctness of the above results. Here we consider the heterogeneous multi-agent systems composed of two second-order agents 1, 2 and three first-order agents 3, 4, 5. The interconnection topology is described in Fig. 1. Without loss of generality, it is assumed that the weights of all edges are 1 and the initial states of five agents are chosen as

$$(x_1(0), y_1(0)) = (-1,-1), (x_2(0), y_2(0)) = (-2,1)$$
$$x_3(0) = 1, x_4(0) = 2, x_5(0) = 1.5.$$

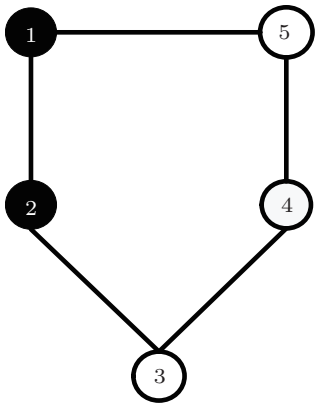


Fig. 1. A weighted, undirected, and connected graph composed of five agents.

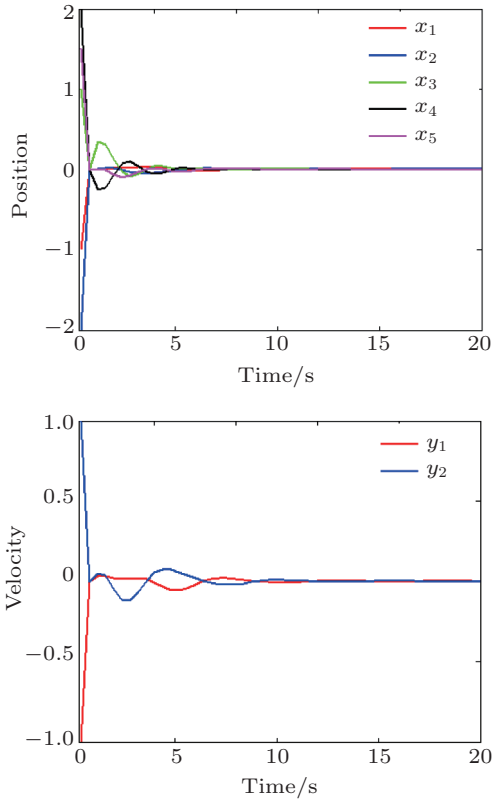


Fig. 2. (color online) Positions and velocities of agents in systems (1) using protocol (3) with $h = 0.2$.

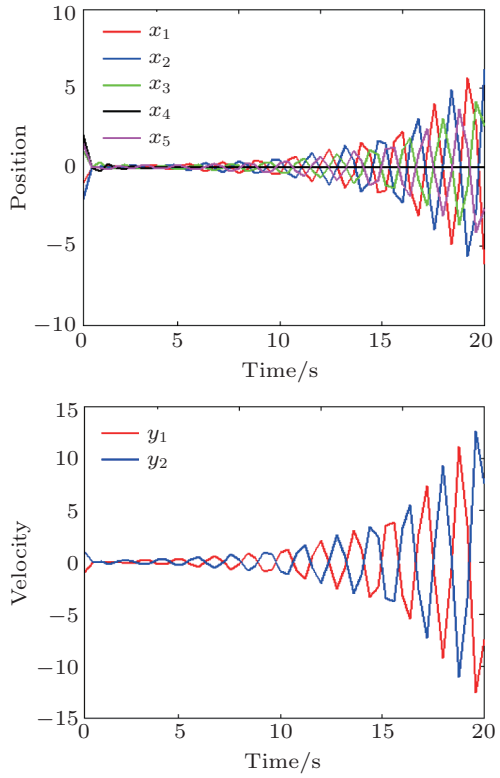


Fig. 3. (color online) Positions and velocities of agents in systems (1) using protocol (3) with $h = 0.8$.

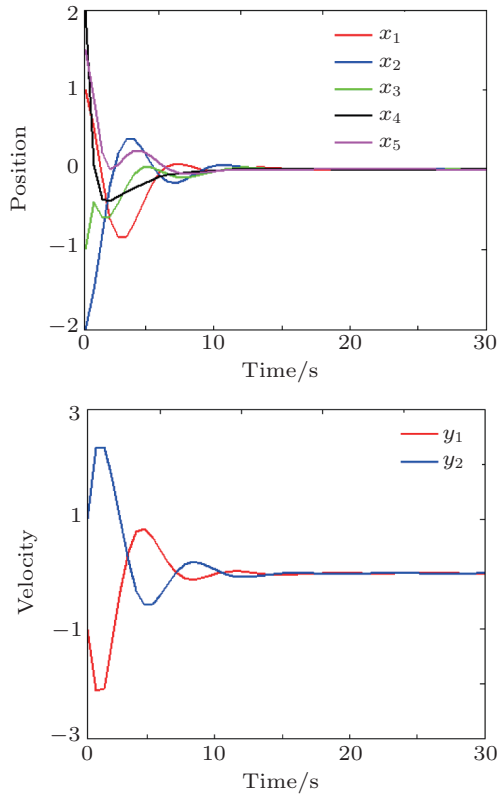


Fig. 4. (color online) Positions and velocities of agents in systems (1) using protocol (24) with $h = 0.2$.

First, we numerically illustrate the correctness of Theorem 1. Here it is assumed that $\tau = 0.15$ and $r = 1.7$. The numerical results are shown in Figs. 2 and 3, respectively. It can be seen from Fig. 2 that when $(\tau, h) = (0.15, 0.2)$, which satisfies the condition given in Theorem 1, the heterogeneous multi-agent system (1) using consensus protocol (3), achieves the stationary consensus. From Fig. 3 we find that when $(\tau, h) = (0.15, 0.8)$, which does not satisfy the condition given in Theorem 1, the heterogeneous multi-agent system (1) using consensus protocol (3), cannot achieve the stationary consensus. Thus, the results of Theorem 1 are numerically verified.

Second, we numerically illustrate the correctness of Corollary 1. Here it is also assumed that $r = 1.7$. The numerical results are shown in Figs. 4 and 5, respectively. It can be seen from Fig. 4 that when $h = 0.2$, which satisfies the condition (25), the heterogeneous multi-agent system (1) using the consensus protocol (24), achieves the stationary consensus. From Fig. 5 we find that when $h = 0.8$, which does not satisfy the condition (25), the heterogeneous multi-agent systems (1) using the consensus protocol (24), cannot achieve the stationary consensus. Thus, the results of Corollary 1 are numerically verified.

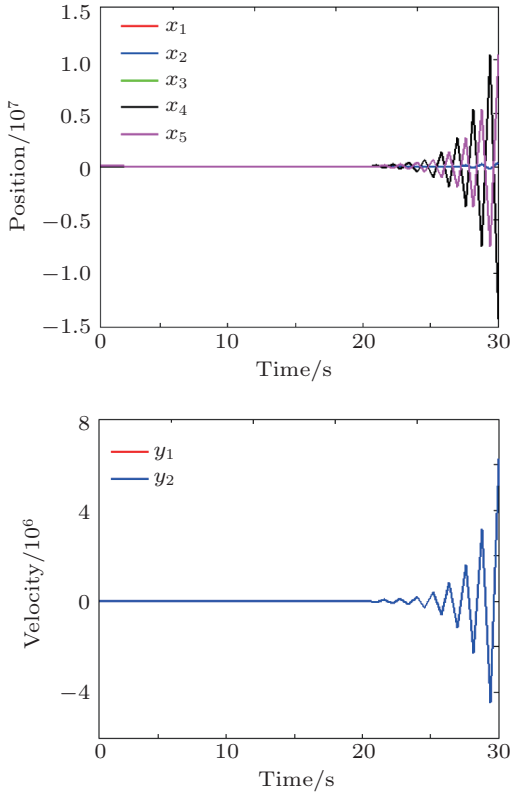


Fig. 5. (color online) Positions and velocities of agents in systems (1) using protocol (24) with $h = 0.8$.

5. Conclusion

In this paper, we have investigated the consensus problems of heterogeneous multi-agent systems based on sampled data with a small sampling delay. Using the algebra graph theory, the matrix method, the stability theory of linear systems,

and some other techniques, we obtained the necessary and sufficient conditions guaranteeing that the heterogeneous multi-agent systems asymptotically achieve the stationary consensus. Future work will include analyzing the effects of the sampling period and the sampling delay on the consensus performance and discussing the consensus problems of heterogeneous multi-agent systems without velocity measurements based on sampled data with a small sampling delay.

References

[1] Su H S, Wang X F and Lin Z L 2009 *IEEE Trans. Autom. Control* **54** 293

[2] Su H S, Chen M Z Q, Wang X F, Chen G R and Wang H W 2013 *IEEE Trans. Cybern.* **43** 394

[3] Vicsek T, Czirok A, Jacob E B, Cohen I and Shochet O 1995 *Phys. Rev. Lett.* **75** 1226

[4] Sun Y Z and Ruan J 2008 *Chin. Phys. Lett.* **25** 3493

[5] Wu Z H, Peng L, Xie L B and Wen J W 2012 *Chin. Phys. B* **21** 128902

[6] Sun Y Z and Ruan J 2008 *Chin. Phys. B* **17** 4137

[7] Bertsekas D P and Tsitsiklis J N 2007 *IEEE Trans. Autom. Control* **52** 968

[8] Cao M, Morse A S and Anderson B D O 2008 *SIAM J. Control Optim.* **47** 575

[9] Cao M, Morse A S and Anderson B D O 2008 *SIAM J. Control Optim.* **47** 601

[10] Xiao F and Wang L 2008 *IEEE Trans. Autom. Control* **53** 1804

[11] Tian Y P and Liu C L 2008 *IEEE Trans. Autom. Control* **53** 2122

[12] Tian Y P and Liu C L 2009 *Automatica* **45** 1347

[13] Zhang W G, Liu J Z, Zeng D L and Hu Y 2013 *Chin. Phys. B* **22** 050511

[14] Park M J, Kwon O M, Park J H, Lee S M and Cha E J 2012 *Chin. Phys. B* **21** 110508

[15] Gao L X, Zhang J J and Chen W H 2012 *Math. Prob. Eng.* **2012** 273140

[16] Gao L X, Yan H J and Jin D 2010 *Chin. Phys. B* **19** 050520

[17] Zhang W G, Zeng D L and Guo Z K 2010 *Chin. Phys. B* **19** 070518

[18] Li J Z 2011 *Chin. Phys. B* **20** 020512

[19] Hong Y G, Chen G and Bushnell L 2008 *Automatica* **44** 846

[20] Wu Z H, Fang H J and She Y Y 2012 *IEEE Trans. Syst., Man, Cybern. B: Cybern.* **42** 1501

[21] Su H S, Chen G R, Wang X F and Lin Z L 2011 *Automatica* **47** 368

[22] Xiao F and Wang L 2007 *IET Control Theory Appl.* **1** 830

[23] Jiang F C, Wang L and Xie G M 2010 *J Control Theory Appl.* **8** 52

[24] He W and Cao J 2011 *IET Control Theory Appl.* **5** 231

[25] Yang T, Jin Y, Wang W and Shi Y 2011 *Chin. Phys. B* **20** 020512

[26] Liu C L and Liu F 2011 *Automatica* **47** 2130

[27] Zheng Y, Zhu Y and Wang L 2011 *IET Control Theory Appl.* **5** 1881

[28] Lee D J and Spong M W 2006 *Proc. Am. Control Conference,* June 14–16, 2006, Minneapolis, Minnesota, p. 756

[29] Zheng Y S and Wang L 2012 *Int. J. Control* **85** 906

[30] Zheng Y S and Wang L 2012 *Syst. Control Lett.* **61** 871

[31] Tian Y P 2011 *Proc. Asian Control Conference,* May 15–18, 2011, Kaohsiung, China, p. 341

[32] Yan J, Guan X P and Luo X Y 2011 *Chin. Phys. B* **20** 048901

[33] Wu Z H, Peng L, Xie L B and Wen J W 2013 *Physica A* **392** 918

[34] Wu Z H, Peng L, Xie L B and Wen J W 2014 *Int. J. Control, Autom. Syst.* **12** 44

[35] Horn R A and Jornson C R 1985 *Matrix Analysis* (New York: Cambridge University)

[36] Kovacs I, Silver D S and Williams S G 1999 *Am. Math. Monthly* **106** 950

[37] Horn R A and Jornson C R 1991 *Topics in Matrix Analysis* (Cambridge: Cambridge University Press)

[38] Ren W and Cao Y C 2008 *Proc. IEEE Conf. Decision and Control*, December 9–11, 2008, Cancun, Mexico, p. 3965