Elastic fields around a nanosized elliptic hole in decagonal quasicrystals*

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Based on the variational principle, a continuum theory of surface elasticity and new boundary conditions for quasicrystals is proposed. The effect of the residual surface stress on a decagonal quasicrystal that is weakened by a nanoscale elliptical hole is considered. The explicit expressions for the hoop stress along the edge of the hole are obtained using the Stroh formalism. The results show that the residual surface stress and the shape of the hole have a significant effect on the elastic state around the hole.

Keywords: quasicrystal, surface elasticity, Stroh formulism, variational method

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1. Introduction

Size-dependent phenomena have been observed experimentally,[1] which has resulted in an increasing amount of academic interest in the mechanical behavior of bodies of small size. The surface elasticity theory that was proposed by Gurtin and Murdoch[2,3] has predicted some size-dependent elastic properties that are in good agreement with certain experimental results or atomic calculations and which have been widely used to study elastic behavior of solids at nano-scales.[4,5]

The analysis of deformation in quasicrystal solid with inhomogeneities has been discussed extensively.[6–13] The classical elastic theory of quasicrystals does not admit size-dependent phenomena since there is no intrinsic length scale involved in the constitutive laws. Recently, a number of nanoscale icosahedral quasicrystalline phase formations have been reported in Zr-based alloys.[14–16] The improvement of mechanical strength is attributed to the effects of size precipitated particles.[15] However, an analysis of the mechanical properties of a nanosized quasicrystal material has not been discussed theoretically with consideration of the effect of surface elasticity. Therefore, it is necessary to develop a size-dependent elastic theory of a quasicrystal, which accounts for the microstructure of the material.

As a general approach, Gurtin and Murdoch[2,3] presented a mathematical framework for the mechanical behavior of material surfaces, including surface elasticity. They modeled the surface as a two-dimensional membrane adhering to an underlying bulk material, which has been successfully extended to nano-dielectrics.[17] In this paper, the classical surface elasticity theory of Gurtin and Murdoch will be developed for a quasicrystal material. As an illustrative example, the effect of surface elasticity on the decagonal quasicrystal material weakened by a nanoscale elliptical hole is examined. Explicit expressions for the hoop stress along the edge of the hole are obtained. When the material becomes isotropic, the hoop stress agrees with the well-known result in the literature.[18]

2. Basic equations of the theory of quasicrystals elasticity

The strain-displacement relations and the equilibrium equations are given by[6]

$$\varepsilon_{ij} = \frac{1}{2}(u_i|_j + u_j|_i), \quad w_{ij} = w_i|_j, \tag{1}$$

$$\sigma_{ij}|_j = 0, \quad H_{ij}|_j = 0, \tag{2}$$

where σ_{ij}, ε_{ij}, and u_i are stress, strain, and displacement of phonon fields, respectively. H_{ij}, w_{ij}, and w_i are the stress, strain, and displacement of phason fields, respectively. A single vertical line denotes the derivative operation. The phonon fields displacement $\boldsymbol{u}$ is analogous to the phonon field in crystals. The elementary excitation associated with phonon mode is propagating. The elementary excitation associated with the phason mode is diffusive. The gradient of phason fields $\boldsymbol{w}$ describes the rearrangements of the unit cells.

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The constitutive equations can be written as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl},$$
$$H_{ij} = R_{klij}\varepsilon_{kl} + K_{ijkl}w_{kl}, \tag{3}$$

in which C_{ijkl} and K_{ijkl} represent elastic constants of phonon fields and phason fields, respectively. R_{ijkl} is the phonon–phason coupling elastic constants.

By extending the elastic energy density to include the phonon field f^u, the phason field f^w, and the phonon–phason coupling term f^{uw}, the elastic energy density f expanded up to the second order takes the following form

$$\begin{aligned} f &= f^u + f^w + f^{uw} \\ &= \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}K_{ijkl}w_{ij}w_{kl} + R_{ijkl}\varepsilon_{ij}w_{kl}. \end{aligned} \tag{4}$$

Under the infinitesimal deformation assumption, the constitutive equations (3) can be re-obtained from the following relations

$$\sigma_{ij} = \frac{\partial f}{\partial \varepsilon_{ij}}, \quad H_{ij} = \frac{\partial f}{\partial w_{ij}}.$$

In general, the boundary surface $\mathscr{S}$ of body $\mathscr{B}$ consists of two parts $\mathscr{S}_t$ and $\mathscr{S}_v$, that is, $\mathscr{S} = \mathscr{S}_t + \mathscr{S}_v$, at $\mathscr{S}_t$ the tractions are given and at $\mathscr{S}_v$ the displacements are prescribed.[7] The stress boundary conditions are expressed as

$$\begin{cases} \sigma_{ij}n_j = T_i, \\ H_{ij}n_j = h_i, \end{cases} \quad \text{on } \mathscr{S}_t, \tag{5}$$

where n_j is the outward unit normal to the boundary surface $\mathscr{S}$, and T_i and h_i represent the traction and generalized traction vectors, respectively. The displacement boundary conditions are given as

$$\begin{cases} u_i = \overline{u}_i, \\ w_i = \overline{w}_i, \end{cases} \quad \text{on } \mathscr{S}_v, \tag{6}$$

where $\overline{u}_i$ and $\overline{w}_i$ are known functions at the boundary.

3. The governing equations and boundary conditions involving surface effects

The surface elasticity theory originally proposed by Gurtin and Murdoch is widely used in the field of nanomechanics. In the following, the surface elasticity theory of quasicrystals will be developed according to the idea of Gurtin and Murdoch.

The essential of surface elasticity is that the surface energy density $\tilde{f}$ depends on the plane strain $\varepsilon^{\mathrm{s}}_{\alpha\beta}$ at the surface. In the quasicrystal nanostructures, it is natural to believe that the surface energy density may also depend on the phason field strain $w^{\mathrm{s}}_{\alpha\beta}$ at the surface. So we assume that the integrated free energy density of the surface layer $\tilde{f}$ can be expressed as

$$\begin{aligned} \tilde{f}(\varepsilon^{\mathrm{s}}_{\alpha\beta}, w^{\mathrm{s}}_{\alpha\beta}) &= \tilde{f}_0 + \tau_0\varepsilon^{\mathrm{s}}_{\alpha\beta} + \frac{1}{2}C^{\mathrm{s}}_{\alpha\beta\lambda\mu}\varepsilon^{\mathrm{s}}_{\alpha\beta}\varepsilon^{\mathrm{s}}_{\lambda\mu} + H_0 w^{\mathrm{s}}_{\alpha\beta} \\ &\quad + \frac{1}{2}K^{\mathrm{s}}_{\alpha\beta\lambda\mu}w^{\mathrm{s}}_{\alpha\beta}w^{\mathrm{s}}_{\lambda\mu} + R^{\mathrm{s}}_{\alpha\beta\lambda\mu}\varepsilon^{\mathrm{s}}_{\alpha\beta}w^{\mathrm{s}}_{\lambda\mu}, \end{aligned} \tag{7}$$

where $\tilde{f}_0$ is the residual surface free energy density, τ_0 is the residual surface phonon field stress, H_0 is the residual surface phason field stress, $C^{\mathrm{s}}_{\alpha\beta\lambda\mu}$ is the surface phonon tensor, $K^{\mathrm{s}}_{\alpha\beta\lambda\mu}$ is the surface phason tensor, and $R^{\mathrm{s}}_{\alpha\beta\lambda\mu}$ is the surface phonon–phason coupling tensor. By the following thermodynamic relations

$$\sigma^{\mathrm{s}}_{\alpha\beta} = \frac{\partial \tilde{f}}{\partial \varepsilon^{\mathrm{s}}_{\alpha\beta}}, \quad H^{\mathrm{s}}_{\alpha\beta} = \frac{\partial \tilde{f}}{\partial w^{\mathrm{s}}_{\alpha\beta}} \tag{8}$$

and Eq. (7), the surface phonon stress tensor $\sigma^{\mathrm{s}}_{\alpha\beta}$ and the surface phason stress tensor $H^{\mathrm{s}}_{\alpha\beta}$ can be obtained as

$$\begin{aligned} \sigma^{\mathrm{s}}_{\alpha\beta} &= \tau_0\delta_{\alpha\beta} + C^{\mathrm{s}}_{\alpha\beta\lambda\mu}\varepsilon^{\mathrm{s}}_{kl} + R^{\mathrm{s}}_{ijkl}w^{\mathrm{s}}_{kl}, \\ H^{\mathrm{s}}_{\alpha\beta} &= H_0\delta_{\alpha\beta} + C^{\mathrm{s}}_{\alpha\beta\lambda\mu}R^{\mathrm{s}}_{klij}\varepsilon^{\mathrm{s}}_{kl} + K^{\mathrm{s}}_{ijkl}w^{\mathrm{s}}_{kl}. \end{aligned} \tag{9}$$

In the following, new boundary conditions governing the surface elasticity are derived by the variational method. Assume that the surface adheres perfectly to the bulk material without slipping, then the energy functional U corresponding to the isothermal free energy principle of a quasicrystal body $\mathscr{B}$ with surface effects consists of two terms: the term due to the bulk free energy F and the surface free energy $\tilde{F}$

$$U = F + \tilde{F}. \tag{10}$$

By the isothermal free energy principle, we have

$$\delta U = \delta F + \delta \tilde{F} = 0, \tag{11}$$

where δ is the variational sign. The variation of the bulk free energy is

$$\begin{aligned} \delta F &= \int_{\mathscr{B}} \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V + \int_{\mathscr{B}} H_{ij}\delta w_{ij}\mathrm{d}V \\ &= \int_{\mathscr{B}} \sigma_{ij}\delta u_i|_j\mathrm{d}V + \int_{\mathscr{B}} H_{ij}\delta w_i|_j\mathrm{d}V \\ &= \int_{\mathscr{S}} \sigma_{ij}n_j\delta u_i\mathrm{d}A - \int_{\mathscr{B}} \sigma_{ij}|_j\delta u_i\mathrm{d}V \\ &\quad + \int_{\mathscr{S}} H_{ij}n_j\delta w_i\mathrm{d}A - \int_{\mathscr{B}} H_{ij}|_j\delta w_i\mathrm{d}V \\ &= \int_{\mathscr{S}_t} \sigma_{\alpha j}n_j\delta u_t\mathrm{d}A - \int_{\mathscr{S}_t} \sigma_{ij}n_in_j\delta u_n\mathrm{d}A \\ &\quad - \int_{\mathscr{B}} \sigma_{ij}|_j\delta u_i\mathrm{d}V + \int_{\mathscr{S}_t} H_{\alpha j}n_j\delta w_t\mathrm{d}A \\ &\quad - \int_{\mathscr{S}_t} H_{ij}n_in_j\delta w_n\mathrm{d}A - \int_{\mathscr{B}} H_{ij}|_j\delta w_i\mathrm{d}V. \end{aligned} \tag{12}$$

The material is assumed to occupy a volume V. The phonon displacement $\boldsymbol{u}$ and the phason displacement $\boldsymbol{w}$ have been decomposed on the surface into a tangential component and a normal component as

$$\boldsymbol{u} = \boldsymbol{u}_t + \boldsymbol{u}_n, \quad \boldsymbol{w} = \boldsymbol{w}_t + \boldsymbol{w}_n,$$

where

$$\boldsymbol{u}_t = \boldsymbol{P}\cdot\boldsymbol{u}, \quad \boldsymbol{u}_n = \boldsymbol{n}\otimes(\boldsymbol{u}^{\mathrm{T}}\cdot\boldsymbol{n}),$$

$$\boldsymbol{w}_t = \boldsymbol{P}\cdot\boldsymbol{w},\ \ \boldsymbol{w}_n = \boldsymbol{n}\otimes(\boldsymbol{w}^{\mathrm{T}}\cdot\boldsymbol{n})$$

with the projector tensor $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n}$, $\boldsymbol{I}$ is the unit tensor, $\boldsymbol{n}$ is the unit normal vector.[2] The variation of the surface free energy is

$$\begin{aligned}\delta\tilde{F} &= \int_{\mathscr{S}} \sigma^{\mathrm{s}}_{\alpha\beta}\delta\varepsilon^{\mathrm{s}}_{\alpha\beta}\,\mathrm{d}A + \int_{\mathscr{S}} H^{\mathrm{s}}_{\alpha\beta}\delta w^{\mathrm{s}}_{\alpha\beta}\,\mathrm{d}A\\ &= -\int_{\mathscr{S}_t}\sigma^{\mathrm{s}}_{\alpha\beta}|_{\beta}\delta u_t\,\mathrm{d}A - \int_{\mathscr{S}_t}\sigma^{\mathrm{s}}_{\alpha\beta}\kappa_{\alpha\beta}\delta u_n\,\mathrm{d}A\\ &\quad - \int_{\mathscr{S}_t} H^{\mathrm{s}}_{\alpha\beta}|_{\beta}\delta w_t\,\mathrm{d}A - \int_{\mathscr{S}_t} H^{\mathrm{s}}_{\alpha\beta}\kappa_{\alpha\beta}\delta w_n\,\mathrm{d}A. \quad (13)\end{aligned}$$

By Eqs. (11)–(13), the new surface traction boundary conditions are obtained as

$$\begin{aligned}&\sigma_{\alpha j}n_j = \sigma^{\mathrm{s}}_{\alpha\beta,\beta},\ \ \sigma_{ij}n_in_j = \sigma^{\mathrm{s}}_{\alpha\beta}\kappa_{\alpha\beta},\\ &H_{\alpha j}n_j = H^{\mathrm{s}}_{\alpha\beta,\beta},\ \ H_{ij}n_in_j = H^{\mathrm{s}}_{\alpha\beta}\kappa_{\alpha\beta}.\end{aligned} \quad (14)$$

4. A decagonal quasicrystal material weakened by a nanoscale elliptical hole

An infinity decagonal quasicrystal material weakened by a nanoscale elliptical hole is considered as an example. The explicit expressions for the hoop stress along the edge of the hole are obtained using the Stroh formalism.

The Stroh formalism for the elastic theory of decagonal quasicrystal will be briefly reviewed as follows. Similar to Ref. [19], we introduce a generalized displacement vector $\boldsymbol{u}$ as

$$\boldsymbol{u} = [u_1, u_2, w_1, w_2]^{\mathrm{T}} = \boldsymbol{a}f(z), \quad (15)$$

where

$$z = x + py,$$

the superscript T represents the transpose, $f(z)$ is an analytic function, p is a complex number, and $\boldsymbol{a}$ is a constant four-element column vector. The equilibrium equations of decagonal quasicrystal elastic theory (if the body force is neglected)[6]

$$\begin{aligned}&C_{ijkl}u_{k,jl} + R_{ijkl}w_{k,jl} = 0,\\ &R_{klij}u_{k,jl} + K_{ijkl}w_{k,jl} = 0\end{aligned} \quad (16)$$

can be satisfied by Eq. (15) for arbitrary $f(z)$ if

$$[\boldsymbol{Q} + p(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p^2\boldsymbol{T}]\boldsymbol{a} = \boldsymbol{0}, \quad (17)$$

where the 4×4 matrices $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{T}$ are given by

$$\begin{aligned}&\boldsymbol{Q} = \begin{bmatrix} C_{i1k1} & R_{i1k1} \\ R^{\mathrm{T}}_{i1k1} & K_{i1k1}\end{bmatrix},\ \ \boldsymbol{R} = \begin{bmatrix} C_{i1k2} & R_{i1k2} \\ R^{\mathrm{T}}_{i1k2} & K_{i1k2}\end{bmatrix},\\ &\boldsymbol{T} = \begin{bmatrix} C_{i2k2} & R_{i2k2} \\ R^{\mathrm{T}}_{i2k2} & K_{i2k2}\end{bmatrix},\ \ i,k = 1,2.\end{aligned} \quad (18)$$

For a nontrivial solution of $\boldsymbol{a}$ we must have

$$|\boldsymbol{Q} + p(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p^2\boldsymbol{T}| = 0, \quad (19)$$

which gives eight roots p_α and $\overline{p}_\alpha$ ($\alpha = 1,2,3,4$), where p_α cannot be real because of the positive definiteness of the strain energy. The associated eigenvector $\boldsymbol{a}$ is determined from Eq. (17). Then, the general solution of Eqs. (3) and (16) can be expressed as

$$\boldsymbol{u} = 2\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(z)\}, \quad (20)$$

in which

$$\begin{aligned}&\boldsymbol{A} = [\boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \boldsymbol{a}_3\ \ \boldsymbol{a}_4],\\ &\boldsymbol{f}(z) = [f_1(z_1)\ \ f_2(z_2)\ \ f_3(z_3)\ \ f_4(z_4)]^{\mathrm{T}},\end{aligned} \quad (21)$$

and $z_\alpha = x + p_\alpha y$, without loss of generality we have let $\mathrm{Im}\,p_\alpha > 0$.

Making use of Eqs. (3), (15), and (18), the phonon stresses and phason stresses can be rewritten as

$$\begin{aligned}&\sigma_{i1} = -pb_if'(z),\ \ \sigma_{i2} = b_if'(z),\\ &H_{i1} = -pb_{i+2}f'(z),\ \ H_{i2} = b_{i+2}f'(z),\end{aligned}\ \ i = 1,2, \quad (22)$$

where, in matrix notation

$$\boldsymbol{b} = (\boldsymbol{R}^{\mathrm{T}} + p\boldsymbol{T})\boldsymbol{a}. \quad (23)$$

Introducing the stress function

$$\begin{aligned}&\sigma_{i1} = -\phi_{i,2},\ \ \sigma_{i2} = \phi_{i,1},\\ &H_{i1} = -\phi_{i+2,2},\ \ H_{i2} = \phi_{i+2,1},\end{aligned}\ \ i = 1,2, \quad (24)$$

then the general solution for stress function ϕ is obtained as

$$\phi = 2\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}(z)\}, \quad (25)$$

in which

$$\boldsymbol{B} = [\boldsymbol{b}_1\ \ \boldsymbol{b}_2\ \ \boldsymbol{b}_3\ \ \boldsymbol{b}_4].$$

In this study, we consider only the case in which the surface stress is independent of the elastic strain. This simplification has been adopted by several authors.[5,18]

Introducing the conformal mapping functions[19]

$$z_\alpha(\zeta_\alpha) = c_\alpha\zeta_\alpha + d_\alpha\zeta_\alpha^{-1}, \quad (26)$$

where

$$c_\alpha = \frac{1}{2}(a - \mathrm{i}p_\alpha b),\ \ d_\alpha = \frac{1}{2}(a + \mathrm{i}p_\alpha b),$$

a and b indicate the major and minor semi axes of the ellipse, maps the exterior of the elliptic region in the z_α-plane into the exterior of a unit circle γ that can be described by $\sigma = \mathrm{e}^{\mathrm{i}\psi}$ ($0 \le \psi \le 2\pi$) in the ζ_α-plane.

By Eq. (14), the boundary condition on the surface of the elliptical hole can be obtained as

$$\sigma_{mm} = \frac{\tau_0}{\rho},\ \ H_{mm} = \frac{H_0}{\rho}, \quad (27)$$

where the subscript m represent the normal direction of the elliptical boundary, and ρ is the curvature radius along the elliptical hole, which can be expressed, in terms of $\zeta = e^{i\psi}$, as

$$\rho = R_0\left(1-\frac{m}{\zeta^2}\right)^{3/2}(1-m\zeta^2)^{3/2}(1-m^2)^{-1},$$

where $R_0 = (a+b)/2$, $m = (a-b)/(a+b)$.[18] Then, the boundary condition (27) along the surface of the elliptical hole can be expressed as

$$\boldsymbol{B}f(\zeta)+\overline{\boldsymbol{B}}\,\overline{f(\zeta)} = \boldsymbol{\Phi}_\gamma, \tag{28}$$

where $\boldsymbol{\Phi}_\gamma$ is given by

$$\boldsymbol{\Phi}_\gamma = \frac{1}{2}\sqrt{\frac{\zeta^2-m}{1-m\zeta^2}}\begin{bmatrix}i\tau_0\\ \tau_0\\ iH_0\\ H_0\end{bmatrix}+\frac{1}{2}\sqrt{\frac{1-m\zeta^2}{\zeta^2-m}}\begin{bmatrix}-i\tau_0\\ \tau_0\\ -iH_0\\ H_0\end{bmatrix}. \tag{29}$$

By using the method of analytic continuation, the expression of $\boldsymbol{f}(\zeta)$ can be obtained as

$$\boldsymbol{f}(\zeta) = \left\langle\sqrt{\frac{1-m\zeta_\alpha^2}{\zeta_\alpha^2-m}}\right\rangle \boldsymbol{B}^{\mathrm{T}}\boldsymbol{L}^{-1}\begin{bmatrix}\tau_0\\ i\tau_0\\ H_0\\ iH_0\end{bmatrix}, \tag{30}$$

where $\langle\ \rangle$ is a 4×4 diagonal matrix and $\boldsymbol{L} = -2i\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}$. The full-field expressions of the displacement and stress function vectors can then be given by substituting Eq. (30) into Eqs. (20) and (25). As a result, the displacement vector along the edge of the elliptical hole can be obtained as

$$\boldsymbol{u}_\gamma = -\mathrm{Re}\left\{\sqrt{\frac{1-me^{2i\psi}}{e^{2i\psi}-m}}(\boldsymbol{L}^{-1}-i\boldsymbol{S}\boldsymbol{L}^{-1})\begin{bmatrix}\tau_0\\ i\tau_0\\ H_0\\ iH_0\end{bmatrix}\right\}, \tag{31}$$

in which $\boldsymbol{S} = i(2\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}}-\boldsymbol{I})$.

The stress of interest to us is that at the edge of hole. The hoop stress vector $\boldsymbol{t}_n$ on the elliptical boundary can be expressed in terms of the displacement vector u_γ and the stress vector Φ_γ as[19]

$$\boldsymbol{t}_n = \boldsymbol{N}_3(\theta_\gamma)\frac{\partial \boldsymbol{u}_\gamma}{\eta\partial\psi}+\boldsymbol{N}_1^{\mathrm{T}}(\theta_\gamma)\frac{\partial\boldsymbol{\Phi}_\gamma}{\eta\partial\psi}, \tag{32}$$

where

$$\begin{aligned}
&\boldsymbol{N}_1(\theta) = -\boldsymbol{T}^{-1}(\theta)\boldsymbol{R}^{\mathrm{T}}(\theta),\\
&\boldsymbol{N}_3(\theta) = \boldsymbol{R}(\theta)\boldsymbol{T}^{-1}(\theta)\boldsymbol{R}^{\mathrm{T}}(\theta)-\boldsymbol{Q}(\theta),\\
&\boldsymbol{Q}(\theta) = \boldsymbol{Q}\cos^2\theta+(\boldsymbol{R}+\boldsymbol{R}^{\mathrm{T}})\sin\theta\cos\theta+\boldsymbol{T}\sin^2\theta,\\
&\boldsymbol{R}(\theta) = \boldsymbol{R}\cos^2\theta+(\boldsymbol{T}-\boldsymbol{Q})\sin\theta\cos\theta-\boldsymbol{R}^{\mathrm{T}}\sin^2\theta,\\
&\boldsymbol{T}(\theta) = \boldsymbol{T}\cos^2\theta-(\boldsymbol{R}+\boldsymbol{R}^{\mathrm{T}})\sin\theta\cos\theta+\boldsymbol{Q}\sin^2\theta,
\end{aligned}$$

and $\eta = \sqrt{a^2\sin^2\psi+b^2\cos^2\psi}$, θ_γ is the angle the tangent to the ellipse makes with the x axis. By Eqs. (29), (31), and (32), the phonon hoop stress σ_{nn} can be given as

$$\sigma_{nn} = -\frac{\tau_0(1-m^2)}{R_0[1+m^2-2m\cos(2\psi)]^{3/2}}, \tag{33}$$

which is just the result given by Ref. [18] for isotropic material. From Eq. (33) we know that the phason field and the residual surface phason field stress H_0 have no effect on the hoop stress of the phonon field for decagonal quasicrystals. By conducting a similar analysis following decagonal quasicrystals, we find that the conclusion for the other kinds of quasicrystal (such as octagonal quasicrystals and icosahedral quasicrystal) is different. The phason hoop stress H_{nn} can be obtained as

$$H_{nn} = -\frac{(1-m^2)(H_0+c\,\tau_0\cos(2\theta_\gamma))}{R_0[1+m^2-2m\cos(2\psi)]^{3/2}} \tag{34}$$

with

$$c = \frac{(K_1-K_2)R}{C_{66}K_1-2R^2}.$$

In general, the phonon–phason coupling elastic constant R is significantly smaller than the phonon and the phason elastic constants. Thus, we can conclude from Eq. (34) that the coupling elastic constant R has little effect on the hoop stress of the phason field.

In order to have a clear and direct picture on how the hole affects the stress, the decagonal Al–Ni–Co is used as an example to numerically demonstrate the hoop stress of a phason field. The material properties of the decagonal quasicrystal are[20]

$$C_{11} = 23.43\ \mathrm{GPa},\ \ C_{12} = 5.741\ \mathrm{GPa},\ \ K_1 = 12.2\ \mathrm{GPa},$$
$$K_2 = 2.4\ \mathrm{GPa},\ \ R = -0.11\ \mathrm{GPa}.$$

Keeping R_0 as constant, the elliptical hole will have different shapes for different values of $e = b/a$. The effects of ratios e on the hoop stress of the phason field is depicted in Fig. 1.

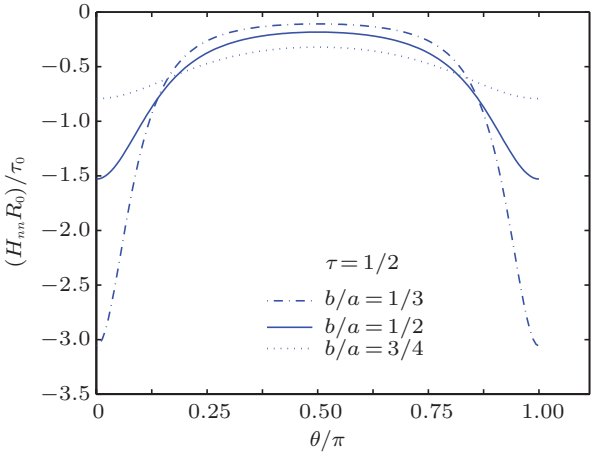


Fig. 1. (color online) Distribution of the hoop stress of the phason field along the elliptical hole.

We have assumed that $\tau = H_0/\tau_0 = 1/2$. From Fig. 1, we see that the hoop stress of the phason field along the elliptical hole is very sensitive to its shape, especially near the end of the major axis of elliptical hole. In addition, the residual surface stress will induce the increase of stress concentration around the hole, as shown in Fig. 2.

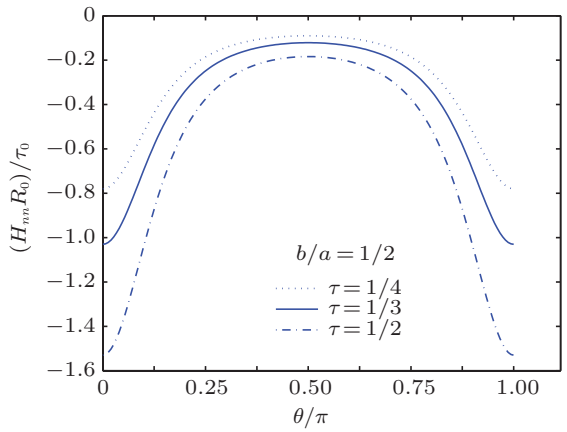


Fig. 2. (color online) The effects of the residual surface stress on the hoop stress of the phason field.

5. Conclusion and discussion

The surface elasticity theory originally proposed by Gurtin and Murdoch is extended to a quasicrystal. The effect of surface elasticity on the decagonal material weakened by a nano-sized elliptical hole is discussed. The expressions of the hoop stresses are derived by the Stroh formalism. The results show that the shape of the hole and the residual surface stress significantly affect the stress state around the elliptical hole. The solutions for the quasicrystals with arbitrary shaped holes at nanoscale can be obtained directly by the present method.

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