Higher-order differential variational principle and differential equations of motion for mechanical systems in event space*

Zhang Xiang-Wu(张相武)[a)†], Li Yuan-Yuan(李院院)[a)], Zhao Xiao-Xia(赵小侠)[a)], and Luo Wen-Feng(罗文峰)[b)]

[a)] *School of Physics and Mechatronic Engineering, Xi'an University of Arts and Science, Xi'an* 710065, *China*
[b)] *School of Electronic Engineering, Xi'an University of Posts and Telecommunications, Xi'an* 710121, *China*

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In this paper we study the higher-order differential variational principle and differential equations of motion for mechanical systems in event space. Based on the higher-order d'Alembert principle of the system, the higher-order velocity energy and the higher-order acceleration energy of the system in event space are defined, the higher-order d'Alembert–Lagrange principle of the system in event space is established, and the parametric forms of Euler–Lagrange, Nielsen and Appell for this principle are given. Finally, the higher-order differential equations of motion for holonomic systems in event space are obtained.

Keywords: event space, the higher-order d'Alembert–Lagrange principle, the higher-order time rate of change of force, the higher-order differential equations of motion

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1. Introduction

It is of importance not only in geometry but also in mechanics to study the dynamics of mechanical systems in event space. In 1960, Synge[1] established the parametric equations of holonomic conservative systems in event space. In 1983, Rumyantsev[2] gave the Lagrange equations with multipliers for the conservative non-holonomic dynamical systems in event space. In 1988–1990, Mei[3–7] studied the variational principle, the parametric equations, and the method of their integration for non-holonomic conservative systems and non-holonomic non-conservative systems in event space. In 1990–1993, Luo[8–11] further studied the variational principle and the parametric equations for non-holonomic non-conservative systems in event space. In 2005, Zhang *et al.*[12] gave the discrete variational principle and the first integrals for holonomic conservative systems in event space. In 2007, Zhang[13] discussed various forms of the differential variational principles of mechanical systems in event space. In 2008, Zhang[14] studied the Pfaff–Birkhoff–d'Alembert principle, Birkhoff's parametric equations, and the first integrals for Birkhoffian systems in event space. In recent years, some important research results of symmetries and conserved quantities of mechanical systems in event space have been obtained. Mei[15–17] studied the Noether symmetry, the Lie symmetry, the Mei symmetry, and the conserved quantities of holonomic or nonholonomic systems in event space. Li *et al.*[18–20] studied the Noether theory of mechanical systems with unilateral constraints in event space. Xu *et al.*[21] gave the symmetries and the Hojman conserved quantity for a holonomic system in event space. Zhang[22–26] studied the perturbations of Lie symmetries and adiabatic invariance for holonomic systems, the various symmetries and the conserved quantities for holonomic systems and Birkhoffian systems in event space. Jia *et al.*[27–31] and Hou *et al.*[32–34] studied the various symmetries and the conserved quantities of nonholonomic systems of non-Chetaev's type with different conditions in event space. Zhang *et al.*[35] studied the symmetry of Lagrangians and the conserved quantity for the holonomic non-conservative system in event space. But all of the above research results only reflect the laws of motion of mechanical systems in event space under the action of forces or generalized forces. Since the forces acting on the actual mechanical systems are generally complex variable forces, the research on the laws of motion of mechanical systems in configuration space under the action of the first-order or higher-order time rate of change of forces has attracted more attention. Mei *et al.*[6] first gave the third-order Lagrange equations for holonomic systems. Ma *et al.*[36–41] studied the third-order Lagrange equations for different mechanical systems, the Lie symmetry and Noether symmetry of the third-order Lagrange equations, and the third-order pseudo-Hamilton's canonical equations. Yang and Ma[42] gave the third-order form invariance and conserved quantity. Zhang[43–46] and Shi[47] further studied the higher-order differential equations of motion for different mechanical systems. Zhao and Ma,[48] and Zhao *et al.*[49] gave the higher-order Hamilton's principle and the higher-order Lagrangian equations for holonomic systems. In the present paper, the laws of motion of mechanical systems in event space under the action of the higher-order time rate of change of forces will be further studied. The higher-order d'Alembert–Lagrange principle of

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†Corresponding author. E-mail: zhxw0215@aliyun.com
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mechanical systems and its parametric forms in event space will be established, and the higher-order differential equations of motion of holonomic systems in event space will be given.

2. Higher-order d'Alembert–Lagrange principle of mechanical system in event space

We consider an ideal mechanical system consisting of N particles. Let m_i be the mass of the i-th particle, $\boldsymbol{r}_i$ the position vector of the i-th particle, s the degrees of freedom of the system, and generalized coordinates q_α $(\alpha=1,2,\ldots,s)$ specify the configuration of the system, i.e., $\boldsymbol{r}_i=\boldsymbol{r}_i(q_\alpha,t)$, then we will construct an $(s+1)$-dimensional event space. The coordinates of space point are q_α $(\alpha=1,2,\ldots,s)$ and time t, which may be regarded as independent variables. We introduce the notation

$$x_1=t,\quad x_{\alpha+1}=q_\alpha,\quad (\alpha=1,2,\ldots,s), \tag{1}$$

then, all the variables x_β $(\beta=1,2,\ldots,s+1)$ may be given as a function of parameter τ. Let $x_\beta=x_\beta(\tau)$ be some curves of class C^2, such that

$$x'_\beta=\frac{\mathrm{d}x_\beta}{\mathrm{d}\tau},\quad (\beta=1,2,\ldots,s+1), \tag{2}$$

and x'_β $(\beta=1,2,\ldots,s+1)$ are not all zero simultaneously, then the expression for the position vector and the virtual displacement of the i-th particle will be given by

$$\boldsymbol{r}_i=\boldsymbol{r}_i(x_\beta), \tag{3}$$

$$\delta\boldsymbol{r}_i=\sum_{\beta=1}^{s+1}\frac{\partial\boldsymbol{r}_i}{\partial x_\beta}\delta x_\beta. \tag{4}$$

Let $\boldsymbol{F}_i$ and $\boldsymbol{R}_i$ be the active force and the constraint force acted on the i-th particle respectively, then based on the d'Alembert principle of the system, the higher-order d'Alembert principle of the system will be written as

$$-m_i\overset{(m+2)}{\boldsymbol{r}}_i+\overset{(m)}{\boldsymbol{F}}_i+\overset{(m)}{\boldsymbol{R}}_i=0,\quad (i=1,2,\ldots,N;m=0,1,2,\ldots). \tag{5}$$

Scalar-multiplying both sides of Eq. (5) by $\delta\boldsymbol{r}_i$, and summing over i, we obtain the result

$$\sum_{i=1}^{N}\left(-m_i\overset{(m+2)}{\boldsymbol{r}}_i+\overset{(m)}{\boldsymbol{F}}_i\right)\cdot\delta\boldsymbol{r}_i=0, \tag{6}$$

where we suppose that the condition of ideal constraint is satisfied, i.e., $\sum\limits_{i=1}^{N}\overset{(m)}{\boldsymbol{R}}_i\cdot\delta\boldsymbol{r}_i=0$. Equation (6) can be called the higher-order d'Alembert–Lagrange principle of mechanical system in event space.

Now, we transform Eq. (6) into parametric form in event space.

From the relation (3), by using successive derivative, the m-th derivative of $\boldsymbol{r}_i$ with respect to τ can be expressed by

$$\begin{aligned}\boldsymbol{r}_i^{(m)}&=\frac{\mathrm{d}\boldsymbol{r}_i^{(m-1)}}{\mathrm{d}\tau}=\sum_{\beta=1}^{s+1}\frac{\partial\boldsymbol{r}_i}{\partial x_\beta}x_\beta^{(m)}\\&\quad+m\sum_{\gamma=1}^{s+1}\sum_{\beta=1}^{s+1}\frac{\partial^2\boldsymbol{r}_i}{\partial x_\beta\partial x_\gamma}x_\beta^{(m-1)}x'_\gamma+\cdots,\end{aligned} \tag{7}$$

where the terms in $\boldsymbol{r}_i^{(m)}$ which do not contain $x_\beta^{(m)}$ nor $x_\beta^{(m-1)}$, are not listed here. Taking the partial derivative of Eq. (7) with respect to $x_\beta^{(m)}$ and $x_\beta^{(m-1)}$, we have

$$\frac{\partial\boldsymbol{r}_i^{(m)}}{\partial x_\beta^{(m)}}=\frac{\partial\boldsymbol{r}_i}{\partial x_\beta},\quad \frac{\partial\boldsymbol{r}_i^{(m)}}{\partial x_\beta^{(m-1)}}=m\frac{\partial\boldsymbol{r}'_i}{\partial x_\beta}. \tag{8}$$

From the relation (3), by performing successive derivative, the m-th derivative of $\boldsymbol{r}_i$ with respect to t can be expressed as follows:

$$\begin{aligned}\overset{(m)}{\boldsymbol{r}}_i&=\frac{\mathrm{d}\overset{(m-1)}{\boldsymbol{r}}_i}{\mathrm{d}\tau}\Big/\frac{\mathrm{d}t}{\mathrm{d}\tau}\\&=\frac{1}{(x'_1)^m}\boldsymbol{r}_i^{(m)}-\frac{m(m-1)}{2}\frac{x''_1}{(x'_1)^{m+1}}\boldsymbol{r}_i^{(m-1)}+\cdots\\&\quad-\frac{mx_1^{(m-1)}}{(x'_1)^{m+1}}\boldsymbol{r}''_i+\frac{m(m+1)}{2}\frac{x''_1x_1^{(m-1)}}{(x'_1)^{m+2}}\boldsymbol{r}'_i\\&\quad-\frac{x_1^{(m)}}{(x'_1)^{m+1}}\boldsymbol{r}'_i,\end{aligned} \tag{9}$$

where the terms in $\overset{(m)}{\boldsymbol{r}}_i$ which do not contain $\boldsymbol{r}_i^{(m)}$, nor $\boldsymbol{r}_i^{(m-1)}$, nor $x_1^{(m)}$, nor $x_1^{(m-1)}$, are not listed here. Taking the partial derivative of Eq. (9) with respect to $\boldsymbol{r}_i^{(m)}$ and $\boldsymbol{r}_i^{(m-1)}$, we obtain

$$\frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial\boldsymbol{r}_i^{(m)}}=\frac{1}{(x'_1)^m},\quad \frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial\boldsymbol{r}_i^{(m-1)}}=-\frac{m(m-1)}{2}\frac{x''_1}{(x'_1)^{m+1}}. \tag{10}$$

Taking the partial derivative of Eq. (9) with respect to $x_1^{(m)}$, $x_{\alpha+1}^{(m)}$, $x_1^{(m-1)}$, and $x_{\alpha+1}^{(m-1)}$ respectively, then in view of Eqs. (8) and (10), we have

$$\begin{aligned}\frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial x_1^{(m)}}&=\frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial\boldsymbol{r}_i^{(m)}}\frac{\partial\boldsymbol{r}_i^{(m)}}{\partial x_1^{(m)}}-\frac{\boldsymbol{r}'_i}{(x'_1)^{m+1}}\\&=\frac{1}{(x'_1)^m}\frac{\partial\boldsymbol{r}_i}{\partial x_1}-\frac{\boldsymbol{r}'_i}{(x'_1)^{m+1}},\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial x_{\alpha+1}^{(m)}}&=\frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial\boldsymbol{r}_i^{(m)}}\frac{\partial\boldsymbol{r}_i^{(m)}}{\partial x_{\alpha+1}^{(m)}}=\frac{1}{(x'_1)^m}\frac{\partial\boldsymbol{r}_i}{\partial x_{\alpha+1}},\\&\quad(\alpha=1,2,\ldots,s),\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial x_1^{(m-1)}}&=\frac{m}{(x'_1)^m}\frac{\partial\boldsymbol{r}'_i}{\partial x_1}-\frac{m(m-1)}{2}\frac{x''_1}{(x'_1)^{m+1}}\frac{\partial\boldsymbol{r}_i}{\partial x_1}\\&\quad-\frac{m}{(x'_1)^{m+1}}\boldsymbol{r}''_i+\frac{m(m+1)}{2}\frac{x''_1}{(x'_1)^{m+2}}\boldsymbol{r}'_i,\end{aligned} \tag{13}$$

$$\frac{\partial \overset{(m)}{\boldsymbol{r}_i}}{\partial x_{\alpha+1}^{(m-1)}} = \frac{\partial \overset{(m)}{\boldsymbol{r}_i}}{\partial \boldsymbol{r}_i^{(m)}}\frac{\partial \boldsymbol{r}_i^{(m)}}{\partial x_{\alpha+1}^{(m-1)}} + \frac{\partial \overset{(m)}{\boldsymbol{r}_i}}{\partial \boldsymbol{r}_i^{(m-1)}}\frac{\partial \boldsymbol{r}_i^{(m-1)}}{\partial x_{\alpha+1}^{(m-1)}} = \frac{m}{(x_1')^m}\frac{\partial \boldsymbol{r}_i'}{\partial x_{\alpha+1}} - \frac{m(m-1)}{2}\frac{x_1''}{(x_1')^{m+1}}\frac{\partial \boldsymbol{r}_i}{\partial x_{\alpha+1}}, \quad (\alpha = 1,2,\ldots,s). \tag{14}$$

Besides, the derivative of $\partial \boldsymbol{r}_i/\partial x_\beta$ with respect to τ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \boldsymbol{r}_i}{\partial x_\beta} = \sum_{\gamma=1}^{s+1}\frac{\partial^2 \boldsymbol{r}_i}{\partial x_\beta \partial x_\gamma}x_\gamma' = \frac{\partial \boldsymbol{r}_i'}{\partial x_\beta}. \tag{15}$$

By utilizing Eqs. (4), (12), (14), and (15), the first term on the left-hand side of Eq. (6) can be rewritten as

$$\begin{aligned}
&-\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}_i}\cdot\delta\boldsymbol{r}_i \\
&= -\sum_{\beta=1}^{s+1}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}_i}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_\beta}\right)\delta x_\beta \\
&= -\frac{1}{x_1'}\sum_{\beta=1}^{s+1}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_\beta}\right) - \sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot\frac{\partial \boldsymbol{r}_i'}{\partial x_\beta}\right\}\delta x_\beta \\
&= -\frac{1}{(x_1')^y}\sum_{\beta=1}^{s+1}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y-1}\frac{\partial \boldsymbol{r}_i}{\partial x_\beta}\right) - \sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y-1}\frac{\partial \boldsymbol{r}_i'}{\partial x_\beta} - (y-1)\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y-2}x_1''\frac{\partial \boldsymbol{r}_i}{\partial x_\beta}\right\}\delta x_\beta \\
&= -\frac{1}{(x_1')^y}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y-1}\frac{\partial \boldsymbol{r}_i}{\partial x_1}\right) - \sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y-1}\frac{\partial \boldsymbol{r}_i'}{\partial x_1} - (y-1)\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y-2}x_1''\frac{\partial \boldsymbol{r}_i}{\partial x_1}\right\}\delta x_1 \\
&\quad - \frac{1}{(x_1')^y}\sum_{\alpha=1}^{s}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial}{\partial x_{\alpha+1}^{(m+1)}}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y+m}\overset{(m+1)}{\boldsymbol{r}_i}\right) - \frac{1}{m+1}\frac{\partial}{\partial x_{\alpha+1}^{(m)}}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y+m}\overset{(m+1)}{\boldsymbol{r}_i}\right) - (y-1+\frac{m}{2})\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{y-2}x_1''\frac{\partial \boldsymbol{r}_i}{\partial x_{\alpha+1}}\right\}\delta x_{\alpha+1}.
\end{aligned} \tag{16}$$

Taking $y = -m/2+1$ and substituting Eqs. (11) and (13) into Eq. (16), we have

$$\begin{aligned}
&-\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}_i}\cdot\delta\boldsymbol{r}_i \\
&= -\frac{1}{x_1'}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}_i}\cdot\boldsymbol{r}_i'\right)\delta x_1 - \frac{1}{(x_1')^{-m/2+1}} \\
&\quad\times\sum_{\beta=1}^{s+1}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{m/2+1}\overset{(m+1)}{\boldsymbol{r}_i}\right) - \frac{1}{m+1}\frac{\partial}{\partial x_\beta^{(m)}}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{m/2+1}\overset{(m+1)}{\boldsymbol{r}_i}\right)\right\}\delta x_\beta.
\end{aligned} \tag{17}$$

Definition 1 We define the m-order velocity energy of the system in event space by

$$D_m^{\mathrm{e}} = \frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot(x_1')^{m/2+1}\overset{(m+1)}{\boldsymbol{r}_i} = (x_1')^{m/2+1}S_m, \tag{18}$$

where

$$S_m = \sum_{i=1}^{N}\frac{1}{2}m_i \overset{(m+1)}{\boldsymbol{r}_i}\cdot\overset{(m+1)}{\boldsymbol{r}_i}$$

is the m-order velocity energy of the system in configuration space.[43]

If $m = 0$, equation (18) can be written as

$$D^{\mathrm{e}} = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\boldsymbol{r}}_i\cdot x_1'\dot{\boldsymbol{r}}_i = x_1'T,$$

which is the kinetic energy of the system in event space.[8]

In accordance with Definition 1, equation (17) can be simplified into

$$\begin{aligned}
&-\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}_i}\cdot\delta\boldsymbol{r}_i \\
&= -\frac{1}{x_1'}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}_i}\cdot\boldsymbol{r}_i'\right)\delta x_1 - \frac{1}{(x_1')^{-m/2+1}} \\
&\quad\times\sum_{\beta=1}^{s+1}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial D_m^{\mathrm{e}}}{\partial x_\beta^{(m+1)}} - \frac{1}{m+1}\frac{\partial D_m^{\mathrm{e}}}{\partial x_\beta^{(m)}}\right\}\delta x_\beta.
\end{aligned} \tag{19}$$

The second term on the left-hand side of Eq. (6) can be written as

$$\begin{aligned}
&\sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}_i}\cdot\delta\boldsymbol{r}_i \\
&= \sum_{\beta=1}^{s+1}\left(\sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}_i}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_\beta}\right)\delta x_\beta \\
&= \left(\sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}_i}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_1}\right)\delta x_1 + \sum_{\alpha=1}^{s}\left(\sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}_i}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_{\alpha+1}}\right)\delta x_{\alpha+1} \\
&= \left(\sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}_i}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_1}\right)\delta x_1 + \sum_{\alpha=1}^{s} Q_\alpha^m\delta x_{\alpha+1},
\end{aligned} \tag{20}$$

where

$$Q_\alpha^m = \sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}_i}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_{\alpha+1}} = \sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}_i}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \quad (\alpha = 1,2,\ldots,s)$$

are the m-order time rates of change of the generalized active forces in configuration space.[43]

Scalar-multiplying both sides of Eq. (5) by $\boldsymbol{r}'_i$, and then summing over i, by using Eq. (7) and the property of ideal constraint $\sum_{i=1}^{N} \overset{(m)}{\boldsymbol{R}}_i \cdot \boldsymbol{r}'_i = 0$, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\left(-m_i \overset{(m+2)}{\boldsymbol{r}}_i + \overset{(m)}{\boldsymbol{F}}_i\right)\cdot \boldsymbol{r}'_i \\ &= -\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \boldsymbol{r}'_i + \sum_{\beta=1}^{s+1}\left(\sum_{i=1}^{N} \overset{(m)}{\boldsymbol{F}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial x_\beta}\right) x'_\beta = 0. \end{aligned} \quad (21)$$

From Eq. (21), we have

$$\begin{aligned}\sum_{i=1}^{N} \overset{(m)}{\boldsymbol{F}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial x_1} &= -\frac{1}{(x'_1)^{-m/2+1}} \sum_{\alpha=1}^{s} (x'_1)^{-m/2} Q^m_\alpha x'_{\alpha+1} \\ &\quad + \frac{1}{(x'_1)^{-m/2+1}} \sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot (x'_1)^{-m/2} \boldsymbol{r}'_i. \end{aligned} \quad (22)$$

Substituting Eq. (22) into Eq. (20), we obtain

$$\begin{aligned}\sum_{i=1}^{N} \overset{(m)}{\boldsymbol{F}}_i \cdot \delta \boldsymbol{r}_i &= \frac{1}{(x'_1)^{-m/2+1}}\left(-\sum_{\alpha=1}^{s} (x'_1)^{-m/2} Q^m_\alpha x'_{\alpha+1} \right. \\ &\quad \left. + \sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot (x'_1)^{-m/2} \boldsymbol{r}'_i\right)\delta x_1 \\ &\quad + \frac{1}{(x'_1)^{-m/2+1}} \sum_{\alpha=1}^{s} (x'_1)^{-m/2+1} Q^m_\alpha \delta x_{\alpha+1}. \end{aligned} \quad (23)$$

Definition 2 We define the m-order time rates of change of generalized active forces in event space by

$$\begin{aligned}P^m_1 &= -\sum_{\alpha=1}^{s} (x'_1)^{-m/2} Q^m_\alpha x'_{\alpha+1}, \\ P^m_{\alpha+1} &= (x'_1)^{-m/2+1} Q^m_\alpha, \quad (\alpha = 1, 2, \ldots, s). \end{aligned} \quad (24)$$

If $m = 0$, equation (24) can be written as $P_1 = -\sum_{\alpha=1}^{s} Q_\alpha x'_{\alpha+1}$, $P_{\alpha+1} = x'_1 Q_\alpha$ $(\alpha = 1, 2, \ldots, s)$, these are generalized active forces in event space.[5]

In accordance with Definition 2, equation (23) can be written as

$$\begin{aligned}\sum_{i=1}^{N} \overset{(m)}{\boldsymbol{F}}_i \cdot \delta \boldsymbol{r}_i &= \frac{1}{x'_1}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \boldsymbol{r}'_i\right)\delta x_1 \\ &\quad + \frac{1}{(x'_1)^{-m/2+1}} \sum_{\beta=1}^{s+1} P^m_\beta \delta x_\beta. \end{aligned} \quad (25)$$

Substituting Eqs. (19) and (25) into Eq. (6), we obtain

$$\sum_{\beta=1}^{s+1}\left\{-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial D^{\mathrm{e}}_m}{\partial x^{(m+1)}_\beta} + \frac{1}{m+1}\frac{\partial D^{\mathrm{e}}_m}{\partial x^{(m)}_\beta} + P^m_\beta\right\}\delta x_\beta = 0. \quad (26)$$

Equation (26) is the parametric form of Euler–Lagrange for the higher-order d'Alembert–Lagrange principle of mechanical system in event space.

Applying Eqs. (11)–(14) and Definition 1 to the first term on the left-hand side of Eq. (6), we have

$$\begin{aligned}&-\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \delta \boldsymbol{r}_i \\ &= -\frac{1}{x'_1}\sum_{\beta=1}^{s+1}\left(\sum_{i=1}^{N} m_i \frac{\mathrm{d}\overset{(m+1)}{\boldsymbol{r}}_i}{\mathrm{d}\tau}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_\beta}\right)\delta x_\beta \\ &= -\frac{1}{x'_1}\left(\sum_{i=1}^{N} m_i \frac{\mathrm{d}\overset{(m+1)}{\boldsymbol{r}}_i}{\mathrm{d}\tau}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_1}\right)\delta x_1 \\ &\quad -\frac{1}{x'_1}\sum_{\alpha=1}^{s}\left(\sum_{i=1}^{N} m_i \frac{\mathrm{d}\overset{(m+1)}{\boldsymbol{r}}_i}{\mathrm{d}\tau}\cdot\frac{\partial \boldsymbol{r}_i}{\partial x_{\alpha+1}}\right)\delta x_{\alpha+1} \\ &= -\frac{1}{(x'_1)^{-m/2+1}}\left\{\sum_{i=1}^{N} m_i \frac{\mathrm{d}\overset{(m+1)}{\boldsymbol{r}}_i}{\mathrm{d}\tau}\cdot(x'_1)^{m/2+1}\right. \\ &\quad \left.\times\left[\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_i}{\partial x^{(m+1)}_1} + \frac{\boldsymbol{r}'_i}{(x'_1)^{m+2}}\right]\right\}\delta x_1 - \frac{1}{(x'_1)^{-m/2+1}} \\ &\quad \times\sum_{\alpha=1}^{s}\left(\sum_{i=1}^{N} m_i \frac{\mathrm{d}\overset{(m+1)}{\boldsymbol{r}}_i}{\mathrm{d}\tau}\cdot(x'_1)^{m/2+1}\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_i}{\partial x^{(m+1)}_{\alpha+1}}\right)\delta x_{\alpha+1} \\ &= -\frac{1}{(x'_1)^{-m/2+1}}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot (x'_1)^{-m/2}\boldsymbol{r}'_i\right)\delta x_1 \\ &\quad -\frac{1}{(x'_1)^{-m/2+1}} \\ &\quad \times\sum_{\beta=1}^{s+1}\left\{\frac{\partial}{\partial x^{(m+1)}_\beta}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}}_i \cdot (x'_1)^{m/2+1}\overset{(m+1)}{\boldsymbol{r}}_i\right)\right. \\ &\quad \left.-\frac{m+2}{m+1}\frac{\partial}{\partial x^{(m)}_\beta}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+1)}{\boldsymbol{r}}_i \cdot (x'_1)^{m/2+1}\overset{(m+1)}{\boldsymbol{r}}_i\right)\right\}\delta x_\beta \\ &= -\frac{1}{x'_1}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \boldsymbol{r}'_i\right)\delta x_1 - \frac{1}{(x'_1)^{-m/2+1}} \\ &\quad \times\sum_{\beta=1}^{s+1}\left\{\frac{\partial}{\partial x^{(m+1)}_\beta}\frac{\mathrm{d}D^{\mathrm{e}}_m}{\mathrm{d}\tau} - \frac{m+2}{m+1}\frac{\partial D^{\mathrm{e}}_m}{\partial x^{(m)}_\beta}\right\}\delta x_\beta. \end{aligned} \quad (27)$$

Substituting Eqs. (27) and (25) into Eq. (6), we obtain

$$\sum_{\beta=1}^{s+1}\left\{-\frac{\partial}{\partial x^{(m+1)}_\beta}\frac{\mathrm{d}D^{\mathrm{e}}_m}{\mathrm{d}\tau} + \frac{m+2}{m+1}\frac{\partial D^{\mathrm{e}}_m}{\partial x^{(m)}_\beta} + P^m_\beta\right\}\delta x_\beta = 0. \quad (28)$$

Equation (28) is the parametric form of Nielsen for the higher-order d'Alembert–Lagrange principle of mechanical system in event space.

Using Eqs. (11) and (12), the first term on the left-hand side of Eq. (6) can also be written as

$$\begin{aligned}&-\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \delta \boldsymbol{r}_i \\ &= -\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial x_1}\right)\delta x_1 \end{aligned}$$

$$
\begin{aligned}
&-\sum_{\alpha=1}^{s}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial x_{\alpha+1}}\right)\delta x_{\alpha+1} \\
&= -\frac{1}{(x_1')^{-m/2+1}}\left\{\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot (x_1')^{m/2+3}\right. \\
&\left.\times\left[\frac{\partial \overset{(m+2)}{\boldsymbol{r}}_i}{\partial x_1^{(m+2)}}+\frac{\boldsymbol{r}_i'}{(x_1')^{m+3}}\right]\right\}\delta x_1 - \frac{1}{(x_1')^{-m/2+1}} \\
&\times\sum_{\alpha=1}^{s}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot (x_1')^{m/2+3}\frac{\partial \overset{(m+2)}{\boldsymbol{r}}_i}{\partial x_{\alpha+1}^{(m+2)}}\right)\delta x_{\alpha+1} \\
&= -\frac{1}{x_1'}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \boldsymbol{r}_i'\right)\delta x_1 - \frac{1}{(x_1')^{-m/2+1}} \\
&\times\sum_{\beta=1}^{s+1}\frac{\partial}{\partial x_\beta^{(m+2)}}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot (x_1')^{m/2+3}\overset{(m+2)}{\boldsymbol{r}}_i\right)\delta x_\beta. \quad (29)
\end{aligned}
$$

Definition 3 We define the m-order acceleration energy of the system in event space by

$$
S_m^{\mathrm{e}} = \frac{1}{2}\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot (x_1')^{m/2+3}\overset{(m+2)}{\boldsymbol{r}}_i = (x_1')^{m/2+3} S_{m+1}, \quad (30)
$$

where

$$
S_{m+1} = \sum_{i=1}^{N}\frac{1}{2} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \overset{(m+2)}{\boldsymbol{r}}_i
$$

is the $(m+1)$-order velocity energy of the system in configuration space.[43]

If $m=0$, equation (30) can be written as

$$
S^{\mathrm{e}} = \frac{1}{2}\sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot (x_1')^3 \ddot{\boldsymbol{r}}_i = (x_1')^3 S,
$$

which is the acceleration energy of the system in event space.[13]

In accordance with Definition 3, equation (29) can be simplified as follows:

$$
\begin{aligned}
-\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \delta \boldsymbol{r}_i &= -\frac{1}{x_1'}\left(\sum_{i=1}^{N} m_i \overset{(m+2)}{\boldsymbol{r}}_i \cdot \boldsymbol{r}_i'\right)\delta x_1 \\
&- \frac{1}{(x_1')^{-m/2+1}}\sum_{\beta=1}^{s+1}\frac{\partial S_m^{\mathrm{e}}}{\partial x_\beta^{(m+2)}}\delta x_\beta. \quad (31)
\end{aligned}
$$

Substituting Eqs. (31) and (25) into Eq. (6), one obtains

$$
\sum_{\beta=1}^{s+1}\left\{-\frac{\partial S_m^{\mathrm{e}}}{\partial x_\beta^{(m+2)}} + P_\beta^m\right\}\delta x_\beta = 0. \quad (32)
$$

Equation (32) is the parametric form of Appell for the higher-order d'Alembert–Lagrange principle of mechanical system in event space.

From Definition 1 and Definition 3, we can see that the m-order acceleration energy of the system is equal to the $(m+1)$-order velocity energy of the system in configuration space, but this is not always true in event space. Therefore, it is necessary in event space to introduce not only the higher-order velocity energy but also the higher-order acceleration energy.

3. Higher-order differential equations of motion for holonomic system in event space

Since δx_β $(\beta = 1,2,\ldots,s+1)$ are independent of each other for the holonomic system, thus each coefficient of δx_β $(\beta = 1,2,\ldots,s+1)$ in Eq. (26) vanishes by itself. We then obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial D_m^{\mathrm{e}}}{\partial x_\beta^{(m+1)}} - \frac{1}{m+1}\frac{\partial D_m^{\mathrm{e}}}{\partial x_\beta^{(m)}} = P_\beta^m, \\
&(\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots). \quad (33)
\end{aligned}
$$

Equation (33) is the higher-order Lagrange equation of motion for the holonomic system in event space.

If some of the higher-order time rates of change of the generalized active force Q_α^m are non-potential, say Q_α^{m*}, and the others, Q_α^m, are derivable from a potential function V_m, according to the definition for the potential of the higher-order time rate of change of the generalized active force,[44] we can write

$$
Q_\alpha^m = -\frac{\partial V_m}{\partial \overset{(m)}{q}_\alpha} + Q_\alpha^{m*} = -\frac{\partial V_m}{\partial \overset{(m)}{x}_{\alpha+1}} + Q_\alpha^{m*}, \quad (34)
$$

where

$$
\begin{aligned}
V_m &= V_m\left(x_1, \overset{(m)}{x}_{\alpha+1}\right) \\
&= V_m\left(x_1, \frac{x_{\alpha+1}^{(m)}}{(x_1')^m} - \frac{m x_1''}{(x_1')^{m+1}}x_{\alpha+1}^{(m-1)} + \cdots - \frac{x_1^{(m)}}{(x_1')^{m+1}}x_{\alpha+1}'\right)
\end{aligned}
$$

is the potential function of Q_α^m. We can then find the following expressions easily:

$$
\begin{aligned}
\frac{\partial V_m}{\partial x_{\alpha+1}^{(m)}} &= \frac{1}{(x_1')^m}\frac{\partial V_m}{\partial \overset{(m)}{x}_{\alpha+1}}, \\
\frac{\partial V_m}{\partial x_1^{(m)}} &= \sum_{\alpha=1}^{s}\frac{\partial V_m}{\partial \overset{(m)}{x}_{\alpha+1}}\frac{\partial \overset{(m)}{x}_{\alpha+1}}{\partial x_1^{(m)}} \\
&= -\frac{1}{(x_1')^{m+1}}\sum_{\alpha=1}^{s}\frac{\partial V_m}{\partial \overset{(m)}{x}_{\alpha+1}}x_{\alpha+1}'. \quad (35)
\end{aligned}
$$

Substituting Eqs. (34) and (35) into the higher-order time rate of change of generalized active force P_1^m and $P_{\alpha+1}^m$, we have

$$
\begin{aligned}
P_1^m &= -(x_1')^{m/2+1}\frac{\partial V_m}{\partial x_1^{(m)}} + P_1^{m*}, \\
P_{\alpha+1}^m &= -(x_1')^{m/2+1}\frac{\partial V_m}{\partial x_{\alpha+1}^{(m)}} + P_{\alpha+1}^{m*}, \quad (\alpha = 1,2,\ldots,s), \quad (36)
\end{aligned}
$$

where

$$
P_1^{m*} = -\sum_{\alpha=1}^{s}(x_1')^{-m/2}Q_\alpha^{m*}x_{\alpha+1}',
$$

$$P^{m*}_{\alpha+1} = (x'_1)^{-m/2+1} Q^{m*}_{\alpha}. \tag{37}$$

Definition 4 We define the m-order Lagrange function of the system in event space by

$$\Lambda_m = D^{\mathrm{e}}_m - (m+1)(x'_1)^{m/2+1} V_m = (x'_1)^{m/2+1} L_m, \tag{38}$$

where $L_m = S_m - (m+1)V_m$ is the m-order Lagrange function of the system in configuration space.[44]

If $m = 0$, equation (38) can be written as $\Lambda = x'_1(T - V) = x'_1 L$, which is the Lagrange function of the system in event space.[6]

Substituting Eq. (36) into Eq. (33), and in accordance with Definition 4, another form of the higher-order Lagrange equations (33) can be shown as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda_m}{\partial x^{(m+1)}_{\beta}} - \frac{1}{m+1}\frac{\partial \Lambda_m}{\partial x^{(m)}_{\beta}} = P^{m*}_{\beta},$$
$$(\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots). \tag{39}$$

Similarly, from Eq. (28), we can obtain the higher-order Nielsen equations of motion for the holonomic system in event space as follows:

$$\frac{\partial}{\partial x^{(m+1)}_{\beta}}\frac{\mathrm{d}D^{\mathrm{e}}_m}{\mathrm{d}\tau} - \frac{m+2}{m+1}\frac{\partial D^{\mathrm{e}}_m}{\partial x^{(m)}_{\beta}} = P^{m}_{\beta},$$
$$(\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots). \tag{40}$$

Substituting Eq. (36) into Eq. (40), considering

$$\frac{\mathrm{d}V_m}{\mathrm{d}\tau} = \frac{\partial V_m}{\partial x_1}x'_1 + x'_1\sum_{\alpha=1}^{s}\frac{\partial V_m}{\partial {}^{(m)}x_{\alpha+1}}{}^{(m+1)}x_{\alpha+1},$$

and noting Definition 4 and Eq. (35), another form of the higher-order Nielsen equations (40) can be written as

$$\frac{\partial}{\partial x^{(m+1)}_{\beta}}\frac{\mathrm{d}\Lambda_m}{\mathrm{d}\tau} - \frac{m+2}{m+1}\frac{\partial \Lambda_m}{\partial x^{(m)}_{\beta}} = P^{m*}_{\beta},$$
$$(\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots). \tag{41}$$

In the same way, from Eq. (32), we can easily obtain the higher-order Appell equations of motion for the holonomic system in event space

$$\frac{\partial S^{\mathrm{e}}_m}{\partial x^{(m+2)}_{\beta}} = P^{m}_{\beta}, \quad (\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots). \tag{42}$$

The higher-order Appell equations (42) can also be written as

$$\frac{\partial S^{\mathrm{e}}_m}{\partial x^{(m+2)}_{\beta}} = -(x'_1)^{m/2+1}\frac{\partial V_m}{\partial x^{(m)}_{\beta}} + P^{m*}_{\beta},$$
$$(\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots). \tag{43}$$

If the holonomic system is conservative, i.e., $Q^{m*}_{\alpha} = 0$ or $P^{m*}_{\beta} = 0$, then from the higher-order differential equations (39), (41), and (43), the higher-order Lagrange equations, Nielsen equations, and Appell equations of the holonomic conservative system in event space can also be obtained respectively as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda_m}{\partial x^{(m+1)}_{\beta}} - \frac{1}{m+1}\frac{\partial \Lambda_m}{\partial x^{(m)}_{\beta}} = 0,$$
$$(\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots), \tag{44}$$
$$\frac{\partial}{\partial x^{(m+1)}_{\beta}}\frac{\mathrm{d}\Lambda_m}{\mathrm{d}\tau} - \frac{m+2}{m+1}\frac{\partial \Lambda_m}{\partial x^{(m)}_{\beta}} = 0,$$
$$(\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots), \tag{45}$$
$$\frac{\partial S^{\mathrm{e}}_m}{\partial x^{(m+2)}_{\beta}} = -(x'_1)^{m/2+1}\frac{\partial V_m}{\partial x^{(m)}_{\beta}},$$
$$(\beta = 1,2,\ldots,s+1; m = 0,1,2,\ldots). \tag{46}$$

If $m = 0$, equations (44), (45), and (46) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_{\beta}} - \frac{\partial \Lambda}{\partial x_{\beta}} = 0, \quad \frac{\partial}{\partial x'_{\beta}}\frac{\mathrm{d}\Lambda}{\mathrm{d}\tau} - 2\frac{\partial \Lambda}{\partial x_{\beta}} = 0,$$
$$\frac{\partial S^{\mathrm{e}}}{\partial x''_{\beta}} = -x'_1\frac{\partial V}{\partial x_{\beta}}, \quad (\beta = 1,2,\ldots,s+1). \tag{47}$$

These are the parametric equations of holonomic conservative system in event space.[3]

4. Conclusions

In this paper, we have established the higher-order d'Alembert–Lagrange principle of mechanical systems and its parametric forms in event space, obtained the higher-order Lagrange equations, higher-order Nielsen equations, and higher-order Appell equations for the holonomic system in event space.

The research results indicate that if $m = 0$, then the results of this paper are consistent with those from the theory of traditional analytical mechanics; if $m \geq 1$, the results of this paper reflect the laws of motion of the system under the action of the higher-order time rate of change of forces, they are a beneficial supplement for the results from the theory of traditional analytical mechanics. Also, if $\tau = t$, then the results in event space will become the ones in configuration space. Therefore, the research results in this paper are of general significance.

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