Hopf bifurcation control for a coupled nonlinear relative rotation system with time-delay feedbacks*

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This paper investigates the Hopf bifurcations resulting from time delay in a coupled relative-rotation system with time-delay feedbacks. Firstly, considering external excitation, the dynamical equation of relative rotation nonlinear dynamical system with primary resonance and 1:1 internal resonance under time-delay feedbacks is deduced. Secondly, the averaging equation is obtained by the multiple scales method. The periodic solution in a closed form is presented by a perturbation approach. At last, numerical simulations confirm that time-delay theoretical analyses have influence on the Hopf bifurcation point and the stability of periodic solution.

Keywords: relative rotation, coupled nonlinear system, bifurcation, time delay

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1. Introduction

The study of rich nonlinear behaviors such as bifurcation and chaos has become an important component in the field of nonlinear system. The research on nonlinear behaviors can be found in dynamical system,[1–3] neural network system,[4–6] financial system,[7,8] and biological system,[9–11] etc. The time delay, as a kind of basis nature of the physical phenomenon, has a great influence on those rich nonlinear behaviors of nonlinear system.[12–14] For an SIS (Strategic Information System) epidemic model[15] with time delays, the influence of time delay on the stability of equilibria is displayed. The action of time delay in a non-autonomous system[16–18] is investigated. It is obtained that time delay may be used as a simple but efficient 'switch' to control motions of a system. There are many remarkable studies focusing on nonlinear behaviors control by using time delay. The band of the saturation control in an auto-parametric dynamical system is investigated by using the linear delayed feedback.[19–22] Considering a forced Holmes–Duffing system, the effects of linear delayed velocity feedback on controlling the fractal eroded safe basin are investigated.[23] The nonlinear delay feedback is applied to move the critical bifurcation point and stabilize the periodic solution in a relative rotation system.[24] Although many remarkable studies on uncoupled nonlinear system have been done as mentioned above, researches on the nonlinear delayed feedback control in nonlinear coupled systems have rarely been proposed. The rolling mill system is a representative mechanical coupled dynamic system and a multi-degree of freedom system in actual industrial production. Many remarkable studies on uncoupled nonlinear system of rolling mill with two masses have been done.[25,26] Wang *et al.*[27] investigated the time-delay neural network with two masses and chaotic control of the complex networks by the method of Open Plus Nonlinear Closed Loop (OPNCL) and the Time-Delay Feedback Control (TDFC) respectively. However, the researches on those dynamical equations with two masses are not applied to the actual well in work conditions of multi-degree of freedom. Therefore, in order to analyze thermodynamical characteristics of this kind of multi-degree of freedom system completely and effectually, we intend to apply a nonlinear time delay to the coupled system with three masses to create a Hopf bifurcation and control the stability of periodic solutions. The research results provide a theoretic basis and a reference for suppressing torsional vibrations of the rolling mill caused by nonlinear time delay. This paper is organized as follows: The dynamical equation of the nonlinear coupled relative rotation system of a three-mass rolling mill is established in Section 2. With the application of the perturbation technique, averaging equations of coupled system with primary and 1:1 internal resonances are presented in Section 3. Section 4 outlines the effect of time-delay parameters on the creation and stability of Hopf bifurcation. Numerical simulations are performed to verify the analytical results in Section 5. Finally, conclusions are given in Section 6.

2. Coupled nonlinear relative-rotation system

Considering a relative rotation system of a three-mass rolling mill, the total kinetic and potential energy of the system

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can be described as

$$E = \sum_{i=1}^{3} \frac{1}{2} J_i \dot{\theta}_i^2 = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} J_3 \dot{\theta}_3^2, \tag{1}$$

$$U = \frac{1}{2} K_{12} (\theta_1 - \theta_2)^2 + \frac{1}{2} K_{23} (\theta_2 - \theta_3)^2. \tag{2}$$

The Lagrange function L and dissipation function F are

$$\begin{aligned} L &= E - U \\ &= \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} J_3 \dot{\theta}_3^2 - \frac{1}{2} K_{12} (\theta_1 - \theta_2)^2 \\ &\quad - \frac{1}{2} K_{23} (\theta_2 - \theta_3)^2, \end{aligned} \tag{3}$$

$$F = \frac{1}{2} C (\dot{\theta}_1 - \dot{\theta}_2)^2. \tag{4}$$

The generalized torque is

$$Q_j = \sum_{i=1}^{3} F_i^i \frac{\partial \theta_i}{\partial q_j}, \quad (j = 1,2,3), \tag{5}$$

where J_1, J_2, J_3 are moments of inertia. θ_i and $\dot{\theta}_i$ $(i = 1,2,3)$ are rotational angle and speed of rotational angle, respectively. k is torsional rigidity, and c is damping coefficient. q denotes generalized coordinate. $F_i^i = T_i + F_i^c$, where T_i is generalized external torque, and F_i^c is generalized damping force. Then substituting Eqs. (3)–(5) into dissipation Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = Q_i \tag{6}$$

yields,

$$\begin{aligned} &T_1 - K_{12} (\theta_1 - \theta_2) - F_1^c = J_1 \ddot{\theta}_1, \\ &K_{12} (\theta_1 - \theta_2) + F_1^c - K_{23} (\theta_2 - \theta_3) - F_2^c = J_2 \ddot{\theta}_2, \\ &K_{23} (\theta_2 - \theta_3) + F_2^c - T_2 = J_3 \ddot{\theta}_3, \end{aligned} \tag{7}$$

where $\dot{\theta}_i$ $(i = 1,2,3)$ is the angular acceleration. The equation (7) can be changed into another form

$$\begin{aligned} &\left(\ddot{\theta}_1 - \ddot{\theta}_2\right) + \frac{J_1 + J_2}{J_1 J_2} K_{12} (\theta_1 - \theta_2) + \frac{J_1 + J_2}{J_1 J_2} F_1^c \\ &= \frac{1}{J_1} T_1 + \frac{1}{J_2} K_{23} (\theta_2 - \theta_3) + \frac{1}{J_2} F_2^c, \\ &\left(\ddot{\theta}_2 - \ddot{\theta}_3\right) + \frac{J_2 + J_3}{J_2 J_3} K_{23} (\theta_2 - \theta_3) + \frac{J_2 + J_3}{J_2 J_3} F_2^c \\ &= \frac{1}{J_3} T_2 + \frac{1}{J_2} K_{12} (\theta_1 - \theta_2) + \frac{1}{J_2} F_1^c. \end{aligned} \tag{8}$$

A kind of nonlinear sliding friction force is considered, supposing

$$\begin{aligned} F_1^c &= m_1 \left(\dot{\theta}_1 - \dot{\theta}_2\right) + n_1 \left(\dot{\theta}_1 - \dot{\theta}_2\right)^3, \\ F_2^c &= m_2 \left(\dot{\theta}_2 - \dot{\theta}_3\right) + n_2 \left(\dot{\theta}_2 - \dot{\theta}_3\right)^3, \end{aligned} \tag{9}$$

where m_1, m_2, n_1, n_2 represent the damping coefficients.

Thus, by substituting Eq. (9) into Eq. (8), one may yield

$$\begin{aligned} &\left(\ddot{\theta}_1 - \ddot{\theta}_2\right) + \frac{J_1 + J_2}{J_1 J_2} K_{12} (\theta_1 - \theta_2) \\ &\quad + \frac{J_1 + J_2}{J_1 J_2} m_1 \left(\dot{\theta}_1 - \dot{\theta}_2\right) + \frac{J_1 + J_2}{J_1 J_2} n_1 \left(\dot{\theta}_1 - \dot{\theta}_2\right)^3 \\ &= \frac{1}{J_1} T_1 + \frac{1}{J_2} K_{23} (\theta_2 - \theta_3) + \frac{1}{J_2} m_2 \left(\dot{\theta}_2 - \dot{\theta}_3\right) \\ &\quad + \frac{1}{J_2} n_2 \left(\dot{\theta}_2 - \dot{\theta}_3\right)^3, \\ &\left(\ddot{\theta}_2 - \ddot{\theta}_3\right) + \frac{J_2 + J_3}{J_2 J_3} K_{23} (\theta_2 - \theta_3) \\ &\quad + \frac{J_2 + J_3}{J_2 J_3} m_2 \left(\dot{\theta}_2 - \dot{\theta}_3\right) + \frac{J_2 + J_3}{J_2 J_3} n_2 \left(\dot{\theta}_2 - \dot{\theta}_3\right)^3 \\ &= \frac{1}{J_3} T_2 + \frac{1}{J_2} K_{12} (\theta_1 - \theta_2) + \frac{1}{J_2} m_1 \left(\dot{\theta}_1 - \dot{\theta}_2\right) \\ &\quad + \frac{1}{J_2} n_1 \left(\dot{\theta}_1 - \dot{\theta}_2\right)^3. \end{aligned} \tag{10}$$

Setting

$$\begin{aligned} &x_1 = \theta_1 - \theta_2, \ x_2 = \theta_2 - \theta_3, \ \omega_1^2 = \frac{J_1 + J_2}{J_1 J_2} K_{12}, \\ &\omega_2^2 = \frac{J_2 + J_3}{J_2 J_3} K_{23}, \ c_1 = \frac{J_1 + J_2}{J_1 J_2} m_1, \ c_2 = \frac{J_2 + J_3}{J_2 J_3} m_2, \\ &d_1 = \frac{J_1 + J_2}{J_1 J_2} n_1, \ F_1 = \frac{1}{J_1} T_1, \ F_2 = \frac{1}{J_3} T_2, \\ &g_1 = \frac{1}{J_2} K_{12}, \ g_2 = \frac{1}{J_2} K_{23}, \ h_1 = \frac{1}{J_2} m_1, \\ &h_2 = \frac{1}{J_2} m_2, \ b_1 = \frac{1}{J_2} n_1, \ b_2 = \frac{1}{J_2} n_2, \end{aligned}$$

the equation (10) can be written as

$$\ddot{x}_1 + \omega_1^2 x_1 - g_2 x_2 + c_1 \dot{x}_1 + d_1 \dot{x}_1^3 - h_2 \dot{x}_2 - b_2 \dot{x}_2^3 = F_1, \tag{11}$$

$$\ddot{x}_2 + \omega_2^2 x_2 - g_1 x_1 + c_2 \dot{x}_2 + d_2 \dot{x}_2^3 - h_1 \dot{x}_1 - b_1 \dot{x}_1^3 = F_2. \tag{12}$$

3. Perturbation analysis of time-delay feedback system

We consider controlling the Hopf bifurcation of system (11) and (12) by a time-delay feedback control law. The goals of control are: (i) to create Hopf bifurcations at the critical point; (ii) to stabilize the Hopf bifurcations. We may explicitly write the controller as follows:

$$F_1 = -\alpha \dot{x}_1 (t - \tau_1) - \beta \dot{x}_1^3 (t - \tau_2). \tag{13}$$

By adding the controller given by Eq. (13) to systems (11) and (12), one may obtain the following system with time-delay feedbacks

$$\begin{aligned} &\ddot{x}_1 + \omega_1^2 x_1 - g_2 x_2 + c_1 \dot{x}_1 + d_1 \dot{x}_1^3 - h_2 \dot{x}_2 - b_2 \dot{x}_2^3 \\ &= -\alpha \dot{x}_1 (t - \tau_1) - \beta \dot{x}_1^3 (t - \tau_2), \end{aligned} \tag{14}$$

$$\begin{aligned} &\ddot{x}_2 + \omega_2^2 x_2 - g_1 x_1 + c_2 \dot{x}_2 + d_2 \dot{x}_2^3 - h_1 \dot{x}_1 - b_1 \dot{x}_1^3 \\ &= f_2 \cos (\Omega_2 t), \end{aligned} \tag{15}$$

where $F_2 = f_2 \cos (\Omega_2 t)$ and it represents the external excitation item.

For Eqs. (14) and (15), the periodic solutions and their stabilities can be analyzed with the aid of averaging equations. Here, the averaging equations are obtained by using the method of multiple scales. Setting

$$T_n = \varepsilon^n t, \quad n = 0, 1, \ldots, \tag{16}$$

ε is a small parameter, and then the derivatives about t become expansions in terms of the partial derivatives according to

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}T_0}{\mathrm{d}t}\frac{\partial}{\partial T_0} + \frac{\mathrm{d}T_1}{\mathrm{d}t}\frac{\partial}{\partial T_1} + \cdots = D_0 + \varepsilon D_1 + \cdots, \tag{17}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2\left(D_1^2 + 2D_0 D_2\right) + \cdots, \tag{18}$$

where D_i denotes $\partial/\partial i$, thus one may seek an approximate solution of Eqs. (14) and (15) in the form

$$x_1 = x_{10}(T_0, T_1) + \varepsilon x_{11}(T_0, T_1) + \cdots, \tag{19}$$

$$x_2 = x_{20}(T_0, T_1) + \varepsilon x_{21}(T_0, T_1) + \cdots. \tag{20}$$

Now substituting Eqs. (17)–(20) into Eqs. (14) and (15), yields

$$\begin{aligned}
&(D_0^2 + 2\varepsilon D_0 D_1)(x_{10} + \varepsilon x_{11}) + \omega_1^2(x_{10} + \varepsilon x_{11}) \\
&= \varepsilon(g_2 x_2 - c_1\dot{x}_1 - d_1\dot{x}_1^3 + h_2\dot{x}_2 + b_2\dot{x}_2^3 \\
&\quad - \alpha\dot{x}_1(t-\tau_1) - \beta\dot{x}_1^3(t-\tau_2)), \\
&(D_0^2 + 2\varepsilon D_0 D_1)(x_{20} + \varepsilon x_{21}) + \omega_1^2(x_{20} + \varepsilon x_{21}) \\
&= \varepsilon(f_2\cos(\Omega_2 t) + g_1 x_1 - c_2\dot{x}_2 - d_2\dot{x}_2^3 + h_1\dot{x}_1 + b_1\dot{x}_1^3).
\end{aligned} \tag{21}$$

Balancing the same powers of ε results in the following ordered perturbation equations:

$$\begin{aligned}
D_0^2 x_{10} + \omega_1^2 x_{10} &= 0, \\
D_0^2 x_{20} + \omega_2^2 x_{20} &= 0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&D_0^2 x_{11} + \omega_1^2 x_{11} \\
&= -2D_0 D_1 x_{10} + g_2 x_{20} - c_1 D_0 x_{10} - d_1 (D_0 x_{10})^3 \\
&\quad + h_2 D_2 x_{20} + b_2 (D_0 x_{20})^3 - \alpha D_0 x_{10}(t-\tau_1) \\
&\quad - \beta [D_0 x_{10}(t-\tau_2)]^3, \\
&D_0^2 x_{21} + \omega_2^2 x_{21} \\
&= -2D_0 D_1 x_{20} + f_2\cos(\Omega_2 t) + g_1 x_{10} - c_2 D_0 x_{20} \\
&\quad - d_2 (D_0 x_{20})^3 + h_1 D_0 x_{10} + b_1 (D_0 x_{10})^3.
\end{aligned} \tag{23}$$

The general solution of Eq. (22) in the complex form is given by

$$\begin{aligned}
x_{10} &= A(T_1)\mathrm{e}^{\mathrm{i}\omega_1 T_0} + \bar{A}(T_1)\mathrm{e}^{-\mathrm{i}\omega_1 T_0}, \\
x_{20} &= B(T_1)\mathrm{e}^{\mathrm{i}\omega_2 T_0} + \bar{B}(T_1)\mathrm{e}^{-\mathrm{i}\omega_2 T_0}.
\end{aligned} \tag{24}$$

Then equation (23) becomes

$$\begin{aligned}
&D_0^2 x_{11} + \omega_1^2 x_{11} \\
&= (-2\mathrm{i}\omega_1 D_1 A - \mathrm{i}c_1 A\omega_1 - 3\mathrm{i}d_1\omega_1^3 A^2\bar{A} \\
&\quad - \mathrm{i}\alpha A\omega_1\mathrm{e}^{-\mathrm{i}\omega_1\tau_1} - \mathrm{i}3\beta\omega_1^3 A^2\bar{A}\mathrm{e}^{-\mathrm{i}\omega_1\tau_2})\mathrm{e}^{\mathrm{i}\omega_1 T_0} \\
&\quad + (g_2 B + \mathrm{i}h_2 B\omega_2 + \mathrm{i}3b_2\omega_2^3 B^2\bar{B})\mathrm{e}^{\mathrm{i}\omega_2 T_0} \\
&\quad + (\mathrm{i}d_1\omega_1^3 A^3 + \mathrm{i}\beta\omega_1^3 A^3\mathrm{e}^{-3\mathrm{i}\omega_1\tau_2})\mathrm{e}^{3\mathrm{i}\omega_1 T_0} \\
&\quad - \mathrm{i}b_2\omega_2^3 B^3\mathrm{e}^{3\mathrm{i}\omega_2 T_0} + \mathrm{cc}, \\
&D_0^2 x_{21} + \omega_2^2 x_{21} \\
&= (-2\mathrm{i}\omega_2 D_1 B - \mathrm{i}c_2 B\omega_2 - 3\mathrm{i}d_2\omega_2^3 B^2\bar{B})\mathrm{e}^{\mathrm{i}\omega_2 T_0} \\
&\quad + f_2\cos(\Omega_2 t) + (g_1 A + \mathrm{i}h_1 A\omega_1 + \mathrm{i}3b_1\omega_1^3 A^2\bar{A})\mathrm{e}^{\mathrm{i}\omega_1 T_0} \\
&\quad + \mathrm{i}d_2\omega_2^3 B^3\mathrm{e}^{\mathrm{i}3\omega_2 T_0} - \mathrm{i}b_1\omega_1^3 A^3\mathrm{e}^{\mathrm{i}3\omega_1 T_0} + \mathrm{cc},
\end{aligned} \tag{25}$$

where cc stands for the complex conjugate of the preceding terms.

We consider the Hopf bifurcation of system equations (14) and (15), when the primary resonance and 1:1 internal resonance occur, and thus let

$$\begin{aligned}
\omega_1 &= \omega_2 + \varepsilon\sigma_1, \\
\Omega_2 &= \omega_2 + \varepsilon\sigma_2,
\end{aligned}$$

where σ_1 and σ_2 are the detuning parameters and $\sigma_i = O(1), (i = 1, 2)$.

Equation (25) can be rewritten as

$$\begin{aligned}
&D_0^2 x_{11} + \omega_1^2 x_{11} \\
&= (-2\mathrm{i}\omega_1 D_1 A + g_2 B\mathrm{e}^{-\mathrm{i}\sigma_1 T_1} - \mathrm{i}c_1 A\omega_1 - 3\mathrm{i}d_1\omega_1^3 A^2\bar{A} \\
&\quad + \mathrm{i}h_2 B\omega_2\mathrm{e}^{-\mathrm{i}\sigma_1 T_1} + 3\mathrm{i}b_2\omega_2^3 B^2\bar{B}\mathrm{e}^{-\mathrm{i}\sigma_1 T_1} - \mathrm{i}\alpha A\omega_1\mathrm{e}^{-\mathrm{i}\omega_1\tau_1} \\
&\quad - 3\mathrm{i}\beta\omega_1^3 A^2\bar{A}\mathrm{e}^{-\mathrm{i}\omega_1\tau_2})\mathrm{e}^{\mathrm{i}\omega_1 T_0} + (\mathrm{i}d_1\omega_1^3 A^3 \\
&\quad - \mathrm{i}b_2\omega_2^3 B^3\mathrm{e}^{-3\mathrm{i}\sigma_1 T_1} - \mathrm{i}\beta\omega_1^3 A^3\mathrm{e}^{-3\mathrm{i}\omega_1\tau_2})\mathrm{e}^{3\mathrm{i}\omega_1 T_0} + \mathrm{cc}, \\
&D_0^2 x_{21} + \omega_2^2 x_{21} \\
&= \left(-2\mathrm{i}\omega_2 D_1 B + \frac{f_2}{2}\mathrm{e}^{\mathrm{i}\sigma_2 T_1} + g_1 A\mathrm{e}^{\mathrm{i}\sigma_1 T_1} - \mathrm{i}c_2 B\omega_2\right. \\
&\quad \left. - 3\mathrm{i}d_2\omega_2^3 B^2\bar{B} + \mathrm{i}h_1 A\omega_1\mathrm{e}^{\mathrm{i}\sigma_1 T_1} + 3\mathrm{i}b_1\omega_1^3 A^2\bar{A}\mathrm{e}^{\mathrm{i}\sigma_1 T_1}\right)\mathrm{e}^{\mathrm{i}\omega_2 T_0} \\
&\quad + (\mathrm{i}d_2\omega_2^3 B^3 - \mathrm{i}b_1\omega_1^3 A^3\mathrm{e}^{\mathrm{i}3\sigma_1 T_0})\mathrm{e}^{3\mathrm{i}\omega_2 T_0} + \mathrm{cc}.
\end{aligned} \tag{26}$$

To eliminate the possible secular terms which may appear in Eq. (26), it is required that

$$\begin{aligned}
&2\omega_1 D_1 A + \mathrm{i}g_2 B\mathrm{e}^{-\mathrm{i}\sigma_1 T_1} + c_1 A\omega_1 + 3d_1\omega_1^3 A^2\bar{A} \\
&- h_2 B\omega_2\mathrm{e}^{-\mathrm{i}\sigma_1 T_1} - 3b_2\omega_2^3 B^2\bar{B}\mathrm{e}^{-\mathrm{i}\sigma_1 T_1} + \alpha A\omega_1\mathrm{e}^{-\mathrm{i}\omega_1\tau_1} \\
&+ 3\beta\omega_1^3 A^2\bar{A}\mathrm{e}^{-\mathrm{i}\omega_1\tau_2} = 0, \\
&2\omega_2 D_1 B + \frac{\mathrm{i}f_2}{2}\mathrm{e}^{\mathrm{i}\sigma_2 T_1} + \mathrm{i}g_1 A\mathrm{e}^{\mathrm{i}\sigma_1 T_1} + c_2 B\omega_2 + 3d_2\omega_2^3 B^2\bar{B} \\
&- h_1 A\omega_1\mathrm{e}^{\mathrm{i}\sigma_1 T_1} - 3b_1\omega_1^3 A^2\bar{A}\mathrm{e}^{\mathrm{i}\sigma_1 T_1} = 0.
\end{aligned} \tag{27}$$

At this point, it is convenient to express the complex functions A and B in the polar form as

$$A = \frac{1}{2}r_1\mathrm{e}^{\mathrm{i}\theta_1}, \tag{28}$$

$$B = \frac{1}{2}r_2\mathrm{e}^{\mathrm{i}\theta_2}. \tag{29}$$

And it is easy to show that

$$D_1 A = \frac{1}{2} e^{i\theta_1} \left(\dot{r}_1 + i r_1 \dot{\theta}_1\right), \tag{30}$$

$$D_1 B = \frac{1}{2} e^{i\theta_2} \left(\dot{r}_2 + i r_2 \dot{\theta}_2\right). \tag{31}$$

Substituting Eqs. (28)–(31) into Eq. (27), yields

$$\begin{aligned}
&\omega_1 \left(\dot{r}_1 + i r_1 \dot{\theta}_1\right) + \frac{i g_2 r_2}{2} e^{i(\theta_2 - \theta_1 - \sigma_1 T_1)} \\
&\quad + \frac{c_1 \omega_1 r_1}{2} + \frac{3 d_1 \omega_1^3 r_1^3}{8} - \frac{h_2 \omega_2 r_2}{2} e^{i(\theta_2 - \theta_1 - \sigma_1 T_1)} \\
&\quad - \frac{3 d_2 \omega_2^3 r_2^3}{8} e^{i(\theta_2 - \theta_1 - \sigma_1 T_1)} + \frac{\alpha}{2} r_1 \omega_1 e^{-i\omega_1 \tau_1} \\
&\quad + \frac{3\beta \omega_1^3 r_1^3}{8} e^{-i\omega_1 \tau_2} = 0, \\
&\omega_2 e^{i\theta_2} \left(\dot{r}_2 + i r_2 \dot{\theta}_2\right) - \frac{i f_2}{2} e^{i\sigma_2 T_1} + \frac{i g_1 r_1}{2} e^{i(\theta_1 + \sigma_1 T_1)} \\
&\quad + \frac{c_2 \omega_2 r_2}{2} e^{i\theta_2} + \frac{3 d_2 \omega_2^3 r_2^3}{8} e^{i\theta_2} - \frac{h_1 \omega_1 r_1}{2} e^{i(\theta_1 + \sigma_1 T_1)} \\
&\quad - \frac{3 b_1 \omega_1^3 r_1^3}{8} e^{i(\theta_1 + \sigma_1 T_1)} = 0.
\end{aligned} \tag{32}$$

Separating real and imaginary parts of Eq. (32), we can obtain the averaging equations of system (14) and (15)

$$\begin{aligned}
\dot{r}_1 &= \frac{g_2 r_2}{2\omega_1} \sin\phi_1 - \frac{c_1 r_1}{2} - \frac{3 d_1 \omega_1^2 r_1^3}{8} \\
&\quad + \frac{4 h_2 \omega_2 r_2 + 3 b_2 \omega_2^3 r_2^3}{8\omega_1} \cos\phi_1 - \frac{\alpha}{2} r_1 \cos(\omega_1 \tau_1) \\
&\quad - \frac{3\beta \omega_1^2 r_1^3}{8} \cos(\omega_1 \tau_2),
\end{aligned} \tag{33}$$

$$\begin{aligned}
r_1 \left(\dot{\phi}_1 + \dot{\phi}_2\right) &= r_1 (\sigma_2 - \sigma_1) + \frac{g_2 r_2}{2\omega_1} \cos\phi_1 \\
&\quad - \frac{4 h_2 \omega_2 r_2 + 3 b_2 \omega_2^3 r_2^3}{8\omega_1} \sin\phi_1 - \frac{\alpha}{2} r_1 \sin(\omega_1 \tau_1) \\
&\quad - \frac{3\beta \omega_1^2 r_1^3}{8} \sin(\omega_1 \tau_2),
\end{aligned} \tag{34}$$

$$\begin{aligned}
\dot{r}_2 &= \frac{f_2}{2\omega_2} \sin\phi_2 - \frac{g_1 r_1}{2\omega_2} \sin\phi_1 - \frac{c_2 r_2}{2} \\
&\quad - \frac{3 d_2 \omega_2^2 r_2^3}{8} + \frac{4 h_1 \omega_1 r_1 + 3 b_1 \omega_1^3 r_1^3}{8\omega_2} \cos\phi_1,
\end{aligned} \tag{35}$$

$$\begin{aligned}
r_2 \dot{\phi}_2 &= r_2 \sigma_2 - \frac{f_2}{2\omega_2} \cos\phi_2 + \frac{g_1 r_1}{2\omega_2} \cos\phi_1 \\
&\quad + \frac{4 h_1 \omega_1 r_1 + 3 b_1 \omega_1^3 r_1^3}{8\omega_2} \sin\phi_1,
\end{aligned} \tag{36}$$

where $\phi_1 = \theta_2 - \theta_1 - \sigma_1 T_1$, $\phi_2 = \sigma_2 T_1 - \theta_2$.

Setting $p_1 = r_1 \cos(\phi_1 + \phi_2)$, $q_1 = r_1 \sin(\phi_1 + \phi_2)$, $p_2 = r_2 \cos\phi_2$, $q_2 = r_2 \sin\phi_2$. Equations (33)–(36) can be rewritten in the Cartesian coordinates as

$$\begin{aligned}
\dot{p}_1 &= -q_2 (\sigma_2 - \sigma_1) - \frac{g_2 q_2}{2\omega_1} - \frac{c_1 p_1}{2} - \frac{3 d_1 \omega_1^2 p_1}{8} \left(p_1^2 + q_1^2\right) \\
&\quad + \frac{h_2 \omega_2 p_2}{2\omega_1} - \frac{3 b_2 \omega_2^3 p_2}{8\omega_1} \left(p_2^2 + q_2^2\right) + \frac{\alpha}{2} [q_1 \sin(\omega_1 \tau_1) \\
&\quad - p_1 \cos(\omega_1 \tau_1)] - \frac{3\beta \omega_1^2}{8} [-p_1 \cos(\omega_1 \tau_2) \\
&\quad + q_1 \sin(\omega_1 \tau_2)] \left(p_1^2 + q_1^2\right), \\
\dot{p}_2 &= -q_2 \sigma_2 - \frac{g_1 q_1}{2\omega_2} - \frac{c_2 p_2}{2} - \frac{3 d_2 \omega_2^2 p_2}{8} \left(p_2^2 + q_2^2\right) \\
&\quad + \frac{h_1 \omega_1 p_1}{2\omega_2} + \frac{3 b_1 \omega_1^3 p_1}{8\omega_2} \left(p_1^2 + q_1^2\right), \\
\dot{q}_1 &= p_1 (\sigma_2 - \sigma_1) + \frac{g_2 p_2}{2\omega_1} - \frac{c_1 q_1}{2} - \frac{3 d_1 \omega_1^2 q_1}{8} \left(p_1^2 + q_1^2\right) \\
&\quad + \frac{h_2 \omega_2 q_2}{2\omega_1} + \frac{3 b_2 \omega_2^3 q_2}{8\omega_1} \left(p_2^2 + q_2^2\right) - \frac{\alpha}{2} [p_1 \sin(\omega_1 \tau_1) \\
&\quad + q_1 \cos(\omega_1 \tau_1)] - \frac{3\beta \omega_1^2}{8} [p_1 \sin(\omega_1 \tau_2) + q_1 \cos(\omega_1 \tau_2)] \\
&\quad \times \left(p_1^2 + q_1^2\right), \\
\dot{q}_2 &= p_2 \sigma_2 + \frac{f_2}{2\omega_2} + \frac{g_1 p_1}{2\omega_2} - \frac{c_2 q_2}{2} - \frac{3 d_2 \omega_2^2 q_2}{8} \left(p_2^2 + q_2^2\right) \\
&\quad + \frac{h_1 \omega_1 q_1}{2\omega_2} + \frac{3 b_1 \omega_1^3 q_1}{8\omega_2} \left(p_1^2 + q_1^2\right).
\end{aligned} \tag{37}$$

Equation (37) is a first-order coupled differential equation which has to be solved simultaneously. The bifurcation and stability analysis for the original system (14) and (15) can now be carried out by using the averaging equations given by Eq. (37), and will be discussed in more details in the next section.

4. Hopf bifurcation control

A represents the Jacobian matrix of Eq. (37) on an equilibrium, and denotes the characteristic polynomial of the Jacobian matrix as

$$D(\lambda) = k_0 \lambda^4 + k_1 \lambda^3 + k_2 \lambda^2 + k_3 \lambda + k_4. \tag{38}$$

Setting

$$\Delta_1 = k_1, \quad \Delta_2 = \begin{vmatrix} k_1 & k_0 \\ k_3 & k_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} k_1 & k_0 & 0 \\ k_3 & k_2 & k_1 \\ 0 & k_4 & k_3 \end{vmatrix},$$

and the Hopf bifurcation conditions can be described as follows:

$$k_i > 0, \tag{39}$$

$$\Delta_1 > 0, \tag{40}$$

$$\Delta_2 > 0, \tag{41}$$

$$\Delta_3 = 0, \tag{42}$$

$$\left.\frac{d\Delta_3}{d\tau_1}\right|_{\tau_1 = \tau_{10}} \neq 0. \tag{43}$$

Here $\tau_1 = \tau_{10}$ is assumed as a desired parameter point of Hopf bifurcation to be created. If system (37) holds for Eqs. (39)–(43), there will be a Hopf bifurcation at $\tau_1 = \tau_{10}$. Equations (39)–(43) are known as the critical criterion of the occurrence of Hopf bifurcations, and the stability of the periodic solutions depends on the higher-order nonlinear terms in system (37). So a proper method is designed to match the bifurcation critical conditions.

Assuming that Hopf bifurcation occurs at the critical point, where a family of limit cycles bifurcates from the equilibriums. We can consider stabilizing the bifurcating limit cycles by using time-delay τ_1 in nonlinear state feedback.

Under the critical conditions Eps. (39)–(43), a Hopf bifurcation of system (37) is created. Further, introducing a linear transformation

$$[p_1,q_1,p_2,q_2]^{\mathrm{T}} = M[y_1,y_2,y_3,y_4]^{\mathrm{T}}, \tag{44}$$

where M is the characteristic matrix of Jacobian matrix A at the critical point. Then converting the system (38) into the form

$$\dot{y} = Jy + F(y), \tag{45}$$

where

$$J = \begin{pmatrix} 0 & -\omega_0 & 0 & 0 \\ \omega_0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

$F(y) = [F_1(y),F_2(y),F_3(y),F_4(y)]^{\mathrm{T}}$ denotes the nonlinear terms of the system. We write the stability condition of Hopf limit circle as

$$\beta_2 = 2\mathrm{Re}\left\{\frac{g_{20}g_{11} - 2|g_{11}|^2 - |g_{02}|^2/3}{2\omega_0}\mathrm{i} + \frac{g_{21}}{2}\right\}, \tag{46}$$

where

$$\begin{aligned}
g_{20} &= \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} + 2\frac{\partial^2 F_2}{\partial y_1 \partial y_2}\right. \\
&\quad \left. + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2\frac{\partial^2 F_1}{\partial y_1 \partial y_2}\right)\right], \\
g_{11} &= \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2}\right)\right], \\
g_{02} &= \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} - 2\frac{\partial^2 F_2}{\partial y_1 \partial y_2}\right. \\
&\quad \left. + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2\frac{\partial^2 F_1}{\partial y_1 \partial y_2}\right)\right], \\
g_{21} &= G_{21} + 2G^1_{110}w^1_{11} + G^1_{101}w^1_{20} + 2G^2_{110}w^2_{11} + G^2_{101}w^2_{20}, \\
G_{21} &= \frac{1}{8}\left[\frac{\partial^3 F_1}{\partial y_1^3} + \frac{\partial^3 F_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 F_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F_2}{\partial y_2^3}\right. \\
&\quad \left. + \mathrm{i}\left(\frac{\partial^3 F_2}{\partial y_1^3} + \frac{\partial^3 F_2}{\partial y_1 \partial y_2^2} - \frac{\partial^3 F_1}{\partial y_1^2 \partial y_2} - \frac{\partial^3 F_1}{\partial y_2^3}\right)\right], \\
G^1_{110} &= \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial y_1 \partial y_3} + \frac{\partial^2 F_2}{\partial y_2 \partial y_3} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_3} - \frac{\partial^2 F_1}{\partial y_2 \partial y_3}\right)\right], \\
G^2_{110} &= \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial y_1 \partial y_4} + \frac{\partial^2 F_2}{\partial y_2 \partial y_4} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_4} - \frac{\partial^2 F_1}{\partial y_2 \partial y_4}\right)\right], \\
G^1_{101} &= \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial y_1 \partial y_3} - \frac{\partial^2 F_2}{\partial y_2 \partial y_3} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_3} + \frac{\partial^2 F_1}{\partial y_2 \partial y_3}\right)\right], \\
G^2_{101} &= \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial y_1 \partial y_4} - \frac{\partial^2 F_2}{\partial y_2 \partial y_4} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial y_1 \partial y_4} + \frac{\partial^2 F_1}{\partial y_2 \partial y_4}\right)\right], \\
w^1_{11} &= -\frac{1}{4\lambda_3}\left(\frac{\partial^2 F_3}{\partial y_1^2} + \frac{\partial^2 F_3}{\partial y_2^2}\right), \\
w^2_{11} &= -\frac{1}{4\lambda_4}\left(\frac{\partial^2 F_4}{\partial y_1^2} + \frac{\partial^2 F_4}{\partial y_2^2}\right), \\
w^1_{20} &= \frac{1}{4(2i\omega_0 - \lambda_3)}\left(\frac{\partial^2 F_3}{\partial y_1^2} - \frac{\partial^2 F_3}{\partial y_2^2} - 2\mathrm{i}\frac{\partial^2 F_3}{\partial y_1 \partial y_2}\right), \\
w^2_{20} &= \frac{1}{4(2i\omega_0 - \lambda_3)}\left(\frac{\partial^2 F_4}{\partial y_1^2} - \frac{\partial^2 F_4}{\partial y_2^2} - 2\mathrm{i}\frac{\partial^2 F_4}{\partial y_1 \partial y_2}\right).
\end{aligned}$$

The stability of the bifurcation solutions in coupled system (14) and (15) can be estimated according to Eq. (46). When $\beta_2(\tau_2) < 0$, the created Hopf circle is stable. On the contrary, if $\beta_2(\tau_2) > 0$, it is unstable. Obviously, β_2 is the function with respect to τ_2, and we can control the stability of limit cycles by time-delay τ_2.

5. Numerical results

In order to improve the computation efficiency of the perturbation technique, nonlinear coupled equations (14) and (15) are solved numerically by using the Runge–Kutta method. The objective is to choose a proper time-delay τ_1 to match the critical Hopf bifurcation conditions and the nonlinear term τ_2 to obtain the stability of the created bifurcation solution.

System parameters can be taken as $\omega_1 = 1.1547$, $b_1 = 0.4$, $c_1 = 0.2667$, $d_1 = 0.2667$, $g_1 = 0.5$, $h_1 = 0.4$, $\omega_2 = 1.095$, $b_2 = 0.4$, $c_2 = 0.82$, $d_2 = 0.24$, $g_2 = 0.5$, $h_2 = 0.9$, $f_2 = 0.01$, $\Omega_2 = 1.295$, $\alpha = 0.3$. By using Eqs. (39)–(43), one may show that a Hopf bifurcation occurs in the equilibrium when $\tau_{10} = 1.145 + 5.48k$ $(k = 0,1,2,\ldots)$. After determining $\tau_1 = \tau_{10}$, we gain the expression of the stability condition with respect to the nonlinear term τ_2, given by

$$\begin{aligned}
\beta_2 &= 0.0294 - 0.1803\beta\cos(\omega_1\tau_2) \\
&\quad - 0.02147\beta\sin(\omega_1\tau_2).
\end{aligned} \tag{47}$$

Choosing $\beta = 0.5$, when $\tau_2 \in (-0.855 + 5.291k, 1.175 + 5.291k)$ $(k = 0,1,2,\ldots)$, we can get $\beta_2 < 0$, and the system produces supercritical Hopf bifurcation. System trajectory is irrelevant to initial values.

When $\tau_2 \in (1.175 + 5.291k, 4.536 + 5.291k)$ with $(k = 0,1,2,\ldots)$, we can get $\beta_2 > 0$, and the system produces subcritical Hopf bifurcation. System trajectory is relevant to initial values. The numerical simulation (Figs. 1–3) can verify the results. τ_2 should be positive.

When $\tau_2 = 1.1 \in (0, 1.175)$ $(k = 0)$, $\beta_2 < 0$. The limit circle of supercritical Hopf bifurcation is stable.

When $\tau_2 = 1.2 \in (1.175, 4.536)$, $\beta_2 > 0$, and system produces subcritical Hopf bifurcation. System trajectory depends on initial values: when initial values are small, equilibrium point is locally asymptotical stable, as shown in Fig. 2; when

initial values are large, the time response jumps to larger amplitude and unstable limit cycle, as shown in Fig. 3. The theoretical analyses that we can control the stability of limit cycles by time-delay τ_2 in Section 4, are confirmed by the above numerical simulations.

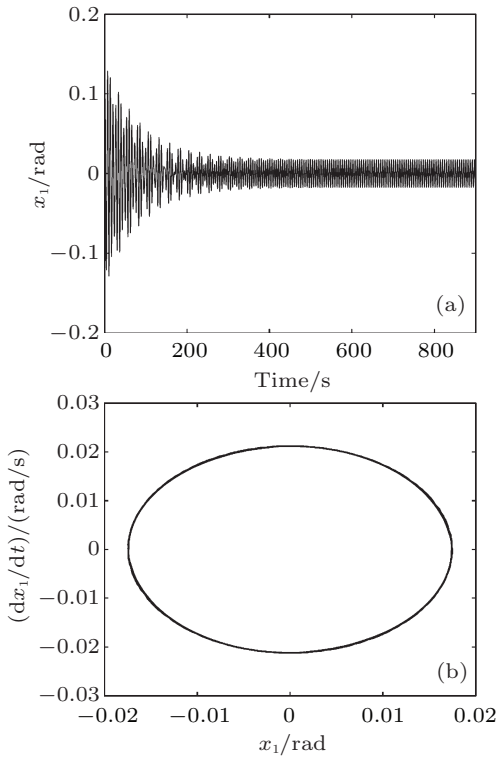


Fig. 1. Stable limit cycle (a) time history, (b) phase diagram.

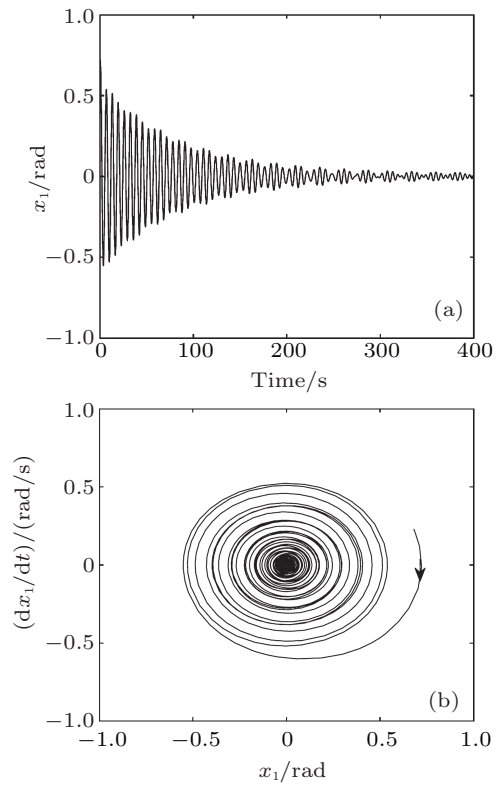


Fig. 2. Stable focus (a) time history, (b) phase diagram.

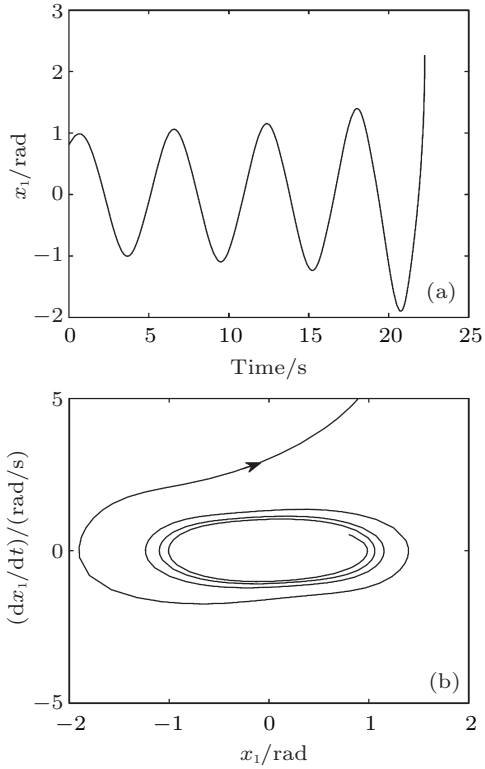


Fig. 3. Unstable limit cycle (a) time history, (b) phase diagram.

6. Conclusions

In this paper, the method of time-delay feedbacks is proposed for controlling the Hopf bifurcation in a coupled nonlinear relative rotation system. The analytical results illustrate the effect of time delay on the creation of Hopf bifurcation as well as on the stability of periodic solutions. We can gain the following conclusions.

(i) Time delay in linear term can affect the creation of Hopf bifurcations.

(ii) Time delay in nonlinear term can be used to stabilize Hopf limit circles. Our results can provide a theoretic basis and a reference for suppressing torsional vibration of rolling mill by nonlinear time delay.

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