Thermodynamics of a two-dimensional charged black hole in the geometric framework

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In this paper, we study the thermodynamic features of a two-dimensional charged black hole. Weinhold curvature and Ruppeiner curvature are explored as information geometry, respectively. Moreover, based on the Legendre invariant proposed by Hernando Quevedo, the geometro-thermodynamics behavior of this black hole is investigated.

Keywords: black hole, thermodynamic curvature, Legendre invariant

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1. Introduction

In 1973, Bekenstein proposed the supposition of a proportional relation between black hole entropy and its horizon area,[1] which is the beginning of what is now called thermodynamics of black holes. Subsequently, Hawking's discovery of the thermal radiation from black holes proved Bekenstein's supposition,[2] and the thermodynamic entropy of black holes was developed by Hawking *et al.*[3,4] by computing the thermodynamic partition function from the viewpoint of a path-integral in the saddle-point approximation. Since this work, some valuable studies have been conducted which have made the black hole thermodynamics become quite a mature subject over the last few decades.[5–10] In 1997, the black hole thermodynamic geometry was first explored by Ferrara *et al.*[11] They investigated the critical points of modulus space by using Weinhold metric and Ruppeiner metric,[12,13] which leads to a geometric approach in which the thermodynamic phase space and its natural contact structure are the basic elements of the construction. As is well known, an equilibrium thermodynamic system has some interesting geometric features. One of the features is that the inner product in the equilibrium thermodynamic state space in the energy representation which was provided by Weinhold can be treated as the Hessian matrix of the internal energy U with respect to the extensive thermodynamic variables N^a, namely[12] $W_{\mu\nu} = \partial_\mu \partial_\nu M(U, N^a)$. However, there was no physical explanation associated with this metric structure. As a modification, Ruppeiner[13] introduced Riemannian metric into the thermodynamic system once more and defined it as the second derivative of entropy S. Here, entropy is a function of internal energy, its extensive variables N^a, and is given by $R_{\mu\nu} = -\partial_\mu \partial_\nu S(U, N^a)$.

Interestingly, the above two metrics are conformally related and the conformal factor is the temperature[11] $W_{\mu\nu} = TR_{\mu\nu}$, $T = \partial M/\partial S$. This method was applied to all kinds of thermodynamic models, for example, the ideal gas, the van der Waals gas, the two-dimensional (2D) Fermi gas, and so on. The results showed that Ruppeiner geometry can overcome the difficulties of covariance and self-consistency of general thermodynamic, and it has phase structure information of the thermodynamic system. Based on Weinhold's theory and Ruppeiner's theory, we find that a different black hole family under various assumptions has numerous interesting results.[14–28] So it is natural to try to describe the phase transitions of black holes in terms of curvature singularities in the space of equilibrium states. Unfortunately, some results are contradictory. For instance, for Reissner–Nordström black hole or the three-dimensional charged-dilaton black hole, the Ruppeiner metric is flat, whereas the Weinhold metric presents a curvature singularity.[24,25] Nevertheless, a simple change of the thermodynamic potential[29] affects Ruppeiner's geometry in such a way that the resulting curvature singularity now corresponds to a phase transition. A dimensional reduction of Ruppeiner's curvature seems to affect its properties too.[30] However, as is well known, ordinary thermodynamics does not depend on the thermodynamic potential.

Recently, Quevedo[31] proposed a formalism of geometro-thermodynamics (GTD) as a geometric approach that incorporates Legendre invariance in a natural way, and allows us to derive Legendre invariant metrics in the space of equilibrium states. Since Weinhold and Ruppeiner metrics are not Legendre-invariant, one of the first results in the context of GTD was the derivation of simple Legendre-invariant generalization of these metrics and their application to black hole thermodynamics.[32–35] The results finished the controversy regarding the application of geometric structures in the black hole thermodynamics. The phase transition structure contained in the heat capacity of the black hole becomes completely integrated in the scalar curvature of the Legendre-invariant metric so that a curvature singularity corresponds to

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a phase transition.

In this paper, we first describe the thermodynamic properties, such as temperature and entropy, of the 2D charged black hole. Then, we reproduce the Weinhold information geometry and the Ruppeiner information geometry of the charged black hole, and also the Weinhold scalar curvature and the Ruppeiner scalar curvature. Moreover, the Geometro-thermodynamics of the charged black hole is also explored. Throughout the paper, the units $G = c = \hbar = 1$ are used.

2. Thermodynamics properties of the 2D charged black hole

The 2D Maxwell-dilaton action induced by the low energy heterotic string theory, which is given by[36,37]

$$I = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{-g}\,\mathrm{e}^{-2\varphi}\left[R + 4(\nabla\varphi)^2 + 4\Lambda^2 - \frac{1}{4}F^2\right], \quad (1)$$

where φ, Λ, and F are the dilaton field, cosmological constant, and Maxwell field tensor respectively. The equations of motion under the variations of the metric, dilaton, and gauge fields are given as follows:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 2\Big[\nabla_\mu\nabla_\nu\varphi + g_{\mu\nu}\Big((\nabla\varphi)^2 - \nabla^2\varphi - \Lambda^2 + \frac{1}{16}F^2\Big)\Big] = 0, \quad (2)$$

$$R + 4\nabla^2\varphi - 4(\nabla\varphi)^2 + 4\Lambda^2 - \frac{1}{4}F^2 = 0, \quad (3)$$

$$\partial_\nu(\sqrt{-g}\,\mathrm{e}^{-2\varphi}F^{\mu\nu}) = 0. \quad (4)$$

In the Schwarzschild gauge, the metric and field tensors are assumed to be

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2, \quad (5)$$

$$F_{rt} = F_{rt}(r). \quad (6)$$

Simple static solutions with suitable boundary conditions are known as follows:

$$\phi(r) = \frac{1}{4}\ln 2 - \Lambda r, \quad (7)$$

$$F_{rt} = \sqrt{2}Q\,\mathrm{e}^{-2\Lambda r}, \quad (8)$$

$$f(r) = 1 - \frac{M}{\Lambda}\mathrm{e}^{-2\Lambda r} + \frac{Q^2}{4\Lambda^2}\mathrm{e}^{-4\Lambda r}. \quad (9)$$

There are two coordinate singularities $r_\pm$ which correspond to the positions of the outer event horizon and the inner Cauchy horizon

$$r_\pm = \frac{1}{2\Lambda}\ln\left[\frac{M}{2\Lambda} \pm \sqrt{\left(\frac{M}{2\Lambda}\right)^2 - \left(\frac{Q}{2\Lambda}\right)^2}\right]. \quad (10)$$

Following Refs. [8] and [9], we find that the Bekenstein–Hawking entropy $S = r_+$, where r_+ is the radius of the outermost Killing horizon, i.e. r_+ satisfies Eq. (10). The mass can be obtained as a function of entropy S and charge Q in the form

$$M = \Lambda\,\mathrm{e}^{2\Lambda S} + \frac{Q^2}{4\Lambda}\mathrm{e}^{-2\Lambda S}. \quad (11)$$

According to the first law of black hole thermodynamics

$$\mathrm{d}M = T\,\mathrm{d}S + \Phi\,\mathrm{d}Q, \quad (12)$$

the expressions for the temperature and the electric potential can easily be computed as $T = \partial M/\partial S$, $\Phi = \partial M/\partial Q$. It is convenient to write the final results by using Eq. (11) as follows:

$$T = -\frac{1}{2}Q^2\,\mathrm{e}^{-2\Lambda S^2} + 2\Lambda^2\,\mathrm{e}^{2\Lambda S}, \quad (13)$$

and

$$\Phi = \frac{Q\,\mathrm{e}^{-2\Lambda\sqrt{S}}}{2\Lambda}. \quad (14)$$

The heat capacity for a fixed charge is

$$C_Q = -\frac{Q^2 - 4\Lambda^2\,\mathrm{e}^{4\Lambda S}}{2\Lambda(Q^2 + 4\Lambda^2\,\mathrm{e}^{\Lambda S})}. \quad (15)$$

Following the fundamentals of black hole thermodynamics as formulated by Davies,[38] the main thermodynamic properties of the charged black hole can be derived from the behavior of its thermodynamic variables M, T, Φ, and C_Q in terms of the extensive variables S and Q. We see that all the thermodynamic variables are well-behaved, except perhaps in the extremal limit $Q^2 = 4\Lambda^2\,\mathrm{e}^{4\Lambda S}$, where the heat capacity vanishes. Moreover, the heat capacity also contains the information about the local stability of the black hole thermodynamics. The fact that the heat capacity is always negative at $Q^2 > 4\Lambda^2\,\mathrm{e}^{4\Lambda S}$ implies that it is an unstable thermodynamics system. However, the heat capacity C_Q changes sign at $Q^2 < 4\Lambda^2\,\mathrm{e}^{4\Lambda S}$.

3. Information geometry of the 2D charged black hole

In this section, we first study Weinhold geometry of a 2D charged black hole. The Weinhold metric can be written as

$$\mathrm{d}s_{\mathrm{W}}^2 = \frac{\partial^2 M}{\partial S^2}\mathrm{d}S^2 + 2\frac{\partial^2 M}{\partial S\partial Q}\mathrm{d}S\mathrm{d}Q + \frac{\partial^2 M}{\partial Q^2}\mathrm{d}Q^2, \quad (16)$$

where the index W denotes the Weinhold information geometry. Here, we have made the choice that the mass M corresponds to the thermodynamic potential, with the extensive variables entropy S and charge Q existing. To describe the Weinhold information geometry, by using the relation between mass and electric charge, Eq. (10), the Weinhold metric components of the charged black hole can be expressed respectively as

$$g_{SS}^W = \Lambda(Q^2\,\mathrm{e}^{-2\Lambda S} + 4\Lambda^2\,\mathrm{e}^{2\Lambda S}), \quad (17)$$

$$g^W_{SQ} = -Q\mathrm{e}^{-2\Lambda S}, \tag{18}$$

$$g^W_{QQ} = \frac{\mathrm{e}^{-2\Lambda S}}{2\Lambda}. \tag{19}$$

Then we obtain the corresponding line element in the space of the Weinhold metric as

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{W}} =& \Lambda(4\Lambda^2\mathrm{e}^{2\Lambda S}+Q^2\mathrm{e}^{-2\Lambda S})\mathrm{d}S^2 \\ &-2Q\mathrm{e}^{-2\Lambda S}\mathrm{d}S\mathrm{d}Q+\frac{\mathrm{e}^{-2\Lambda S}}{2\Lambda}\mathrm{d}Q^2, \end{aligned} \tag{20}$$

from which it can be shown that the Weinhold scalar curvature in the entropy representation becomes

$$\mathfrak{R}_{\mathrm{W}} = -\frac{16\Lambda\,\mathrm{e}^{6\Lambda S}(16\Lambda^4(\Lambda^2+1)\mathrm{e}^{8\Lambda S}+(\Lambda^2+2)Q^4-4\Lambda^2(2\Lambda^2-7)Q^2\mathrm{e}^{4\Lambda S}}{(Q^2-4\Lambda^2\mathrm{e}^{4\Lambda S})^4}. \tag{21}$$

Obviously, the scalar curvature $\mathfrak{R}_{\mathrm{W}}$ naively diverges at the extreme limit of the charged black hole where $Q^2 = 4\Lambda^2\mathrm{e}^{4\Lambda S}$, the heat capacity vanishes, with the first-order transition points reproduced. It coincides with the result of analyzing the heat capacity as described above.

Now we turn to the Ruppeiner information geometry of the 2D charged black hole. Recalling the conformal relation between the Ruppeiner information geometry and the Weinhold information geometry, we obtain the Ruppeiner metric

$$\mathrm{d}s^2_{\mathrm{R}} = T^{-1}g^{\mathrm{W}} = \frac{\Lambda(2\Lambda\sqrt{S}-1)}{2S^2}\left(\Lambda^2-\frac{Q^2}{4}\right)\mathrm{d}S^2. \tag{22}$$

A straightforward calculation of the Ruppeiner scalar curvature for this metric is $\mathfrak{R}_{\mathrm{R}} = 0$. This behavior indicates that phase transition does not occur.

4. Geometro-thermodynamics of the 2D charged black hole

We now turn to use the recent geometric formulation of extended thermodynamic behavior of the 2D charged black hole. From the point of view of GTD,[31] equation (11) represents the fundamental relation $M = (S, Q)$, from which all the thermodynamic information can be obtained. Therefore, we consider a five-dimensional phase space $\mathfrak{I}$ with coordinates $\{M,\ S,\ T,\ Q,\ \Phi\}$, a contact 1-form

$$\Theta = \mathrm{d}M - T\,\mathrm{d}S - \Phi\,\mathrm{d}Q. \tag{23}$$

The space of equilibrium states is then defined by the smooth mapping

$$\begin{aligned} G =& (\mathrm{d}M - T\,\mathrm{d}S - \Phi\,\mathrm{d}Q)^2 \\ &+(TS+\Phi Q)(-\mathrm{d}T\,\mathrm{d}S+\mathrm{d}\Phi\,\mathrm{d}Q). \end{aligned} \tag{24}$$

The triplet $(\mathfrak{I}, \Theta, G)$ defines a contact Riemannian manifold that plays an auxiliary role in GTD. It is used to properly handle the invariance with respect to Legendre transformations. In fact, for the charged black hole, a Legendre transformation involves in general all the thermodynamic variables M, S, Q, T, and Φ so that they must be independent of each other as they are in the phase space. We introduce also the geometric structure of the space of equilibrium states $\mathcal{E}$ in the following manner: $\mathcal{E}$ is a 2D submanifold of $\mathfrak{I}$ that is defined by the smooth embedding map φ: $\mathcal{E} \mapsto \mathfrak{I}$, satisfying the condition that the "projection" of the contact form Θ on $\mathcal{E}$ vanishes, namely $\varphi^*(\Theta) = 0$, where φ^* is the pullback of φ, and that G induces a Legendre-invariant metric g on $\mathcal{E}$ by means of $\mathcal{E}$. In principle, any 2D subset of the set of coordinates of $\mathfrak{I}$ can be applied to coordinative $\mathcal{E}$. For simplicity, we will use the set of extensive variables S and Q which in ordinary thermodynamics corresponds to the energy representation. Then, the embedding map for this specific choice is

$$\varphi : \{S, Q\} \mapsto \{M(S,Q), S, Q, T(S,Q)\}. \tag{25}$$

The condition $\varphi^*(\Theta) = 0$ is equivalent to Eq. (12) (the first law of thermodynamics) and Eq. (11), Eq. (13) (the conditions of thermodynamic equilibrium), the induced metric is obtained as

$$g = \left(S\frac{\partial M}{\partial S}+Q\frac{\partial M}{\partial Q}\right)\left(-\frac{\partial^2 M}{\partial S^2}\mathrm{d}S^2+\frac{\partial^2 M}{\partial Q^2}\mathrm{d}Q^2\right). \tag{26}$$

This metric determines all the geometric properties of the equilibrium space $\mathcal{E}$. We see that in order to obtain the explicit form of the metric it is only necessary to specify the thermodynamic potential M as a function of S and Q.

As for the 2D charged black hole, the fundamental equation $M = M(S, Q)$ is obtained from the area-entropy relationship which is given in Eq. (11). Then, it is easy to compute the explicit form of the thermodynamic metric (26), using the expressions for S and Q, it can be rewritten as

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{L}} =& \frac{1}{2}\mathrm{e}^{-4\Lambda S}(Q^2+4\Lambda^2\mathrm{e}^{4\Lambda S}) \\ &\times(Q^2(S\Lambda-1)-4S\Lambda^3\mathrm{e}^{4\Lambda S})\mathrm{d}S^2 \\ &+\frac{1}{4\Lambda^2}\mathrm{e}^{-4S\Lambda}(4S\Lambda^2\mathrm{e}^{4S\Lambda}-Q^2(S\Lambda-1))\mathrm{d}Q^2. \end{aligned} \tag{27}$$

We obtain its curvature scalar, which can be expressed in terms of the entropy and electric charge as

$$\mathfrak{R}_{\mathrm{L}} = \frac{\mathbb{N}}{16\Lambda^2(Q^2+4\Lambda^2\mathrm{e}^{4\Lambda S})(4\Lambda^3 S\mathrm{e}^{4\Lambda S}+(1-\Lambda S)Q^2)^2}, \tag{28}$$

where

$$\mathbb{N} = \mathrm{e}^{-4\Lambda S}(-(\Lambda S-1)Q^{10}-1024\Lambda^{10}\mathrm{e}^{20\Lambda S}$$

$$\times(6\Lambda^3S^3+\Lambda^2S^2+6\Lambda S+1)-128\Lambda^6Q^4\,\mathrm{e}^{12\Lambda S}$$
$$\times(8\Lambda^3S^3-4\Lambda^2S^2-4\Lambda S-9)-4\Lambda^2Q^8\,\mathrm{e}^{4\Lambda S}$$
$$\times(10\Lambda^3S^3-41\Lambda^2S^2+50\Lambda S-19)+32\Lambda^4Q^6\,\mathrm{e}^{8\Lambda S}$$
$$\times(10\Lambda^3S^3-24\Lambda^2S^2+8\Lambda S+5)+256\Lambda^8Q^2\,\mathrm{e}^{16\Lambda S}$$
$$\times(12\Lambda^3S^3+\Lambda^2S^2-2\Lambda S+11)). \quad (29)$$

Obviously, the curvature scalar is rather cumbersome and cannot give a compact form. The general behavior of this curvature shown in the metric (27) describes a space of negative curvature which, in $Q^2\neq 4\Lambda^2\,\mathrm{e}^{4\Lambda S}$ and $4\Lambda^3S\,\mathrm{e}^{4\Lambda S}\neq Q^2-\Lambda SQ^2$, is free of singularities: this behavior indicates that the phase transitions do not occur.

5. Discussion and conclusions

For the 2D charged black hole, we reproduced the thermodynamic properties such as temperature and entropy, and derive the Weinhold metric and the Ruppeiner metric for the space of equilibrium states of the charged black hole. In all these cases, our results show that the thermodynamic curvature is in general different, indicating the presence of thermodynamic interaction. For instance, the scalar curvature $\Re_{\mathrm{W}}$ indicates the presence of the first-order transition points in $Q^2=4\Lambda^2\,\mathrm{e}^{4\Lambda S}$. Nevertheless, the thermodynamic curvatures $\Re_{\mathrm{R}}$ and $\Re_{\mathrm{L}}$ indicate that no phase transitions occur, and the scalar curvature $\Re_{\mathrm{R}}$ loses the information about charge Q, besides, its curvature must be non-zero except at the point $2\Lambda\sqrt{S}=1$.

In addition, the thermodynamic metric proposed in this paper is applied to the case of black hole configurations in four dimensions with and without the cosmological constant. It is shown that this thermodynamic metric correctly describes the thermodynamic behavior of the corresponding black hole configurations. One additional advantage of this thermodynamic metric is the invariance with respect to total Legendre transformations. This means that the results are independent of the thermodynamic potential used to generate the thermodynamic metric. A very interesting result is that it can recreate the lost information in the Ruppeiner metric by using Legendre transformation. We expect that this unified geometry description may give more information about a thermodynamic system.

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