Higher-order solitons in amplitude-disordered waveguide arrays*

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We investigate the existence and stability of different families of spatial solitons in optical waveguide arrays whose amplitudes obey a disordered distribution. The competition between focusing nonlinearity and linearly disordered refractive index modulation results in the formation of spatial localized nonlinear states. Solitons originating from Anderson modes with few nodes are robust during propagation. While multi-peaked solitons with in-phase neighboring components are completely unstable, multipole-mode solitons whose neighboring components are out-of-phase can propagate stably in wide parameter regions provided that their power exceeds a critical value. Our findings, thus, provide the first example of stable higher-order nonlinear states in disordered systems.

Keywords: disordered lattices, higher-order solitons, stability

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1. Introduction

Since the theoretical prediction of the Anderson localization,[1] wave propagation in various fields involving disordered configurations, including condensed-matter physics, solid-state physics, and optics, has attracted intense attention.[2–11] The renewed interest in the Anderson localization is due to the experimental realization of localization of optical waves in disordered media.[3,4] For a review of early works, see e.g., Refs. [5] and [6], and the references therein.

Nonlinearity and disorder in discrete configurations also stir up broad interests in a variety of fields of physics. Photonic lattices or waveguide arrays allow investigation of them in an optical context. Recently, Efremidis *et al.* reported the dynamics of solitons in a disordered lattice embedded in a focusing/defocusing medium.[7] The width and separation between successive waveguide sites are uncorrelated random variables obeying uniform distributions. Solitons originating from Anderson modes and resonant interactions between solitons and Anderson modes were predicted. Different families of solitons characterized by a constant number of nodes follow a cascade of Hopf-like bifurcations.[8] Following these works, two-dimensional solitons in similar disordered lattices were also reported.[9] In addition to the above progress, light propagation in spatially random nonlinear optical media[3,10] and expansion of a Bose–Einstein condensate in a random optical potential[11] were observed experimentally.

However, until now, no attention has been paid to the existence and stability of higher-order solitons in disordered systems. Besides the random spatial spacing or scale modulation of a periodic potential, random amplitude modulation of a periodic potential is also a frequently encountered problem in practice. Considering the fact that, in nonlinear cubic media modulated by periodic lattices, higher-order solitons are usually unstable, it is natural to ask: Whether higher-order solitons can exist in disordered systems? If yes: what forms should they exhibit? Can they propagate stably for a very long distance?

To answer these questions, we study the propagation dynamics of spatial solitons in optical waveguide arrays (lattices) whose amplitudes are uncorrelated random variables and obey uniform distributions. Different branches of solitons bifurcating from the associating lower-order Anderson modes are found to be stable in their existence domains. Resonant interaction between solitons and certain Anderson modes happen as long as their profiles have spatial overlaps. Furthermore, we explore the evolutions of higher-order solitons in the form of multi-peaked solitons with in-phase neighboring components and multipole-mode solitons with out-of-phase neighboring components. Linear stability analysis and direct numerical propagation simulations demonstrate that multi-peaked solitons are always unstable and multipole-mode solitons can propagate stably provided that their power exceeds a critical value. Such results indicate that disordered lattices can significantly suppress the instability of multipole solitons, in sharp contrast to the cases in periodic lattices. Thus, we report, for the first time, the stable evolutions of higher-order solitons in disordered optical potentials.

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2. Theoretical model

We start our analysis by considering beam propagation in an amplitude-disordered waveguide array embedded in a focusing cubic medium. The evolution of laser beam can be described by the scaled nonlinear Schrödinger equation for optical electric field amplitude q:

$$\mathrm{i}\frac{\partial q}{\partial z} = -\frac{\partial^2 q}{\partial x^2} - |q|^2 q - R(x)V(x)q, \tag{1}$$

where x and z are the normalized transverse and longitudinal coordinates, respectively; $V(x)$ is a periodic step potential modeling a waveguide array whose amplitudes are normalized as 1. We fix the spacing between the successive waveguide centers $L = 2$ and the width of each waveguide $W = 1$. The amplitudes of each waveguide are uncorrelated random variables obeying uniform distributions that take values in the ranges $10 - 2r \le R(x) \le 10 + 2r$. For a single waveguide site, $R(x)$ is a constant. The parameter $r \in [0, 1]$ describes the disordered degree of lattice. The amplitude-disordered lattice degenerates into a periodic one if $r = 0$. The nonlinear system described by Eq. (1) admits several conserved quantities, including the energy flow or power $P = \int_{-\infty}^{\infty} |q|^2 \mathrm{d}x$ and the Hamiltonian $H = \int_{-\infty}^{\infty} [|q_x|^2 + RV|q|^2 - (1/2)|q|^4] \mathrm{d}x$.

We search for stationary solutions of Eq. (1) in the form $q(x,z) = w(x)\exp(-\mathrm{i}bz)$, where w and b represent the profile of light field and propagation constant respectively. After substituting the above expression into Eq. (1), one obtains the following nonlinear ordinary differential equation:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + bw + RVw + |w|^2 w = 0, \tag{2}$$

which can be solved by a Newton iterative method or by the squared operator iteration method.[12] To elucidate the stability of nonlinear modes, we search for perturbed solutions of Eq. (1) in the form $q(x) = [w(x) + u(x,z) + \mathrm{i}v(x,z)]\exp(-\mathrm{i}bz)$, where u, v are real and imaginary parts of perturbation which may grow with complex rate δ upon propagation. Linearization of Eq. (1) around w yields the eigenvalue problem:

$$\begin{aligned} \delta u &= -\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - bv - RVv - w^2 v, \\ \delta v &= \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + bu + RVu + 3w^2 u. \end{aligned} \tag{3}$$

The coupled equations can be solved numerically by a finite-difference method. Stability of a soliton is determined by the spectrum of the above linearization operator. Solitons can propagate stably only when all real parts of eigenvalues δ equal zero.

3. Anderson modes and solitons originating from them

Before we discuss the dynamics of solitons, it is instructive to understand the band-gap structure of amplitude-disordered lattices. In the following discussion, we limit our analysis on a lattice with random amplitude $8 \le R(x) \le 12$ ($r = 1$) including 40 waveguide sites. In the linear limit and in the case of a periodic lattice ($r = 0$), equation (2) has infinite eigenfunctions w and the corresponding eigenvalues b which form a band-gap structure. Extended periodic Floquet–Bloch modes reside in the bands and no linear modes exist in the gaps. When the nonlinearity is introduced, nonlinear localized modes (gap solitons) are allowed in the gaps. If the lattices are disordered ($r \neq 0$), the extended Floquet–Bloch modes become localized Anderson modes. The eigenvalues corresponding to Anderson modes can also form a band-gap structure. To elucidate this, we plot the first 120 eigenvalues (enumerated in increasing b order) in Fig. 1(a).

Similar to the cases in width and separation-disordered lattices,[7] the shaded area in Fig. 1(a) separates the bound modes of each waveguide ($-\max(|R(x)|) \le b \le 0$) from the corresponding radiation modes ($b > 0$). However, two well-defined bandgaps between Anderson modes $40, 41$ and $80, 81$ open up. The open gaps make it possible to investigate the dynamics of optical solitons in defocusing media. Figure 1(b) displays the dependence of band-gap on the parameter r. The two gaps shrink slowly with the growth of r and will be closed if r reaches 2.5. In other words, the gaps will disappear if the degree of disorder exceeds a critical value.

Typical profiles of Anderson modes are shown in Figs. 1(c)–1(f). Anderson modes residing in the first band are well localized. When the eigenvalue enters into the second band, the delocalization of the corresponding Anderson modes becomes stronger [Fig. 1(f)]. By comparing the localization length of Anderson modes in width and separation-disordered lattices,[7] one will find that the amplitude-disordered lattice can enhance the localization of Anderson modes, especially in the first band. Anderson modes will extend further and even cover the whole lattice if their eigenvalues approach to the upper edge of the second band or enter into the third band.

When the nonlinearity cannot be ignored, localized nonlinear states or spatial solitons may exist in the bandgap of disordered lattice. Since the nonlinearity is focusing, the propagation constants of solitons can only reside in the semi-infinite gap. As stated in Ref. [7], solitons in disordered systems can originate from different Anderson modes and resonant interactions between a soliton and an Anderson mode may take place. Due to the nature of focusing nonlinearity, the power of solitons bifurcating from different Anderson modes decreases monotonically with the growth of propagation constant [Figs. 2(a) and 2(c)]. The solitons bifurcating from Anderson modes have no threshold power, i.e., the power of solitons can be arbitrary small, which is in contrast with the purely nonlinear solitons in periodic lattices.[13]

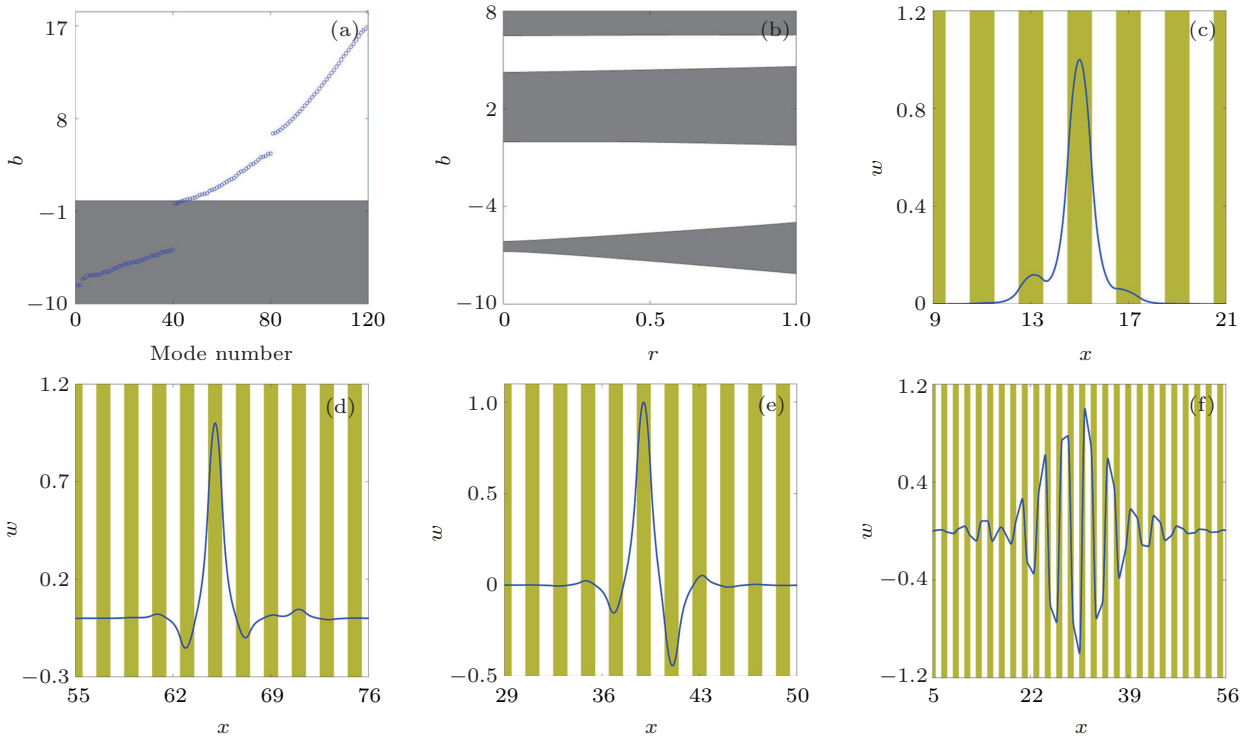


Fig. 1. (color online) (a) Eigenvalues of the Anderson modes (enumerated in increasing b order) forming a band structure. (b) Band-gap structure of disordered lattices as a function of r. Bands are marked by shaded regions; gaps are shown white. (c)–(f) Anderson modes 1, 22, 40, and 44, with eigenvalues $b = -8.226, -6.072, -4.756, 0.001$ respectively. Shaded regions in panels (c)–(f) correspond to $RV \geq 0$.

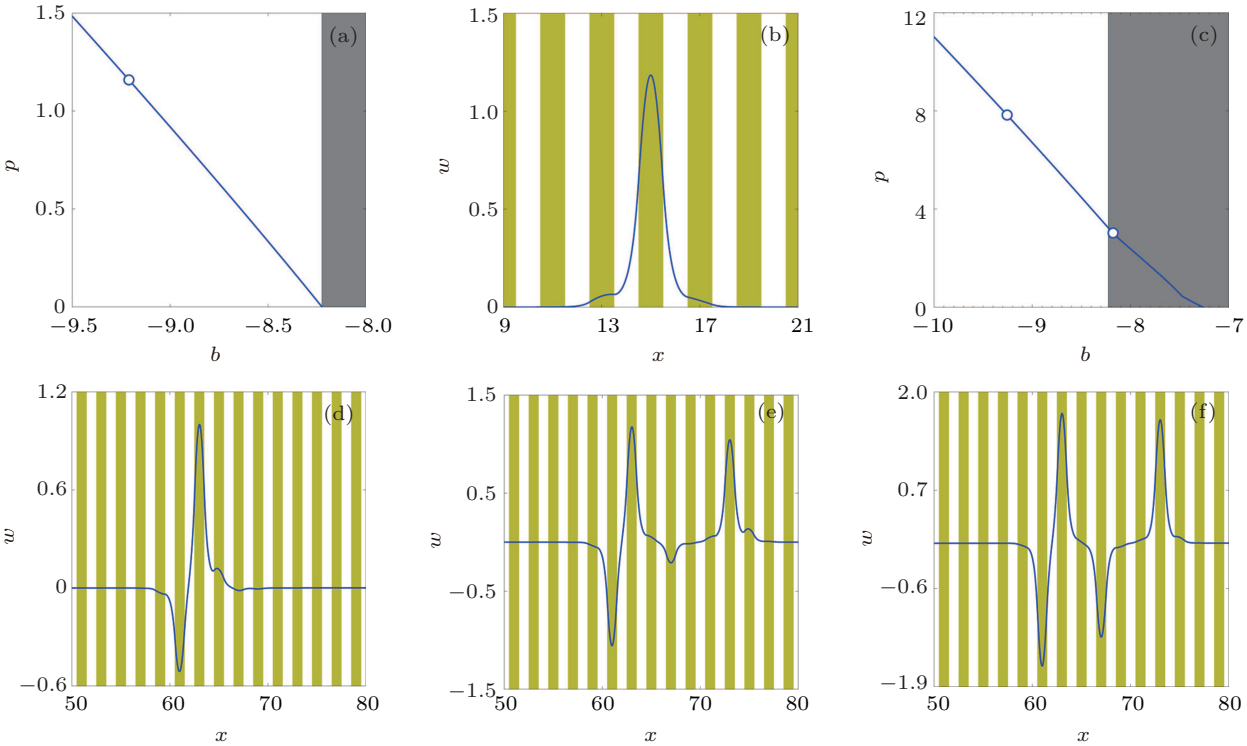


Fig. 2. (color online) (a) and (c) Power versus propagation constant for solitons originating from the 1st and 5th Anderson modes respectively. In each subfigure, shaded region represents the first band. (b) Profile of soliton marked by circle in panel (a) with $b = -9.211$. (d) Profile of the 5th Anderson mode. (e) and (f) Solitons marked by circle in panel (c) with $b = -8.178, -9.257$ respectively.

Solitons bifurcating from the first Anderson mode can be viewed as fundamental solitons, since there are no nodes across the soliton profiles [Fig. 2(b)]. With the decrease of propagation constant, soliton becomes narrower and its peak value becomes higher. Similarly, solitons originating from other Anderson modes can be regarded as higher-order nonlinear states [Fig. 2(f)]. To elucidate the resonant interaction between a soliton and an Anderson mode, in Figs. 2(d)–2(f), we

show the profiles of the fifth Anderson mode and solitons originating from it at different propagation constants. The fifth Anderson mode has two main poles at the centers of waveguide sites 31 and 32. The locations of the maximum amplitudes of Anderson modes $4, 3, 2$ reside on the centers of waveguide sites $37, 31, 34$ respectively. These Anderson modes have small or large spatial overlaps with the solitons originating from the Anderson mode 5. Thus, with the increase of soliton power, the small dips on the solitons will resonate with Anderson mode at the same lattice sites and be enlarged when the propagation constant decreases further. Note that Anderson mode 3 with $b_3 = -7.595$ overlaps with one pole of soliton, the resonate effect cannot be distinguished obviously. Thus, one can conclude that the resonant interaction leads to a significant broadening of the soliton profile [Fig. 2(f)]. On the other hand, the soliton does not resonate with Anderson mode 1 with $b_1 = -8.226$, just because its maximum amplitude reside on the 8th waveguide site and the spatial separation between them is too large. We stress that, similar to Ref. [7], resonance can take place only when the profiles of solitons and Anderson modes have spatial overlaps.

Due to the relatively large localization lengths of higher-order Anderson modes, solitons originating from them will resonate with more Anderson modes and result in the formation of solitons with many peaks. Moreover, the cascade resonance may occur for solitons originating from Anderson modes with higher node numbers.

To study the stability of disordered lattice solitons originating from Anderson modes, we conduct linear-stability analysis on the stationary solutions according to Eqs. (3). Numerical results demonstrate that, for solitons originating from Anderson modes with a few nodes, all real parts of instability growth rates equal 0, which implies that solitons belonging to these branches are robust when they propagate over a very long distance. For solitons bifurcating from Anderson modes with more nodes or larger b, oscillatory instability arises and destroys the propagation of them.

4. Higher-order solitons in amplitude-disordered waveguide arrays

Now, we examine the properties of higher-order solitons in amplitude-disordered lattices. First, we consider the multi-peaked solitons with in-phase neighboring components. Figure 3(a) shows the dependence of power of solitons with different numbers of peaks on propagation constant. These multi-peaked solitons are purely nonlinear states since they have lower cutoffs of power. That is to say, to excite such solitons, the power of input beam must exceed a critical value.

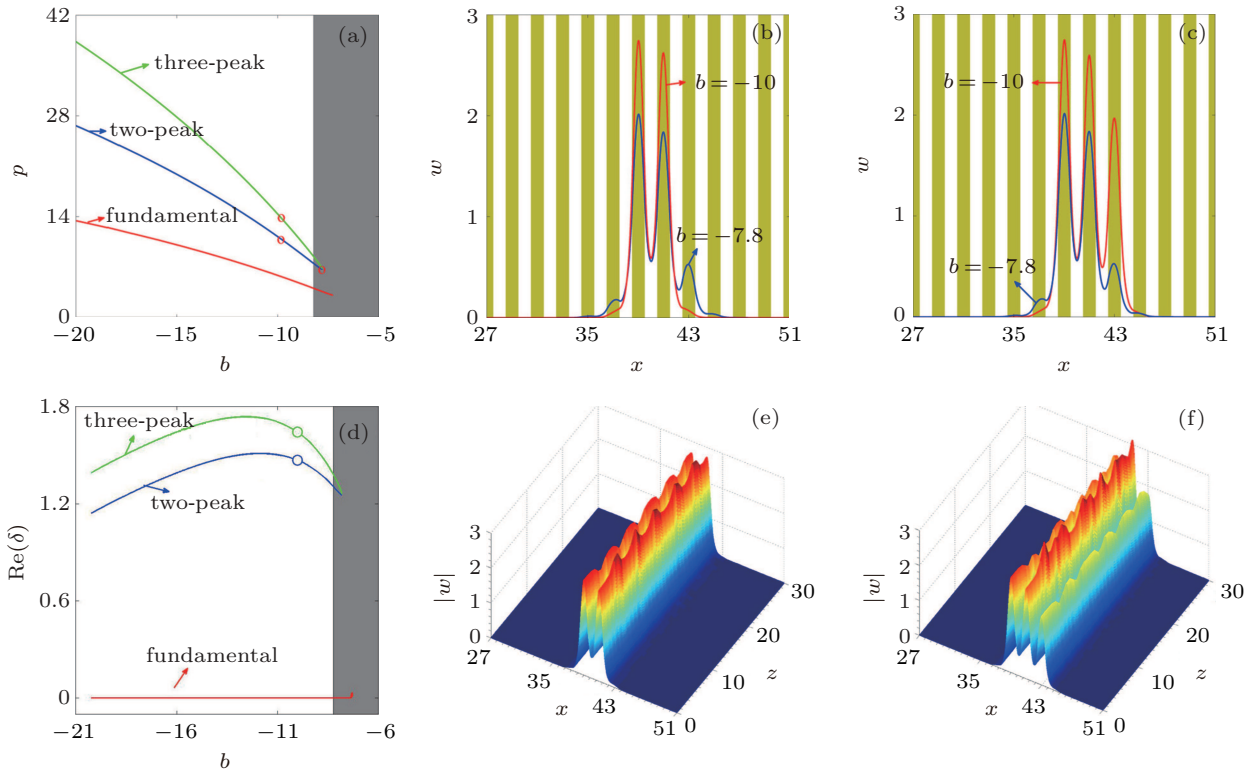


Fig. 3. (color online) (a) P–b curves of fundamental and multi-peaked solitons with in-phase neighboring components. Profiles of two-peak (b) and three-peak (c) solitons with $b = -7.8$ and -10. Fundamental, two-peak, and three-peak solitons reside on waveguide 20, 20–21, and 20–22 respectively. (d) Perturbation growth rate versus propagation constant for fundamental, two-peak, and three-peak solitons. (e) and (f) Unstable propagations of multi-peaked solitons plotted in panels (b) and (c) with $b = -10$.

Several representative profiles of multi-peaked solitons are illustrated in Figs. 3(b) and 3(c). We make five comments here. (i) Due to the fact that the amplitudes of the neighboring lattice sites are uncorrelated, unlike solitons in periodic lattices, the profiles of multi-peaked solitons here are asymmetrical. (ii) With the growth of propagation constant, the peaks of solitons decrease and solitons become more delocalized. Small lobes around the main peaks emerge for solitons with lower power. (iii) With the decrease of propagation constant, the small lobes may grow up or be suppressed, which leads to the formation of different branches of solitons with different numbers of peaks. For example, two- and three-peak solitons bifurcate from the same nonlinear mode at the cutoff of propagation constant ($b_{\rm co} = -7.799$). We also find that solitons with four peaks can also bifurcate from the same soliton solution. In this process, the left lobe of two-peak soliton at $b_{\rm co}$ also grows up. (iv) Unlike solitons in periodic lattices or in width- and separation-disordered lattices, the propagation constant of fundamental and multi-peaked solitons penetrates slightly into the first band of lattice. The reason may be found in the fact that the introduction of nonlinearity leads to a slight shift of bandgap of linear system. (v) Different branches of multi-peaked solitons can exist anywhere in the disordered lattices, which differ from the solitons originating from Anderson modes.

To comprehensively understand the stability characteristics of multi-peaked solitons, we perform linear-stability analysis on the stationary solutions with different numbers of peaks. Representative instability growth rates of fundamental, two-peak, and three-peak solitons are illustrated in Fig. 3(d) as a function of propagation constant. Fundamental solitons are unstable in a very narrow region near the cutoff of propagation constant. On the contrary, solitons with more peaks are completely unstable in their whole existence domains. The instability may be attributed to the repulsive forces between the neighboring soliton components, just similar to the multi-peaked solitons in periodic lattices imprinted in a focusing cubic medium. Instability analysis results are verified by extensive propagation simulations by the split-step Fourier method. Typical examples are displayed in Figs. 3(e) and 3(f). Oscillatory instability occurs for solitons with two and three peaks. The energy in neighboring soliton components exchanges during propagation. At the same time, a relatively small portion of energy radiates away from the solitons, which eventually destroys the distribution of light fields.

Next, we study the stability properties of higher-order solitons with out-of-phase neighboring components, i.e., multipole-mode solitons. The power of multipole solitons is also a decreasing function of propagation constant. They have lower threshold values and the corresponding propagation constants reside mainly in the semi-infinite gap of disordered lattice. By comparing with the aforementioned multi-peaked solitons, the power curves of dipole, tripole, and quadrupole solitons do not converge to a common propagation constant. It indicates that different branches of multipole solitons cannot bifurcate from a same nonlinear state. Typical examples of soliton profiles are shown in Figs. 4(b)–4(d). The unequal peak values of soliton poles are again due to the random amplitude distribution of lattice sites. Other properties, such as variance of soliton profiles with propagation constant, are similar to those of multi-peaked solitons.

Figure 4(e) shows the instability growth rate of multipole solitons versus propagation constant, which constitutes the central finding in this paper. Multipole solitons are completely

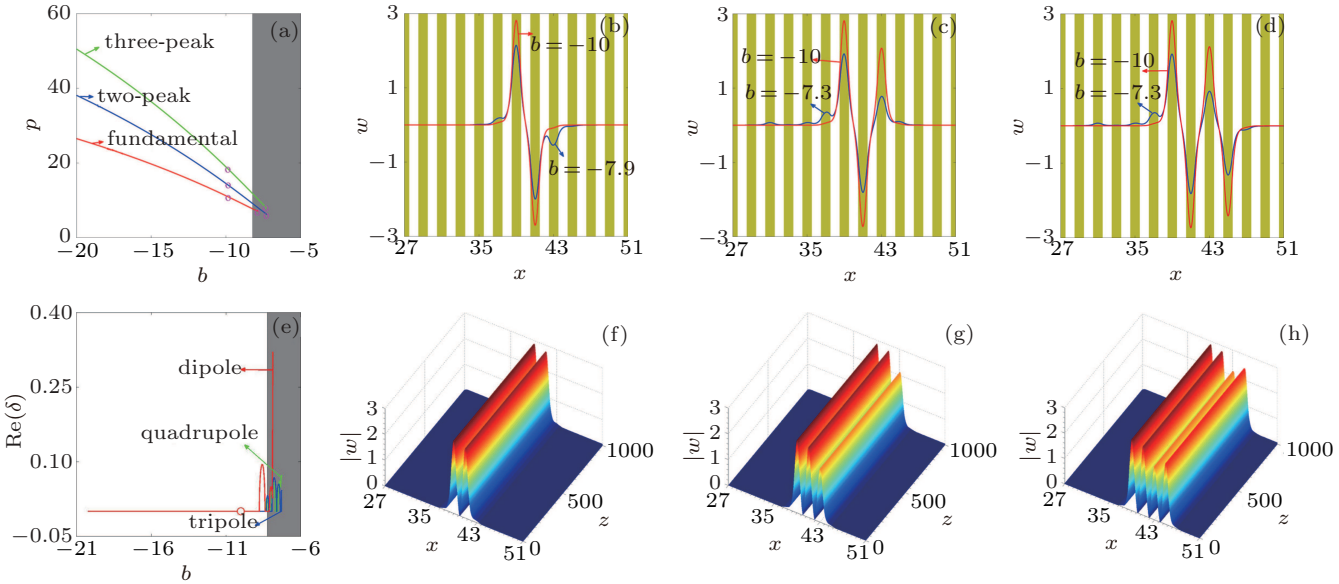


Fig. 4. (color online) (a) P–b curves of multipole solitons with out-of-phase neighboring components. Profiles of dipole (b), tripole (c), and quadrupole (d) solitons. Dipole, tripole, and quadrupole solitons reside on waveguide 20–21, 20–22, and 20–23, respectively. (e) Perturbation growth rate versus propagation constant for multipole solitons. (f)–(h) Stable propagations of multipole solitons plotted in panels (b)–(d) with $b = -10$.

stable provided that their power exceeds a critical value. Solitons are unstable only when their propagation constants approach to the cutoff values. Moreover, the precise structure of instability regions is rather complicated. There exist multiple narrow stability 'windows' near $b_{\rm co}$ [Fig. 4(e)]. By comparing Fig. 3(d) with Fig. 4(e), one finds that the stability of multi-peaked solitons with in-phase neighboring components is very different from that of multipole solitons with out-of-phase neighboring components. Recalling the fact that stable multipole solitons are impossible in periodic lattices embedded in a focusing cubic medium,[14] our stability analysis results here indicate that the disordered distribution of lattice amplitudes plays a key role for the suppression of instability of multipole solitons.

To confirm the above conclusions, we conduct extensive propagation simulations of multipole solitons by two ways, i.e., split-step Fourier algorithm and pseudo-spectral method. A very good agreement was found between the results obtained by two methods. The stability of solitons is only determined by the spectrum of the above linearization operator. In other words, the stable solitons will propagate without shape deformation no matter whether white noises are added into the initial inputs or not. Typical stable propagations of dipole, tripole, and quadrupole solitons at $b = -10$ are illustrated in Figs. 4(f)–4(h). The solitons are shape-preserved and without any distortions during propagation. One can also expect that solitons with pole number larger than 4 can propagate stably if their propagation constants are below certain critical values. We emphasize that, in disordered lattices, stable evolutions of higher-order solitons have not yet been reported.

Finally, although the disordered waveguide array discussed above is a specific configuration, the main properties of solitons in other disordered lattices obeying uncorrelated uniform distributions can be characterized by them. We have examined the propagation dynamics of higher-order solitons in other disordered lattices and found the following results: i) The variation of lattice distribution leads to a slight shift or deformation of the band-gap structure; ii) The position of Anderson modes and their corresponding eigenvalues are slightly different; iii) The positions of resonant modes and bifurcation points of solitons will be slightly different. However, bifurcation of solitons from Anderson modes, resonant interactions between solitons and Anderson modes still occur. Moreover, the stability analysis results are also similar to the above discussion. We conclude that these changes influence only the propagation dynamics of solitons quantitatively. Thus, one can grasp the main characters of disordered solitons through a typical configuration.

5. Summary

In conclusion, we studied the existence, stability, and propagation of solitons in an amplitude-disordered lattice imprinted in a focusing cubic medium. Band-gap structure of amplitude-disordered lattice and the corresponding linear Anderson modes in the bands were found. Different branches of solitons can bifurcate from different Anderson modes and resonate with certain other Anderson modes when their profiles have spatial overlaps. In particular, we investigated the dynamics of higher-order solitons in the form of multi-peaked solitons and multipole-mode solitons. Linear stability analysis verified by direct numerical propagation simulations reveals that multi-peaked solitons are unstable in their existence domains and multipole solitons can propagate stably if their power exceeds a critical value. This finding implies that the instability of multipole solitons can be remarkably suppressed by the amplitude-disordered potentials. Our findings, thus, provide the first example of stable higher-order nonlinear states in disordered configurations.

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