Generalized projective synchronization of the fractional-order chaotic system using adaptive fuzzy sliding mode control*

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An adaptive fuzzy sliding mode strategy is developed for the generalized projective synchronization of a fractional-order chaotic system, where the slave system is not necessarily known in advance. Based on the designed adaptive update laws and the linear feedback method, the adaptive fuzzy sliding controllers are proposed via the fuzzy design, and the strength of the designed controllers can be adaptively adjusted according to the external disturbances. Based on the Lyapunov stability theorem, the stability and the robustness of the controlled system are proved theoretically. Numerical simulations further support the theoretical results of the paper and demonstrate the efficiency of the proposed method. Moreover, it is revealed that the proposed method allows us to manipulate arbitrarily the response dynamics of the slave system by adjusting the desired scaling factor λ_i and the desired translating factor η_i, which may be used in a channel-independent chaotic secure communication.

Keywords: generalized projective chaos synchronization, fuzzy sliding mode control, fractional order chaotic system

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1. Introduction

Even though the fractional calculus has more than a 300-year history of being regarded as a mathematical topic, its application to physics and engineering has attracted great attention only in recent years. The fractional-order differential equation refers to an equation with the fractional order or any order, thus it is a generalization of the integer order differential equation. The fractional-order differential equation not only provides a new mathematical tool but also allows us to describe and model a real object more accurately than the traditional integer-order differential equation. It has been found that many systems or phenomena in the natural and social science can be described by the fractional-order differential equations, such as an economical system,[1] a viscoelastic medium,[2] solid materials with memory,[3] dielectric polarization[4] and electrode–electrolyte polarization.[5]

As is well known, the synchronization and the generalized projective synchronization (GPS) of a fractional-order chaotic system are more complicated than those of an integer-order system due to the complexity of the fractional-order differential dynamical system. Recently, the synchronization and the GPS of the fractional-order chaotic system have become an active research field, and the various control techniques have been reported for the synchronization and the GPS of a fractional-order chaotic system. For example, several scholars[6–8] used the active control technique for synchronizing the fractional-order chaotic system. Zhou and Zhu[9] investigated the function-projective synchronization and Zhou *et al.*[10] studied the muti-drive-one response synchronization for the fractional-order chaotic systems. Peng[11] and Matouk[12] adopted a linear feedback control scheme to synchronize two identical fractional-order chaotic systems. Zhang and Yang,[13] Zhou and Ding,[14] and Odibat[15] designed the adaptive controllers for the synchronization of two chaotic fractional-order systems, respectively. Xin *et al.*[16] proposed a linear controller for the GPS of the fractional-order energy resource demand-supply system. In Refs. [17] and [18], the state-observer-based approach was proposed to realize the GPS of the fractional-order chaotic system. Furthermore, a sliding model technique was also adopted to design a controller for the GPS of the fractional-order chaotic system.[19–21]

Among these above-mentioned methods, the fractional-order chaotic system is assumed to be fully known and the external disturbance is neglected. Thus, the above-mentioned methods can be used only to solve the problems of the known fractional-order chaotic systems with no external disturbances. Despite the fact that several scholars[22–24] have investigated the synchronization or the GPS of the uncertain fractional-order chaotic system with the external disturbances recently, it should be worth noting that these studies are valid only on condition that part of the parameters in the fractional-order chaotic system are unknown. However, from a practical point of view, the master system or the slave system is likely to be entirely unknown. In addition, one important purpose of the

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synchronization or the GPS is to use the chaos for achieving the secure communication.[25,26] Since the master system and the slave system are always in a varying environment and disturbed by some unknown factors from the environment, any small perturbation to the chaotic system may result in a drastic change in the behavior of the master system or the slave system. Thus, it is significant to consider the effects of the external disturbances on the synchronization or the GPS of the fractional-order chaotic systems. However, as far as we know, few reports are related to the GPS of the entirely unknown fractional-order chaotic system with the external disturbances. Therefore, it is still a great challenge to design a novel controller for the GPS of the entirely unknown fractional-order chaotic system.

Inspired by the above work, in this paper we investigate the GPS between a known fractional-order system (defined as the master system) and an entirely unknown fractional-order system (defined as the slave system) with external disturbances. The main difficulty in reaching this goal is how to overcome the negative effects of little knowledge about the slave system and the external disturbances on the GPS. An adaptive fuzzy sliding mode controller is designed to cope with the above difficulty and construct the GPS of the entirely unknown fractional-order chaotic system. First, we propose the novel fractional-order sliding mode surfaces. Then, based on the fractional-order Lyapunov stability theory, an adaptive fuzzy sliding mode control law is derived to guarantee the occurrence of the sliding motion. Numerical simulations demonstrate the efficiency and the wider applicability of the proposed method and support the theoretical results of the paper. The main highlights of this paper are as follows: 1) investigating the GPS for the fractional-order systems when the slave system is entirely unknown and subjected to the external disturbances; 2) considering the effects of the scaling factor λ_i and the translating factor η_i on the GPS; 3) manipulating arbitrarily the response dynamics of the slave system by adjusting the desired scaling factor and the translating factor, which can be used to build a channel-independent chaotic secure communication system;[27,28] and 4) verifying the good robustness of the closed-loop system for the time-varying complicated external disturbances.

The rest of this paper is organized as follows. At first, some definitions and the main theoretical results are presented in Section 2. Subsequently, the main simulation results and discussion will be given in Section 3. Finally, the conclusions are drawn from the present study in the last section.

2. Generalized projective synchronization problem formulation

2.1. Definitions of fractional derivatives

The fractional calculus plays an important role in modern science. Both the Reimann–Liouville and the Caputo fractional operators are used as the main tools. The Caputo fractional derivative is defined in this paper as[29]

$$D^{\alpha}x(t) = J^{m-\alpha}x^{(m)}(t), \quad \alpha \in (0,1),$$

where $m = [\alpha]$, i.e. m is the smallest integer that is not less than α, $x^{(m)}$ is the m-order derivative in the usual sense, and J^{β} $(\beta > 0)$ is the β-order Reimann–Liouville integral operator given by the following equation:

$$J^{\beta}y(t) = \frac{1}{\Gamma(s)}\int_0^t (t-x)^{\beta-1}y(x)\,\mathrm{d}x, \quad \beta > 0,$$

where $\Gamma(\cdot)$ is a Gamma function.

2.2. Main results

2.2.1. Generalized projective synchronization error system

Consider a class of fractional-order chaotic system described by

$$D_t^{\boldsymbol{q}}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x},t) \quad \boldsymbol{x} \in R^{n\times 1}, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is the state vector of the system. $\boldsymbol{f}(\boldsymbol{x},t) = [f_1(\boldsymbol{x},t), f_2(\boldsymbol{x},t), \ldots, f_n(\boldsymbol{x},t)]^{\mathrm{T}} \in R^{n\times 1}$, $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)^{\mathrm{T}}$, $0 < q_i < 1$ $(i = 1, 2, \ldots, n)$.

Now we define the master system as Eq. (1), and the slave system as follows:

$$D_t^{\boldsymbol{q}}\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{y},t) + b\boldsymbol{u} + d, \quad \boldsymbol{y} \in R^{n\times 1}, \tag{2}$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in R^{n\times 1}$ is the state vector of the slave system. $\boldsymbol{g}(\boldsymbol{y},t) = [g_1(\boldsymbol{y},t), g_2(\boldsymbol{y},t), \ldots, g_n(\boldsymbol{y},t)]^{\mathrm{T}} \in R^{n\times 1}$. $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in R^{n\times 1}$ is a control input, $\boldsymbol{b} = \mathrm{diag}(b_1, b_2, \ldots, b_n)$ is the coefficients matrix of the control input $\boldsymbol{u}$. The diag$(\cdot)$ is a diagonal function. $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)^{\mathrm{T}} \in R^{n\times 1}$ is the external disturbance. It is assumed that $|d_i| \le \bar{d} < +\infty$ for all t, where $\bar{d}$ is not necessarily known in advance.

The GPS error between Eqs. (1) and (2) is defined as follows:

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{\lambda}\boldsymbol{x} - \boldsymbol{\eta}, \tag{3}$$

where $\boldsymbol{e} = (e_1, e_2, \ldots, e_n)^{\mathrm{T}} \in R^{n\times 1}$, $\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^{n\times n}$, and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)^{\mathrm{T}} \in R^{n\times 1}$.

Then, from Eq. (3), we can obtain the error dynamic equation as follows:

$$D_t^{\boldsymbol{q}}\boldsymbol{e} = \boldsymbol{g}(\boldsymbol{y},t) - \boldsymbol{\lambda}\boldsymbol{f}(\boldsymbol{x},t) + b\boldsymbol{u} + \boldsymbol{d}. \tag{4}$$

The object of this paper is that for any given chaotic system with the same dimensions, such as Eqs. (1) and (2), the adaptive fuzzy sliding mode controllers are designed in spite of external disturbances, so that the asymptotic stability of system (4) can be achieved in the sense that

$$\lim_{t\to+\infty} \boldsymbol{e} = 0 \ \text{ and } \ \lim_{t\to+\infty} D_t^{\alpha}\boldsymbol{e} = 0. \tag{5}$$

2.2.2. Design of adaptive fuzzy slide mode controller

For the error variables $\boldsymbol{e}$ and $D_t^{\alpha}\boldsymbol{e}$, we need to design a proper switching surface to ensure the stability of system (4) in the sliding modes. To reach this goal, the switching sliding surface is defined as

$$s_i(t) = J_t^{1-q_i} e_i(t) + c_i \int_0^t e_i(\tau)\,\mathrm{d}\tau, \ \ (i = 1,2,\ldots,n), \tag{6}$$

where c_i is a positive constant. Then the first derivative of $s_i(t)$ with respect to time t is

$$\begin{aligned} \dot{s}_i(t) \ &= \ g_i(\boldsymbol{y},t) - \lambda_i f_i(\boldsymbol{x},t) + b_i u_i \\ &\quad + d_i + c_i e_i, \ \ (i = 1,2,\ldots,n). \end{aligned} \tag{7}$$

As is well known, when the system operates in the sliding mode, it satisfies the following equations:[30]

$$s_i(t) = 0 \ \text{ and } \ \dot{s}_i(t) = 0, \ \ (i = 1,2,\ldots,n).$$

Therefore, the equivalent controller is obtained by

$$u_i = b_i^{-1}(-g_i + \lambda_i f_i - c_i e_i - d_i), \ \ (i = 1,2,\ldots,n). \tag{8}$$

However, in real-world applications, if both the slave system and the external disturbance d_i are unknown, the equivalent controller (8) fails. To control and identify the unknown slave system (2) and the unknown external disturbances, the fuzzy systems $\hat{g}_i(\boldsymbol{y},t)$, $\hat{b}_i$, and $\hat{d}_i$ are used to approach the functions $g_i(\boldsymbol{y},t)$, b_i, and d_i, respectively. To do so, the fuzzy rules of the fuzzy systems are defined as follows:

$R(j)$: If y_i is $A_1^{(j)}$ and $D_t^{q_i} y_i$ is $A_2^{(j)}$,
then $\hat{g}_i$ is $G^{(j)}$ and $\hat{b}_i$ is $B^{(j)}$;
If s_i is $C^{(j)}$, then $\hat{h}_i$ is $H^{(j)}$.

Based on the singleton fuzzifier, the product inference, and the weighted average defuzzifier,[31] the fuzzy systems $\hat{g}_i(y_i, D_t^{q_i} y_i)$, $\hat{b}_i(y_i, D_t^{q_i} y_i)$, and $\hat{h}_i(s_i)$ are respectively described by

$$\hat{g}_i = \alpha_i^{\mathrm{T}} \boldsymbol{\xi}\left(y_i, D_t^{q_i} y_i\right), \ \hat{b}_i = \beta_i^{\mathrm{T}} \boldsymbol{\xi}\left(y_i, D_t^{q_i} y_i\right), \text{ and } \hat{h}_i = \gamma_i^{\mathrm{T}} \boldsymbol{\varphi}(s_i),$$

where α_i, β_i, and γ_i vary with the designed adaptive update laws. Both $\boldsymbol{\xi}(\cdot)$ and $\phi(\cdot)$ are the fuzzy vectors and defined respectively as follows:

$$\begin{aligned} &\boldsymbol{\xi}(y_i, D_t^{q_i} y_i) \\ &= \left(\xi^{(1)}(y_i, D_t^{q_i} y_i), \xi^{(2)}(y_i, D_t^{q_i} y_i), \ldots, \xi^{(N)}(y_i, D_t^{q_i} y_i)\right)^{\mathrm{T}} \end{aligned}$$

and

$$\phi(s_i) = \left(\phi^{(1)}(s_i), \phi^{(2)}(s_i), \ldots, \phi^{(N)}(s_i)\right)^{\mathrm{T}},$$

where

$$\xi^{(j)}(y_i, D_t^{q_i} y_i) = \frac{\mu_{A_1^{(j)}}(y_i)\mu_{A_2^{(j)}}(D_t^{q_i} y_i)}{\sum_{j=1}^{N} \mu_{A_1^{(j)}}(y_i)\mu_{A_2^{(j)}}(D_t^{q_i} y_i)},$$

$$\phi^{(j)}(s_i) = \frac{\varphi_{C^{(j)}}(s_i)}{\sum_{j=1}^{N} \varphi_{C^{(j)}}(s_i)},$$

with $\mu_{A_1^{(j)}}(y_i)$, $\mu_{A_2^{(j)}}(D_t^{q_i} y_i)$, and $\varphi_{C^{(j)}}(s_i)$ being respectively the membership functions of y_i, $D_t^{q_i} y_i$, and s_i, and N is the number of the fuzzy rules.

Thus, according to the linear feedback method, the controller is redesigned as follows:

$$u_i = \hat{b}_i^{-1}(-\hat{g}_i + \lambda_i f_i - c_i e_i - \hat{h}_i), \ \ (i = 1,2,\ldots,n). \tag{9}$$

To describe the new design and analysis, the following optimization parameters are needed

$$\alpha_i^* \ = \ \underset{\alpha_i\in\Omega_g}{\operatorname{argmin}} \left[\sup_{y_i\in R} |\hat{g}_i[(y_i, D_t^{q_i} y_i)|\alpha_i] - g(\boldsymbol{y},t) \right], \tag{10a}$$

$$\beta_i^* \ = \ \underset{\beta_i\in\Omega_b}{\operatorname{argmin}} \left[\sup_{y_i\in R} |\hat{b}_i[(y_i, D_t^{q_i} y_i)|\beta_i] - b_i \right], \tag{10b}$$

$$\gamma_i^* \ = \ \underset{\gamma_i\in\Omega_\gamma}{\operatorname{argmin}} \left[\sup_{s_i\in R} |\hat{h}_i(s_i\,|\gamma_i) - (\bar{d}+\delta)\operatorname{sgn}(s_i) \right], \tag{10c}$$

where $i = 1,2,\ldots,n$; Ω_g, Ω_b, and Ω_γ are the sets of α_i, β_i, and γ_i, respectively; δ is a small positive number; $\operatorname{sgn}(\cdot)$ is a symbol function.

The minimum approaching error is defined as follows:

$$w_i = g_i - \hat{g}_i^* + \left(b_i - \hat{b}_i^*\right) u_i,$$

where $\hat{g}_i^* = \hat{g}_i\left[(y_i, D_t^{q_i} y_i)\,|\alpha_i^*\right]$ and $\hat{b}_i^* = \hat{b}_i\left[(y_i, D_t^{q_i} y_i)\,|\beta_i^*\right]$. Then equation (7) can be rewritten as

$$\begin{aligned} \dot{s}_i \ &= \ g_i - \lambda_i f_i + \hat{b}_i u_i + \left(b_i - \hat{b}_i\right) u_i + d_i + c_i e_i \\ &= \ g_i - \hat{g}_i + \left(b_i - \hat{b}_i\right) u_i - \hat{h}_i + d_i \\ &= \ g_i - \hat{g}_i^* + \left(b_i - \hat{b}_i^*\right) u_i + \hat{g}_i^* - \hat{g}_i + \left(\hat{b}_i^* - \hat{b}_i\right) u_i \\ &\quad + \hat{h}_i^* - \hat{h}_i - \hat{h}_i^* + d_i \\ &= \ w_i + \left(\alpha_i^{*\mathrm{T}} - \alpha_i^{\mathrm{T}}\right) \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) + \left(\beta_i^{*\mathrm{T}} - \beta_i^{\mathrm{T}}\right) \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) u_i \\ &\quad + \ \left(\gamma_i^{*\mathrm{T}} - \gamma_i^{\mathrm{T}}\right) \phi(s_i) - \hat{h}_i^* + d_i \\ &= \ w_i + \tilde{\alpha}_i^{\mathrm{T}} \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) + \tilde{\beta}_i^{\mathrm{T}} \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) u_i \\ &\quad + \tilde{\gamma}_i^{\mathrm{T}} \phi(s_i) - \hat{h}_i^* + d_i, \end{aligned}$$

where $\tilde{\alpha}_i = \alpha_i^* - \alpha_i$, $\tilde{\beta}_i = \beta_i^* - \beta_i$, and $\tilde{\gamma}_i = \gamma_i^* - \gamma_i$.

Theorem 1 When $g_i(\boldsymbol{y},t)$, b_i, and d_i are unknown, and $\hat{h}_i(s_i|\gamma_i^*) = (\bar{d}+\delta)\operatorname{sgn}(s_i)$, $(\delta \geq 0)$ and $|d_i| \leq \bar{d}$, $(i =$

$1,2,\ldots,n$), if the controllers are designed as Eq. (9) and the adaptive update laws satisfy the following equations:

$$\begin{cases} \dot{\alpha}_i = r_i s_i \boldsymbol{\xi}\left(y_i, D_t^{q_i} y_i\right), \\ \dot{\beta}_i = p_i s_i \boldsymbol{\xi}\left(y_i, D_t^{q_i} y_i\right) u_i, \\ \dot{\gamma}_i = \rho_i s_i \phi\left(s_i\right), \end{cases} \tag{11}$$

then systems (1) and (2) will obtain the GPS.

Proof Construct a Lyapunov error function in the following form:

$$V = \sum_{i=1}^{n} V_i, \tag{12}$$

where

$$V_i = \frac{1}{2} s_i^2 + \frac{1}{2r_i} \tilde{\alpha}_i^{\mathrm{T}} \tilde{\alpha}_i + \frac{1}{2p_i} \tilde{\beta}_i^{\mathrm{T}} \tilde{\beta}_i + \frac{1}{2\rho_i} \tilde{\gamma}_i^{\mathrm{T}} \tilde{\gamma}_i. \tag{13}$$

Then the first derivative of V_i with respect to time t is

$$\begin{aligned} \dot{V}_i &= s_i \dot{s}_i + \frac{1}{r_i} \tilde{\alpha}_i^{\mathrm{T}} \dot{\tilde{\alpha}}_i + \frac{1}{p_i} \tilde{\beta}_i^{\mathrm{T}} \dot{\tilde{\beta}}_i \\ &\quad + \frac{1}{\rho_i} \tilde{\gamma}_i^{\mathrm{T}} \dot{\tilde{\gamma}}_i \\ &= s_i [w_i + \tilde{\alpha}_i^{\mathrm{T}} \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) + \tilde{\beta}_i^{\mathrm{T}} \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) u_i + \tilde{\gamma}_i^{\mathrm{T}} \phi(s_i) \\ &\quad - \hat{h}_i^* + d_i] + \frac{1}{r_i} \tilde{\alpha}_i^{\mathrm{T}} \dot{\tilde{\alpha}}_i + \frac{1}{p_i} \tilde{\beta}_i^{\mathrm{T}} \dot{\tilde{\beta}}_i + \frac{1}{\rho_i} \tilde{\gamma}_i^{\mathrm{T}} \dot{\tilde{\gamma}}_i \\ &= s_i w_i + \tilde{\alpha}_i^{\mathrm{T}} \left[s_i \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) + \frac{1}{r_i} \dot{\tilde{\alpha}}_i \right] \\ &\quad + \tilde{\beta}_i^{\mathrm{T}} \left[s_i \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) u_i + \frac{1}{p_i} \dot{\tilde{\beta}}_i \right] + \tilde{\gamma}_i^{\mathrm{T}} \left[s_i \phi(s_i) + \frac{1}{\rho_i} \dot{\tilde{\gamma}}_i \right] \\ &\quad - \left(\bar{d}_i + \delta\right) |s_i| + d_i s_i \\ &= s_i w_i + \tilde{\alpha}_i^{\mathrm{T}} \left[s_i \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) - \frac{1}{r_i} \dot{\alpha}_i \right] \\ &\quad + \tilde{\beta}_i^{\mathrm{T}} \left[s_i \boldsymbol{\xi}(y_i, D_t^{q_i} y_i) u_i - \frac{1}{p_i} \dot{\beta}_i \right] + \tilde{\gamma}_i^{\mathrm{T}} \left[s_i \phi(s_i) - \frac{1}{\rho_i} \dot{\gamma}_i \right] \\ &\quad - \left(\bar{d}_i + \delta\right) |s_i| + d_i s_i, \end{aligned} \tag{14}$$

where $\dot{\tilde{\alpha}}_i = -\dot{\alpha}_i$, $\dot{\tilde{\beta}}_i = -\dot{\beta}_i$, and $\dot{\tilde{\gamma}}_i = -\dot{\gamma}_i$.

Substituting Eq. (11) into Eq. (14), we yield the following equation:

$$\dot{V}_i = s_i w_i - \left(\bar{d}_i + \delta\right) |s_i| + d_i s_i \leq s_i w_i + \left(|d_i| - \bar{d}_i - \delta\right) |s_i|.$$

According to a fuzzy approaching theory,[31] an adaptive update fuzzy system can accurately approach any unknown system, so the minimum approaching error w_i can approach zero.

Therefore

$$\dot{V}_i \leq s_i w_i + \left(|d_i| - \bar{d}_i - \delta\right) |s_i| < 0,$$

then

$$\dot{V} = \sum_{i=1}^{n} \dot{V}_i \leq 0.$$

The $\dot{V} \leq 0$ confirms the presence of the reaching condition,[32] and means that s_i approaches the switching surface $s_i = 0$ ($i = 1,2,\ldots,n$). Furthermore, since the sliding modes are asymptotically stable, it is clearly shown that the error dynamic responses, i.e. $\lim_{t\to+\infty} \boldsymbol{e} = 0$ and $\lim_{t\to+\infty} D_t^q \boldsymbol{e} = 0$ are satisfied.[33] Hence, the proof is achieved completely.

3. Numerical simulations

In this section, in order to show the effectiveness of the proposed scheme in the preceding section, the numerical examples about the GPS of the fractional-order chaotic system will be provided. When numerically solving fractional-order differential equations, we adopt the predictor–corrector method.[34]

Five kinds of membership degree functions are used to haze the error variables y_i and $D_t^{q_i} y_i$ ($i = 1,2,\ldots,n$) and respectively defined as follows:

$$\begin{aligned} \mu_{NM}(z) &= \exp\left[-\left(z + \frac{\pi}{6}\right)^2 / \frac{\pi^2}{24^2} \right], \\ \mu_{NS}(z) &= \exp\left[-\left(z + \frac{\pi}{12}\right)^2 / \frac{\pi^2}{24^2} \right], \\ \mu_{ZO}(z) &= \exp\left[-z^2 / \frac{\pi^2}{24^2} \right], \\ \mu_{PS}(z) &= \exp\left[-\left(z - \frac{\pi}{12}\right)^2 / \frac{\pi^2}{24^2} \right], \\ \mu_{PM}(z) &= \exp\left[-\left(z - \frac{\pi}{6}\right)^2 / \frac{\pi^2}{24^2} \right]. \end{aligned}$$

The membership degree functions of the switching sliding surface s_i ($i = 1,2,3$) are defined as follows:

$$\varphi_{NM}(z) = 1/\left[1 + \exp(5z + 15)\right], \quad \varphi_{ZO}(z) = \exp\left(-z^2\right),$$
$$\text{and} \quad \varphi_{PM}(z) = 1/\left[1 + \exp(5z - 15)\right].$$

In the numerical simulations, α and β are both defined as 25×1 vectors, and γ is defined as 3×1 vectors, the initial values of α, β, and γ are all set to be 0.1. The initial conditions of the master system and the slave system are selected as $(x_1(0), x_2(0), x_3(0)) = (0.3, 0.5, 0.6)$ and $(y_1(0), y_2(0), y_3(0)) = (0.1, -0.1, -0.5)$, respectively. $(c_1, c_2, c_3) = (2, 5.5, 4)$. The coefficient $\boldsymbol{b}$ is set as a unity matrix, and the coefficients of the adaptive update laws are set as $(r_1, r_2, r_3) = (2, 10, 30)$, $(p_1, p_2, p_3) = (1000, 200, 30)$, and $(\rho_1, \rho_2, \rho_3) = (1000, 200, 80)$, respectively.

3.1. Generalized projective synchronization between two identical chaotic oscillators

Both the known master system and the unknown slave system are the fractional-order chaotic Lorenz systems,[35] which are described by the following equations:

$$D_t^{q_1} x_1 = 10(x_2 - x_1),$$

$$D_t^{q_2}x_2 = 4x_1 + 5x_2 - x_1x_3,$$
$$D_t^{q_3}x_3 = x_2x_3 - 8/3x_3. \quad (15)$$

Through the numerical simulations, one can find that system (15) will exhibit the chaotic behaviors when $(q_1,q_2,q_3) = (0.97,0.92,0.96)$, which is shown in Fig. 1.

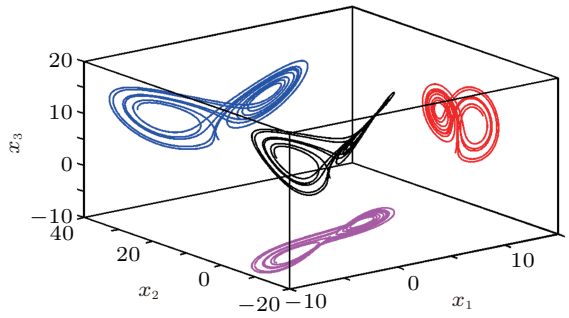


Fig. 1. (color online) Chaotic attractor and projections of the fractional-order Lorenz chaotic system with $(q_1,q_2,q_3) = (0.97,0.92,0.96)$.

3.1.1. Effectiveness and robustness of the proposed scheme

To illustrate the effectiveness of the proposed methods, figure 2(a) shows the temporal evolution of the GPS errors between the known master system and the unknown slave system. One can find that the GPS asymptotically approaches zero, which implies that the GPS between the known master system and the unknown slave system can be indeed realized even though the slave system is fully unknown. The time responses of the sliding surfaces s_i $(i = 1,2,3)$ are plotted in Fig. 2(b). It is seen that the sliding surfaces asymptotically converge to zero. The temporal evolutions of the applied control inputs u_i $(i = 1,2,3)$ are depicted in Fig. 2(c). Obviously, the control inputs are feasible in practical applications because they do not nearly contain harmful and frequent oscillations as chattering.

Assume the time-varying external disturbances to be as follows:

$$d_i(t) = \begin{cases} 0, & t \in [0,3.5)\cup[6,7)\cup[9,10)\cup[11,12], \\ 5\,\mathrm{e}^{-2(t-10)^2}, & t \in [3.5,6), \\ 5\sin 0.2t, & t \in [7,9), \\ 5, & t \in [10,11), \end{cases}$$
$$(i = 1,2,3). \quad (16)$$

To verify and demonstrate the robustness of the controlled system for the external disturbances, the time-varying external disturbances (16) (as shown in Fig. 3(b)) are used to disturb the unknown slave system. The GPS errors between the known master system and the unknown slave system are depicted in Fig. 3(a). By comparing Fig. 3(a) with Fig. 2(a), one can find that the GPS errors are hardly affected by the external disturbances and asymptotically converge to zero, which implies that the GPS between the known Lorenz system and the unknown Lorenz system can be well realized even though the unknown slave system is subjected to complicated external disturbances. The temporal evolutions of the applied control inputs are depicted in Fig. 3(c). Obviously, by comparing Fig. 3(c) with Fig. 3(b), one can find that the control inputs u_i $(i = 1,2,3)$ can be adaptively adjusted with the external disturbances (16). Therefore the designed controllers (9) and (11) can resist all kinds of external disturbances and the GPS is well achieved. By the way, the GPSs shown in Figs. 2 and 3 are indeed the synchronization.

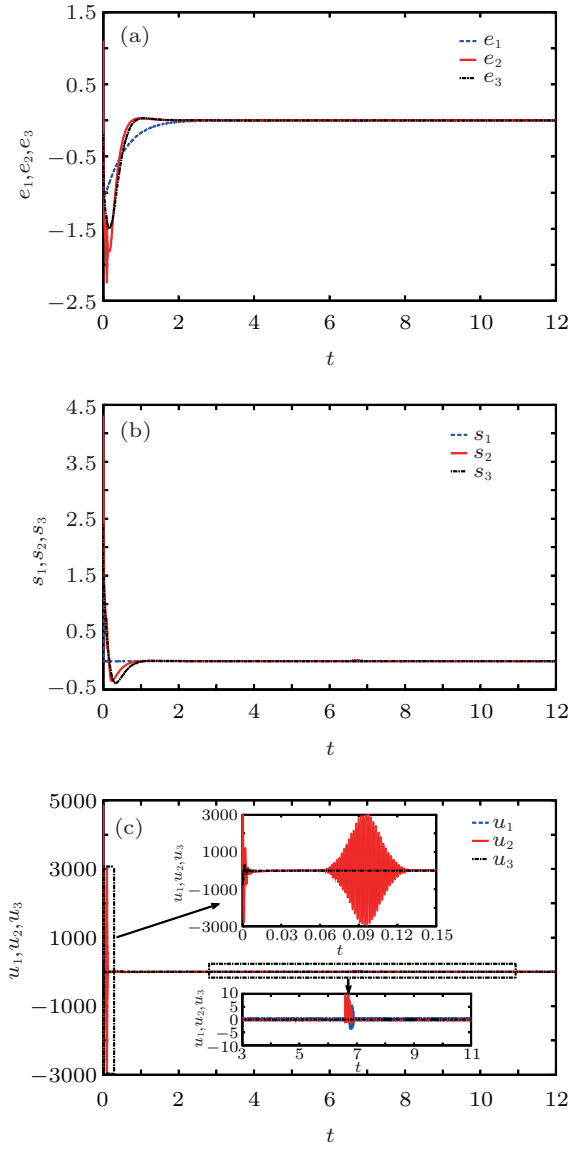


Fig. 2. (color online) Without external disturbances, the GPS errors when $\boldsymbol{\lambda} = (1,1,1)^{\mathrm{T}}$ and $\eta = (0,0,0)^{\mathrm{T}}$.

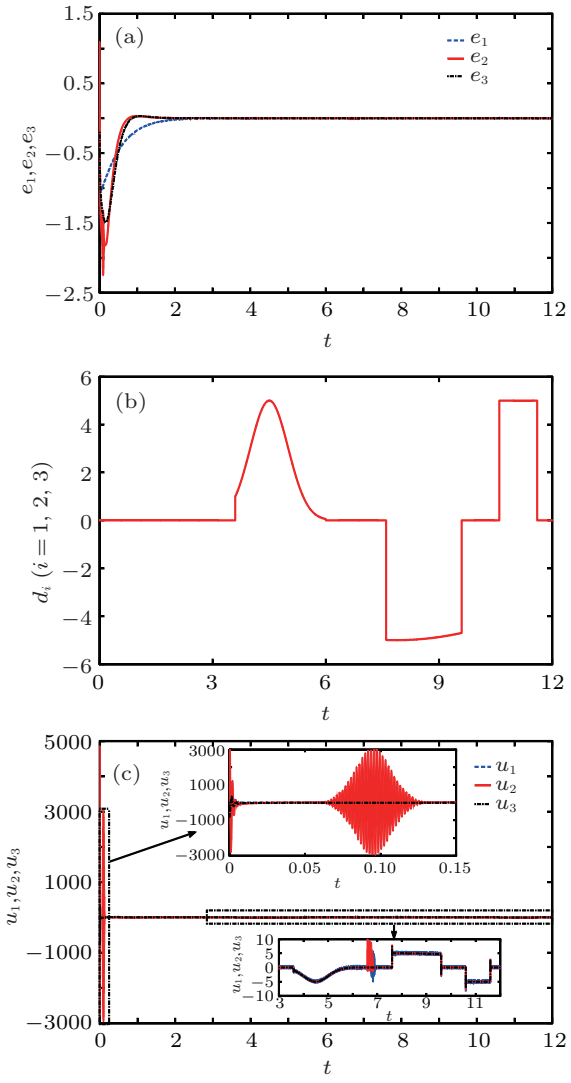


Fig. 3. (color online) With external disturbances, the GPS errors between the known master system and the unknown slave system when $\boldsymbol{\lambda} = (1,1,1)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (0,0,0)^{\mathrm{T}}$.

3.1.2. Effects of $\boldsymbol{\lambda}$ and $\boldsymbol{\eta}$ on GPS

From Figs. 2 and 3, one can easily observe that the unknown slave system efficiently synchronizes with the known master system through the proposed controllers (9) and (11) when $\boldsymbol{\lambda} = (1,1,1)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (0,0,0)^{\mathrm{T}}$. Another natural problem is whether the GPS between the known master system and the unknown slave system is well achieved if we set the scaling factor λ_i and the translating factor η_i to be other values. Thus, one can obtain the results in Fig. 4 (transient evolution is not plotted here). Figure 4 shows two cases for two identical fractional-order chaotic Lorenz systems, where the GPS is controlled to the desired factors as $\boldsymbol{\lambda} = (-2,-1,2)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (-4,0,-5)^{\mathrm{T}}$ in Fig. 4(a) and $\boldsymbol{\lambda} = (-1,-1,1)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (0,0,0)^{\mathrm{T}}$ in Fig. 4(b), respectively. From Fig. 4, one can see that the state vectors of two synchronized systems evolve in the following rules: (i) the absolute value of the scaling factor λ_i implies the ratio of the amplitudes of the vector y_i of the slave system to the amplitudes of the vector x_i of the master system; (ii) the state vector y_i of the slave system and the state vector x_i of the master system evolve in the same (or opposite) direction for the case $\lambda_i > 0$ (or $\lambda_i < 0$); and (iii) the absolute value of the translating factor η_i implies the displacement of the slave system relative to the master system in the positive (or negative) direction of y_i for $\eta_i > 0$ (or $\eta_i < 0$). Incidentally, the GPS between the vector y_i of the slave system and the vector x_i of the master system actually is synchronization (or anti-synchronization) when $\lambda_i = 1$ and $\eta_i = 0$ (or $\lambda_i = -1$ and $\eta_i = 0$), which is shown in Fig. 4(b). Since the scaling factor λ_i is independent of the translating factor η_i, the proposed method allows us to manipulate arbitrarily the response dynamics of the unknown slave system by adjusting the desired scaling factor λ_i and the desired translating factor η_i. Therefore, this feature can be used to build a channel-independent chaotic secure communication system.[27,28]

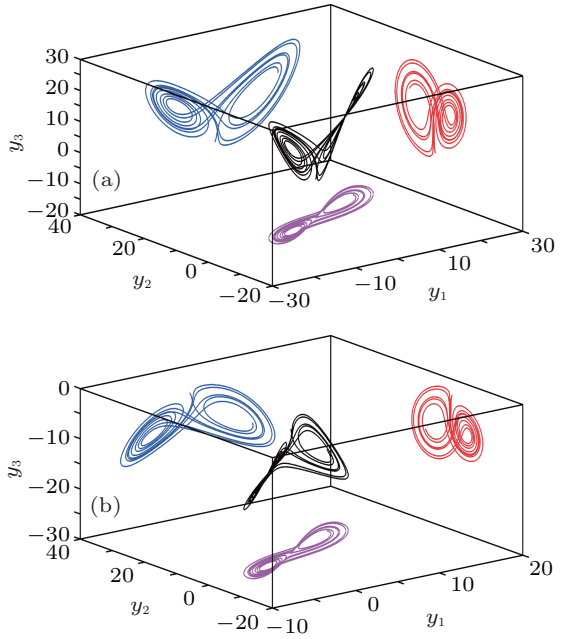


Fig. 4. (color online) Without external disturbances, chaotic attractors and projections of the unknown slave system: (a) $\boldsymbol{\lambda} = (-2,-1,2)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (-4,0,-5)^{\mathrm{T}}$; (b) $\lambda = (-1,-1,1)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (0,0,0)^{\mathrm{T}}$.

3.2. Generalized projective synchronization between two different chaotic oscillators

In order to illustrate further the effectiveness of the proposed scheme, in this subsection, we consider the GPS between two different chaotic oscillators. To do so, the fractional-order chaotic Rossler system[36] is introduced, which is described by the following equations:

$$\begin{aligned} D_t^{q_1} x_1 &= -(x_2 - x_3), \\ D_t^{q_2} x_2 &= x_1 + 0.4x_2, \\ D_t^{q_3} x_3 &= -10x_3 + x_1 x_3 + 0.2. \end{aligned} \tag{17}$$

Through the numerical simulations, one can find that system (17) will exhibit the chaotic behaviors when $(q_1, q_2, q_3) = (0.97, 0.92, 0.96)$, which is shown in Fig. 5.

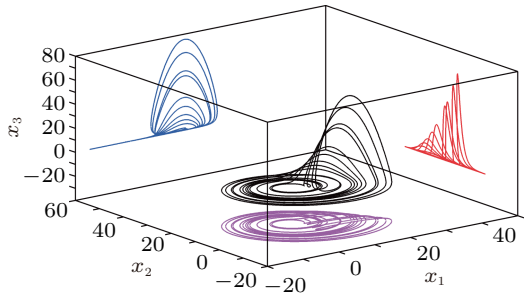


Fig. 5. (color online) Chaotic attractor and projections of the fractional-order Rossler chaotic system with $(q_1, q_2, q_3) = (0.97, 0.92, 0.96)$.

Figure 6 shows two cases with the different adjusting factors (transient evolution is not plotted here), where the known master system is defined as the fractional-order chaotic Rossler system (17) and the slave system is defined as the entirely unknown fractional-order chaotic Lorenz system (15). One can easily observe that the GPS is controlled to the desired adjusting factors as $\boldsymbol{\lambda} = (-0.5, -1, 2)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (2, 3, -1)^{\mathrm{T}}$ in Fig. 6(a) and $\lambda = (-1, -1, 1)^{\mathrm{T}}$ and $\eta = (0, 0, 0)^{\mathrm{T}}$ in Fig. 6(b), respectively. These results show that the proposed method in this paper is also valid for the GPS between two different fractional-order chaotic systems, which implies the wider application scope of the proposed method.

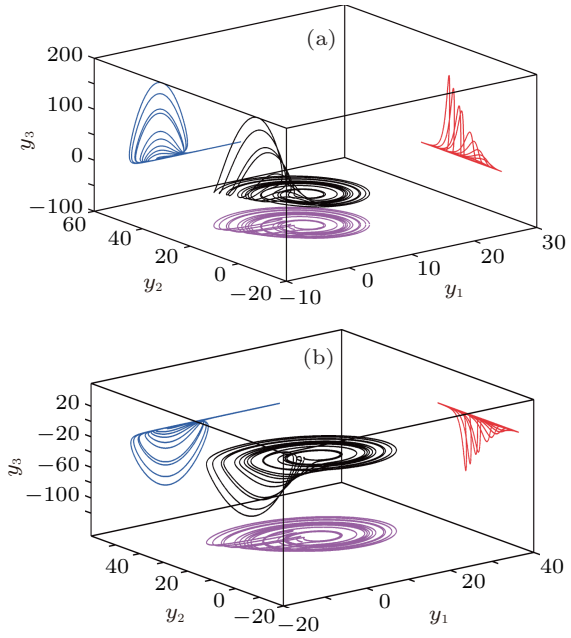


Fig. 6. (color online) Without external disturbances, chaotic attractors and projections of the unknown slave system: (a) $\boldsymbol{\lambda} = (-0.5, -1, 2)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (2, 3, -1)^{\mathrm{T}}$; (b) $\boldsymbol{\lambda} = (-1, -1, 1)^{\mathrm{T}}$ and $\boldsymbol{\eta} = (0, 0, 0)^{\mathrm{T}}$.

4. Conclusions

In this paper, we studied the GPS of the fractional-order chaotic system, where the slave system is not necessarily known in advance. Based on the designed adaptive update laws, the fuzzy system and the sliding mode strategy have been combined into an adaptive fuzzy sliding mode control strategy. Based on the Lyapunov stability theorem, the stability and the robustness of the controlled system were proved theoretically. To illustrate further the effectiveness and feasibility of the proposed method, we applied it to the fractional-order Lorenz system and the fractional-order Rossler system. In comparison with the previous methods, the proposed scheme in this paper is remarkable. It is valid for the GPS of the fractional-order systems even though the fractional-order slave system is entirely unknown. Besides, it allows us to manipulate arbitrarily the response dynamics of the slave system by adjusting the desired scaling factors and the desired translating factors. These prominent features can be used to build a channel-independent chaotic secure communication system. Therefore, the results of this paper are also very applicable and representative. We hope our work can contribute to the GPS of the unknown fractional-order chaotic system. As for a terminal technology applied to the GPS of the fully unknown fractional-order chaotic systems, we will discuss elsewhere.

References

[1] Dadras S and Momeni H R 2010 *Physica A* **389** 2434
[2] Ingman D and Suzdalnitsky J 2002 *J. Vib. Acoust.* **124** 642
[3] Koeller R C 2007 *Acta Mech.* **191** 125
[4] Sun H H, Abdelwahab A A and Onaral B 1984 *IEEE Trans. Automat. Control* **29** 441
[5] Ichise M, Nagayanagi Y and Kojima T 1971 *J. Electroanal. Chem.* **33** 253
[6] Bhalekar S and Daftardar-Gejji V 2010 *Commun. Nonlinear Sci. Numer. Simul.* **15** 3536
[7] Pan L, Zhou W N, Zhou L and Sun K H 2011 *Commun. Nonlinear Sci. Numer. Simul.* **16** 2628
[8] Song L, Yang J Y and Xu S Y 2010 *Nonlinear Anal.* **72** 2326
[9] Zhou P and Zhu W 2011 *Nonlinear Anal.* **12** 811
[10] Zhou P, Ding R and Cao Y X 2012 *Nonlinear Dyn.* **70** 1263
[11] Peng G J 2007 *Phys. Lett. A* **363** 426
[12] Matouk A E 2011 *Commun. Nonlinear. Sci. Numer. Simul.* **16** 975
[13] Zhang R X and Yang S P 2011 *Chin. Phys. B* **20** 110506
[14] Zhou P and Ding R 2012 *Indian J. Phys.* **86** 497
[15] Odibat Z M 2010 *Nonlinear Dyn.* **60** 479
[16] Xin B G, Chen T and Liu Y Q 2011 *Commun. Nonlinear Sci. Numer. Simul.* **16** 4479
[17] Cafagna D and Grassi G 2012 *Nonlinear Dyn.* **68** 117
[18] Lu J G 2006 *Physica A* **359** 107
[19] Wang X Y, Zhang X P and Ma C 2012 *Nonlinear Dyn.* **69** 511
[20] Liu D and Yan X M 2009 *Acta Phys. Sin.* **58** 3747 (in Chinese)
[21] Huang L L and Qi X 2013 *Acta Phys. Sin.* **62** 080507 (in Chinese)
[22] Wang S and Yu Y G 2012 *Chin. Phys. Lett.* **29** 020505
[23] Zhang R X and Yang S P 2008 *Chin. Phys. B* **17** 4073
[24] Yu Y G and Li H X 2010 *Nonlinear Anal.* **11** 2456
[25] Wu C W, Yang T and Chua L O 1996 *Int. J. Bifur. Chaos* **6** 455
[26] Yang T and Chua L O 1996 *IEEE Trans. Circuits Syst. I* **43** 817
[27] Yang T and Chua L O 1996 *Int. J. Bifur. Chaos* **6** 2653
[28] Yang T, Wu C W and Chua L O 1997 *IEEE Trans. Circuits Syst. I* **44** 469
[29] Li Y, Chen Y Q and Podlubny I 2010 *Comput. Math. Appl.* **59** 1810
[30] Liu J K 2005 *Intelligence Control* (Beijing: Publishing House of Electronic Industory) (in Chinese)
[31] Toshir T, Kiyoji A and Michio S 1992 *Fuzzy Systems Theory and Its Applications* (Boston: Academic Press)
[32] Slotine J J E and Li W P 1991 *Applied Nonlinear Control* (Englewood Cliffs: Prentice-Hall)
[33] Konishi K, Hirai M and Kokame H 1998 *Phys. Lett. A* **245** 511
[34] Deng W H 2007 *J. Comput. Phys.* **227** 1510
[35] Grigorenko I and Grigorenko E 2003 *Phys. Rev. Lett.* **91** 034101
[36] Li C G, Liao X F and Yu J B 2003 *Phys. Rev. E* **68** 067203