Relativistic symmetries of Deng–Fan and Eckart potentials with Coulomb-like and Yukawa-like tensor interactions

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Relativistic symmetries of the Dirac equation under spin and pseudo-spin symmetries are investigated and a combination of Deng–Fan and Eckart potentials with Coulomb-like and Yukawa-like tensor interaction terms are considered. The energy equation is obtained by using the Nikiforov–Uvarov method and the corresponding wave functions are expressed in terms of the hypergeometric functions. The effects of the Coulomb and Yukawa tensor interactions are numerically discussed as well.

Keywords: Deng–Fan potential, Eckart potential, Nikiforov–Uvarov method, spin and pseudo-spin symmetries

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1. Introduction

The recent advances in the search for the solutions of the Dirac equation with physical potentials have led to the discovery of some new phenomena.[1] A particular example of such concepts is the the so-called spin and pseudo-spin symmetries[1] which have acceptably accounted for the experimental observation of the quasi-degeneracy in single-nucleon doublets between normal parity orbitals $(n,l,j=l+1/2)$ and $(n-1,l+2,j=l+3/2)$, where n, l and j represent the radial, orbital and total angular momentum quantum numbers, respectively. These symmetries have been used to form an effective nuclear shell model[2] and to analyze other different physical phenomena in the nuclear structure such as deformation, super-deformation, magnetic moments and identical bands.[3] Ginocchio[4] established the relationship between the pseudo-spin symmetry and the Dirac equation by recognizing that the pseudo-orbital angular momentum $\tilde{l}=l+1$ is in fact the usual orbital angular momentum l of the lower component of the Dirac spinor. His studies also revealed that in the Dirac theory, the spin symmetry occurs when the difference between the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant, that is, $\Delta(r)=V(r)-S(r)=$ const. On the other hand, the pseudo-spin symmetry arises when the sum of the repulsive Lorentz vector potential $V(r)$ and the attractive Lorentz scalar potential $S(r)$ is a constant, that is, $\Sigma(r)=V(r)+S(r)=$ const. In addition, the investigation of relativistic effects in the Dirac theory is not restricted to such cases and plays an important role in studying other quantum mechanical systems such as heavy atoms and heavy ion doping.[5] The symmetry limits of the Dirac equation have been already studied with a number of potentials including Woods–Saxon,[6] Manning–Rosen,[7] Eckart potential,[8] harmonic oscillator,[9] Hyllerass potential,[10] Deng–Fan potential,[11] and so on.[12] In such studies, a variety of analytical techniques such as the asymptotic iteration method,[13] the Nikiforov–Uvarov (NU) method[14] and the supersymmetric quantum mechanics,[15,16] have been used.[16] On the other hand, we are now almost sure that these symmetries have significant roles in nuclear and hadronic spectroscopy.[17–20] In such studies, the choice of the tensor interaction is a crucial point as the tensor term removes the degeneracy between the doublets.[21] In most of the studies, due to the mathematical structure of the problem, the tensor interaction is considered as Coulomb or Cornell potentials.[22] Hassanabadi *et al.*, for the first time, introduced the Yukawa potential as the tensor term.[23] Here, we consider a combination of Deng–Fan and Eckart potentials including Yukawa and Coulomb-like tensor terms. The Deng–Fan potential, which resembles the modified Morse and Manning–Rosen potentials, has been used to describe diatomic molecular energy spectra and electromagnetic transitions.[24,25] We intend to solve the problem in an approximate analytical manner and we first introduce our tool, i.e. the NU technique.

2. The Nikiforov–Uvarov method

The NU method can solve a second-order differential equation in the form of[14]

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most, of second order, and $\tilde{\tau}(s)$ is a first-order polynomial. To make the application of the NU method simpler and more direct, we introduce

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a more compact presentation of the idea. In order to do so, we rewrite Eq. (1) as[26]

$$\psi_n''(s)+\left(\frac{c_1-c_2 s}{s(1-c_3 s)}\right)\psi_n'(s)+\left(\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-c_3 s)^2}\right)\psi_n(s)=0, \tag{2}$$

in which

$$\psi_n(s)=\phi(s)y_n(s). \tag{3}$$

Comparing Eq. (1) with Eq. (2), we obtain the following identifications:

$$\tilde{\tau}(s)=c_1-c_2 s,\ \ \sigma(s)=s(1-c_3 s),\ \ \tilde{\sigma}(s)=-\xi_1 s^2+\xi_2 s-\xi_3. \tag{4}$$

Following the NU method,[14,26] we obtain the following required parameters:

(i) the relevant constants

$$\begin{aligned}
&c_4=\frac{1}{2}(1-c_1),\quad c_5=\frac{1}{2}(c_2-2c_3),\\
&c_6=c_5^2+\xi_1,\quad c_7=2c_4c_5-\xi_2,\\
&c_8=c_4^2+\xi_3,\quad c_9=c_3c_7+c_3^2c_8+c_6,\\
&c_{10}=c_1+2c_4+2\sqrt{c_8},\quad c_{11}=c_2-2c_5+2(\sqrt{c_9}+c_3\sqrt{c_8}),\\
&c_{12}=c_4+\sqrt{c_8},\quad c_{13}=c_5-(\sqrt{c_9}+c_3\sqrt{c_8});
\end{aligned} \tag{5}$$

(ii) the essential polynomial functions

$$\pi(s)=c_4+c_5 s-[(\sqrt{c_9}+c_3\sqrt{c_8})s-\sqrt{c_8}], \tag{6}$$

$$k=-(c_7+2c_3c_8)-2\sqrt{c_8c_9}, \tag{7}$$

$$\tau(s)=c_1+2c_4-(c_2-2c_5)s-2[(\sqrt{c_9}+c_3\sqrt{c_8})s-\sqrt{c_8}], \tag{8}$$

$$\tau'(s)=-2c_3-2(\sqrt{c_9}+c_3\sqrt{c_8})<0; \tag{9}$$

(iii) the energy equation

$$\begin{aligned}
&c_2 n-(2n+1)c_5+(2n+1)(\sqrt{c_9}+c_3\sqrt{c_8})+n(n-1)c_3\\
&+c_7+2c_3c_8+2\sqrt{c_8c_9}=0;
\end{aligned} \tag{10}$$

(iv) the wave functions

$$\rho(s)=s^{c_{10}}(1-c_3 s)^{c_{11}}, \tag{11}$$

$$\varphi(s)=s^{c_{12}}(1-c_3 s)^{c_{13}},\quad c_{12}>0,\quad c_{13}>0, \tag{12}$$

$$y_n(s)=\mathrm{P}_n^{(c_{10},c_{11})}(1-2c_3 s),\quad c_{10}>-1,\quad c_{11}>-1, \tag{13}$$

$$\begin{aligned}
\psi_{n\kappa}(s)&=N_{n\kappa}s^{c_{12}}(1-c_3 s)^{-c_{12}-c_{13}/c_3}\\
&\times\mathrm{P}_n^{(c_{10}-1,c_{11}/c_3-c_{10}-1)}(1-2c_3 s),
\end{aligned} \tag{14}$$

where $\mathrm{P}_n^{(\mu,\nu)}(x)$, $\mu>-1$, $\nu>-1$, and $x\in[-1,1]$ are Jacobi polynomials with

$$\mathrm{P}_n^{(\alpha,\beta)}(1-2s)=\frac{(\alpha+1)_n}{n!}{}_2\mathrm{F}_1(-n,1+\alpha+\beta+n;\alpha+1;s), \tag{15}$$

and $N_{n\kappa}$ is a normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function via

$$\begin{aligned}
\psi_{n\kappa}(s)\ &=\ N_{n\kappa}s^{c_{12}}(1-c_3 s)^{c_{13}}\\
&\times{}_2\mathrm{F}_1(-n,1+c_{10}+c_{11}+n;c_{10}+1;c_3 s),
\end{aligned} \tag{16}$$

with $c_{12}>0$, $c_{13}>0$ and $s\in[0,1/c_3]$, $c_3\neq 0$. This method has been extensively used to solve various second-order differential equations in quantum mechanics such as Schrödinger, Klein–Gordon, Duffin–Kemmer–Petiau, spinless-Salpeter and Dirac equations.[17–21]

3. Theory of the Dirac equation

The Dirac equation for spin-1/2 particles moving in an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$, and a tensor potential $U(r)$ in the relativistic unit ($\hbar=c=1$) is[20]

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta(M+S(r))-\mathrm{i}\beta\boldsymbol{\alpha}\cdot\hat{r}U(r)]\,\psi(r)=[E-V(r)]\,\psi(r), \tag{17}$$

where E is the relativistic energy of the system; $\boldsymbol{p}=-\mathrm{i}\nabla$ is the three-dimensional momentum operator; M is the mass of the fermionic particle; $\boldsymbol{\alpha}$ and β are the 4×4 Dirac matrices given as

$$\boldsymbol{\alpha}=\begin{pmatrix}0&\boldsymbol{\sigma}_i\\ \boldsymbol{\sigma}_i&0\end{pmatrix},\quad \beta=\begin{pmatrix}I&0\\0&-I\end{pmatrix}, \tag{18}$$

with I being the 2×2 unitary matrix, and $\boldsymbol{\sigma}_i$ the Pauli three-vector matrices:

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{19}$$

The eigenvalues of the spin–orbit coupling operator are $\kappa=(j+1/2)>0$, $\kappa=-(j+1/2)<0$ for unaligned $j=l-1/2$ and the aligned spin $j=l+1/2$, respectively. The set $\left(H,K,J^2,J_z\right)$ forms a complete set of conserved quantities. Thus, we can write the spinors as,[21]

$$\psi_{n\kappa}(r)=\frac{1}{r}\begin{pmatrix}F_{n\kappa}(r)\mathrm{Y}_{jm}^{l}(\theta,\phi)\\ \mathrm{i}G_{n\kappa}(r)\mathrm{Y}_{jm}^{\tilde{l}}(\theta,\phi)\end{pmatrix}, \tag{20}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ represent the upper and lower components of the Dirac spinors respectively, $\mathrm{Y}_{jm}^{l}(\theta,\varphi)$ and $\mathrm{Y}_{jm}^{\tilde{l}}(\theta,\varphi)$ are the spin and pseudo-spin spherical harmonics respectively, and m is the projection on the z axis. Using the well-known identities,[24]

$$\begin{aligned}
&(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B})=\boldsymbol{A}\cdot\boldsymbol{B}+\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}),\\
&\boldsymbol{\sigma}\cdot\boldsymbol{p}=\boldsymbol{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\boldsymbol{p}+\mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}\right),
\end{aligned} \tag{21}$$

as well as the relations

$$\begin{aligned}
&(\boldsymbol{\sigma}\cdot\boldsymbol{L})\,\mathrm{Y}_{jm}^{\tilde{l}}(\theta,\varphi)=(\kappa-1)\mathrm{Y}_{jm}^{\tilde{l}}(\theta,\varphi),\\
&(\boldsymbol{\sigma}\cdot\boldsymbol{L})\,\mathrm{Y}_{jm}^{l}(\theta,\varphi)=-(\kappa-1)\mathrm{Y}_{jm}^{l}(\theta,\varphi),\\
&(\boldsymbol{\sigma}\cdot\hat{r})\,\mathrm{Y}_{jm}^{l}(\theta,\varphi)=-\mathrm{Y}_{jm}^{\tilde{l}}(\theta,\varphi),\\
&(\boldsymbol{\sigma}\cdot\hat{r})\,\mathrm{Y}_{jm}^{\tilde{l}}(\theta,\varphi)=-\mathrm{Y}_{jm}^{l}(\theta,\varphi),
\end{aligned} \tag{22}$$

we find the following two coupled first-order equations:[20]

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)=(M+E_{n\kappa}-\Delta(r))\,G_{n\kappa}(r), \quad (23)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r)=(M-E_{n\kappa}+\Sigma(r))\,F_{n\kappa}(r), \quad (24)$$

where,

$$\Delta(r)=V(r)-S(r), \quad (25)$$

$$\Sigma(r)=V(r)+S(r). \quad (26)$$

Eliminating one component in favor of the other, we find the second-order Schrödinger-like equations

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa U(r)}{r}-\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)-(M+E_{n\kappa}-\Delta(r))\,(M-E_{n\kappa}+\Sigma(r))+\frac{\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)}{(M+E_{n\kappa}-\Delta(r))}\right\}F_{n\kappa}(r)=0, \quad (27)$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa U(r)}{r}+\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)-(M+E_{n\kappa}-\Delta(r))\,(M-E_{n\kappa}+\Sigma(r))+\frac{\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)}{(M-E_{n\kappa}+\Sigma(r))}\right\}G_{n\kappa}(r)=0, \quad (28)$$

where $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$, $\kappa(\kappa+1)=l(l+1)$.

4. Symmetries with Coulomb-like tensor interaction

In this section, we will work on the Coulomb term as the tensor interaction.

4.1. Pseudo-spin symmetry with Coulomb-like tensor interaction

The pseudo-spin symmetry occurs in the Dirac equation when $\mathrm{d}\Sigma(r)/\mathrm{d}r=0$ or equivalently $\Sigma(r)=C_{\mathrm{ps}}=\text{const.}$[2–4] We consider the sum of the scalar and vector potentials as,[27]

$$\Delta(r)=D\left(1-\frac{b}{\mathrm{e}^{\alpha r}-1}\right)^2-\frac{V_1\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})}+\frac{V_2\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2},$$
$$b=\mathrm{e}^{\alpha r_c}-1, \quad (29)$$

where $r\in(0,\infty)$ and the other parameters D, r_{e}, V_1, V_2, and α denote the dissociation energy, the equilibrium inter-nuclear distance, the depths of the potential well, and the range of the potential well, respectively. For the tensor term, we consider

$$U(r)=-\frac{H}{r}, \quad r\geq R_{\mathrm{e}}, \quad (30)$$

with

$$H=\frac{z_a z_b\,\mathrm{e}^2}{4\pi\varepsilon_0}, \quad (31)$$

where R_{e} is the Coulomb radius, z_a and z_b denote the charges of the projectile nucleus and the target nucleus, respectively.[21]

Substituting the above equations into Eq. (27), we find

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\Lambda_\kappa(\Lambda_\kappa-1)}{r^2}-\left(M+E^{\mathrm{ps}}_{n,\kappa}\right)\left(M-E^{\mathrm{ps}}_{n,\kappa}+C_{\mathrm{ps}}\right)+\left(M-E^{\mathrm{ps}}_{n,\kappa}+C_{\mathrm{ps}}\right)\left[D\left(1-\frac{b}{\mathrm{e}^{\alpha r}-1}\right)^2-\frac{V_1\,\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}+\frac{V_2\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}\right]\right\}G^{\mathrm{ps}}_{n,\kappa}(r)=0, \quad (32)$$

where $\Lambda_\kappa=(\kappa+H)$.

As is well known, the above equation cannot be solved exactly due to the presence of the centrifugal term r^{-2}. In order to obtain the arbitrary-state solution, we make use of the following approximation:[28]

$$\frac{1}{r^2}\approx\frac{\omega\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}+\frac{\lambda\,\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}, \quad (33)$$

where ω and λ are the adjustable parameters. Substituting Eq. (33) into Eq. (32) and using the transformation $s=\mathrm{e}^{-\alpha r}$, equation (32) becomes

$$\frac{\mathrm{d}^2G^{\mathrm{ps}}_{n,\kappa}}{\mathrm{d}r^2}+\frac{(1-s)}{s(1-s)}\frac{\mathrm{d}G^{\mathrm{ps}}_{n,\kappa}}{\mathrm{d}r}+\frac{1}{s^2(1-s)^2}\left[-\chi^{\mathrm{ps}}_1 s^2+\chi^{\mathrm{ps}}_2 s-\chi^{\mathrm{ps}}_3\right]G^{\mathrm{ps}}_{n,\kappa}=0, \quad (34)$$

where

$$\chi^{\mathrm{ps}}_1=\frac{1}{\alpha^2}\left(\varepsilon^2_{\mathrm{ps}}+\lambda\Lambda_\kappa(\Lambda_\kappa-1)-\beta_{\mathrm{ps}}D(1+b)^2-\beta_{\mathrm{ps}}V_1\right), \quad (35)$$

$$\chi^{\mathrm{ps}}_2=\frac{1}{\alpha^2}\left(2\varepsilon^2_{\mathrm{ps}}+\beta_{\mathrm{ps}}V_2-\omega\Lambda_\kappa(\Lambda_\kappa-1)-2D\beta_{\mathrm{ps}}(1+b)-\beta_{\mathrm{ps}}V_1\right), \quad (36)$$

$$\chi^{\mathrm{ps}}_3=\frac{1}{\alpha^2}\left(\varepsilon^2_{\mathrm{ps}}-\beta_{\mathrm{ps}}D\right), \quad (37)$$

$$\varepsilon^2_{\mathrm{ps}}=\left(M+E^{\mathrm{ps}}_{n,\kappa}\right)\beta_{\mathrm{ps}}, \quad \beta_{\mathrm{ps}}=\left(M-E^{\mathrm{ps}}_{n,\kappa}+C_{\mathrm{ps}}\right). \quad (38)$$

4.2. Spin symmetry with Coulomb-like tensor interaction

In the spin symmetry limit of this sub-section, we use

$$\Sigma(r)=D\left(1-\frac{b}{\mathrm{e}^{\alpha r}-1}\right)^2-\frac{V_1\,\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}+\frac{V_2\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2},$$
$$\Delta(r)=C_{\mathrm{s}}, \quad U(r)=-\frac{H}{r}. \quad (39)$$

Substitution of Eq. (39) into Eq. (27) yields,

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\eta_\kappa(\eta_\kappa-1)}{r^2}-\varepsilon_{\rm s}^2-\beta_{\rm s}\left[D\left(1-\frac{b\,\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}\right)^2-\frac{V_1\,\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}+\frac{V_2\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}\right]\right\}F^{\rm s}_{n,k}(r)=0, \quad (40)$$

$$\eta_\kappa=(\kappa+H+1),$$
$$\varepsilon_{\rm s}^2=\left(M-E^{\rm s}_{n,k}\right)\left(M+E^{\rm s}_{n,k}-C_{\rm s}\right),$$
$$\beta_{\rm s}=\left(M+E^{\rm s}_{n,k}-C_{\rm s}\right). \quad (41)$$

By applying the approximation of Eq. (33) to the centrifugal term in Eq. (40) and applying $z=\mathrm{e}^{-\alpha r}$, we have

$$\frac{\mathrm{d}^2F^{\rm s}_{n,k}}{\mathrm{d}r^2}+\frac{(1-z)}{z(1-z)}\frac{\mathrm{d}F^{\rm s}_{n,k}}{\mathrm{d}r}+\frac{1}{z^2(1-z)^2}\times\left[-Q^{\rm s}_1z^2+Q^{\rm s}_2z-Q^{\rm s}_3\right]F^{\rm s}_{n,k}(z)=0, \quad (42)$$

where,

$$Q^{\rm s}_1=\left(\frac{\varepsilon_{\rm s}^2}{\alpha^2}+\frac{\beta_{\rm s}D(1+b)^2}{\alpha^2}+\frac{\eta_\kappa(\eta_\kappa-1)\lambda}{\alpha^2}+\frac{\beta_{\rm s}V_1}{\alpha^2}\right), \quad (43)$$

$$Q^{\rm s}_2=\left(\frac{2\varepsilon_{\rm s}^2}{\alpha^2}+\frac{2\beta_{\rm s}D(1+b)}{\alpha^2}+\frac{\beta_{\rm s}V_1}{\alpha^2}-\frac{\eta_\kappa(\eta_\kappa-1)\omega}{\alpha^2}-\frac{\beta_{\rm s}V_2}{\alpha^2}\right), \quad (44)$$

$$Q^{\rm s}_3=\left(\frac{\varepsilon_{\rm s}^2}{\alpha^2}+\frac{\beta_{\rm s}D}{\alpha^2}\right). \quad (45)$$

4.3. Approximate analytical solutions with Coulomb-like tensor interaction

In this section, we will solve Eqs. (34) and (42) by using the parametric generalization of the NU method.[26]

4.3.1. Pseudo-spin symmetry solution with Coulomb-like tensor interaction

Comparing Eq. (2) with Eq. (34), our first required parameters are determined as

$$c_1=1,\ c_2=1,\ c_3=1,\ \xi_1=\chi^{\rm s}_1,\ \xi_2=\chi^{\rm s}_2,\ \xi_3=\chi^{\rm s}_3, \quad (46)$$

and the other parameters can be obtained from Eq. (5) as

$$c_4=0,\ c_5=-\frac{1}{2},\ c_6=\frac{1}{4}+\chi^{\rm ps}_1,\ c_7=-\chi^{\rm ps}_2,$$
$$c_8=\chi^{\rm ps}_3,\ c_9=\frac{1}{4}+\chi^{\rm ps}_1+\chi^{\rm ps}_3-\chi^{\rm ps}_2,\ c_{10}=1+2\sqrt{\chi^{\rm ps}_3},$$
$$c_{11}=2+2\left[\sqrt{\frac{1}{4}+\chi^{\rm ps}_1+\chi^{\rm ps}_3-\chi^{\rm ps}_2}+\sqrt{\chi^{\rm ps}_3}\right],\ c_{12}=\sqrt{\chi^{\rm ps}_3},$$
$$c_{13}=\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\chi^{\rm ps}_1+\chi^{\rm ps}_3-\chi^{\rm ps}_2}+\sqrt{\chi^{\rm ps}_3}\right). \quad (47)$$

Substituting Eqs. (46) and (47) into Eq. (10), we find the energy relation as

$$n^2+(n+\frac{1}{2})+(2n+1)\left[\sqrt{\frac{1}{4}+\chi^{\rm ps}_1+\chi^{\rm ps}_3-\chi^{\rm ps}_2}+\sqrt{\chi^{\rm ps}_3}\right]-\chi^{\rm ps}_2+2\chi^{\rm ps}_2+2\sqrt{\chi^{\rm ps}_3\left(\frac{1}{4}+\chi^{\rm ps}_1+\chi^{\rm ps}_3-\chi^{\rm ps}_2\right)}=0. \quad (48)$$

From Eqs. (14) and (15) the lower component of the wave function is

$$G^{\rm ps}_{n,\kappa}(r)=N^{\rm ps}_{n,\kappa}\,\mathrm{e}^{-\alpha\sqrt{\chi^{\rm ps}_3}r}\left(1-\mathrm{e}^{-\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\chi^{\rm ps}_1+\chi^{\rm ps}_3-\chi^{\rm ps}_2}}\times P_n^{\left(2\sqrt{\chi^{\rm ps}_3},2\sqrt{\frac{1}{4}+\chi^{\rm ps}_1+\chi^{\rm ps}_3-\chi^{\rm ps}_2}\right)}(1-2\,\mathrm{e}^{-\alpha r}), \quad (49)$$

and the other component can be simply obtained from

$$F^{\rm ps}_{n,\kappa}(r)=\frac{1}{M-E^{\rm ps}_{n,\kappa}+C_{\rm ps}}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}-\frac{H}{r}\right)G^{\rm ps}_{n,\kappa}(r), \quad (50)$$

where $N^{\rm ps}_{n,\kappa}$ is the normalization constant and $E^{\rm ps}_{n,\kappa}\neq M+C_{\rm ps}$.

4.3.2. Spin symmetry solution with Coulomb-like tensor interaction

In order to avoid repetition, we simply mention that we follow the same steps from Eq. (34) to Eq. (42). Then, the energy equation and the wave function respectively are

$$n^2+\left(n+\frac{1}{2}\right)+(2n+1)\left[\sqrt{\frac{1}{4}+Q^{\rm s}_1+Q^{\rm s}_3-Q^{\rm s}_2}+\sqrt{Q^{\rm s}_3}\right]-Q^{\rm s}_2+2Q^{\rm s}_3+2\sqrt{Q^{\rm s}_3\left(\frac{1}{4}+Q^{\rm s}_1+Q^{\rm s}_3-Q^{\rm s}_2\right)}=0, \quad (51)$$

$$F^{\rm s}_{n,\kappa}(r)=N^{\rm s}_{n,\kappa}\,\mathrm{e}^{-\alpha\sqrt{Q^{\rm s}_3}r}\left(1-\mathrm{e}^{-\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+Q^{\rm s}_1+Q^{\rm s}_3-Q^{\rm s}_2}}\times P_n^{\left(2\sqrt{Q^{\rm s}_3},2\sqrt{\frac{1}{4}+Q^{\rm s}_1+Q^{\rm s}_3-Q^{\rm s}_2}\right)}(1-2\,\mathrm{e}^{-\alpha r}), \quad (52)$$

where $N^{\rm s}_{n,\kappa}$ is the normalization constant and the other component of the Dirac spinor is obtained from

$$G^{\rm s}_{n,\kappa}(r)=\frac{1}{M+E^{\rm s}_{n,\kappa}-C_{\rm s}}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}+\frac{H}{r}\right)F^{\rm s}_{n,\kappa}(r). \quad (53)$$

5. Pseudo-spin and spin symmetry limits with Yukawa-like tensor interaction

In this section, we intend to investigate the Dirac equation with the combination of Deng–Fan and Eckart potentials in the presence of a Yukawa-like tensor interaction.

5.1. Pseudo-spin symmetry with Yukawa tensor interaction

In this part we consider

$$\Delta(r)=D\left(1-\frac{b}{\mathrm{e}^{2\alpha r}-1}\right)^2-\frac{V_1\,\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}+\frac{V_2\,\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}, \quad (54)$$

$$U(r)=-\frac{V_0\,\mathrm{e}^{-\alpha r}}{r}. \quad (55)$$

Inserting Eqs. (54) and (55) into Eq. (28) we find the differential equation

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}-\frac{2\kappa V_0\,\mathrm{e}^{-\alpha r}}{r^2}+\frac{\alpha V_0\,\mathrm{e}^{-\alpha r}}{r}+\frac{V_0\,\mathrm{e}^{-\alpha r}}{r^2}\right.$$

$$-\frac{V_0^2\,\mathrm{e}^{-2\alpha r}}{r^2}\Big\}G^{\rm ps}_{n,\kappa}(r)+\left(-\varepsilon^2_{\rm ps}-\beta_{\rm ps}\left[-D\left(1-\frac{b\,\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right)^2\right.\right.$$
$$\left.\left.+\frac{V_1\,\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}-\frac{V_2\,\mathrm{e}^{-2\alpha r}}{\left(1-\mathrm{e}^{-2\alpha r}\right)^2}\right]\right)G^{\rm ps}_{n,\kappa}(r)=0, \tag{56}$$

where $\varepsilon^2_{\rm ps}=\left(M+E^{\rm ps}_{n,\kappa}\right)\left(M-E^{\rm ps}_{n,\kappa}+C_{\rm ps}\right)$ and $\beta_{\rm ps}=\left(M-E^{\rm ps}_{n,\kappa}+C_{\rm ps}\right)$. Just as indicated in the previous section, equation (56) cannot be exactly solved and we ought to use the approximations[29]

$$\frac{1}{r^2}\approx\frac{4\alpha^2\,\mathrm{e}^{-2\alpha r}}{\left(1-\mathrm{e}^{-2\alpha r}\right)^2}, \tag{57}$$

$$\frac{1}{r^2}\approx\frac{4\alpha^2\,\mathrm{e}^{-\alpha r}}{\left(1-\mathrm{e}^{-2\alpha r}\right)^2}. \tag{58}$$

We show the behaviour of the above approximations in Fig. 1, where we can see the acceptable agreement for small α values. Substituting Eqs. (57) and (58) into Eq. (40) and using the transformation $s=\mathrm{e}^{-2\alpha r}$, we obtain

$$\frac{\mathrm{d}G^{\rm ps}_{n,\kappa}}{\mathrm{d}s^2}+\frac{(1-s)}{s(1-s)}\frac{\mathrm{d}G^{\rm ps}_{n,\kappa}}{\mathrm{d}s}+\frac{1}{s^2(1-s)^2}$$
$$\times\left[\rho^{\rm ps}_1 s^2+\rho^{\rm ps}_2 s+\rho^{\rm ps}_3\right]G^{\rm ps}_{n,\kappa}(s)=0, \tag{59}$$

where

$$\rho^{\rm ps}_1=\left(-\frac{\varepsilon^2_{\rm ps}}{4\alpha^2}+\frac{\beta_{\rm ps}D(1+b)^2}{4\alpha^2}+\frac{\beta_{\rm ps}V_1}{4\alpha^2}-V_0\left(V_0+\frac{1}{2}\right)\right), \tag{60}$$

$$\rho^{\rm ps}_2=\left(-\kappa(\kappa-1)-\left(2\kappa-\frac{3}{2}\right)V_0+\frac{\beta_{\rm ps}V_2}{4\alpha^2}\right.$$
$$\left.-\frac{\beta_{\rm ps}D(1+b)}{\alpha^2}+\frac{\varepsilon^2_{\rm ps}}{2\alpha^2}-\frac{\beta_{\rm ps}V_1}{4\alpha^2}\right), \tag{61}$$

$$\rho^{\rm ps}_3=\left(\frac{\beta_{\rm ps}D}{4\alpha^2}-\frac{\varepsilon^2_{\rm ps}}{4\alpha^2}\right). \tag{62}$$

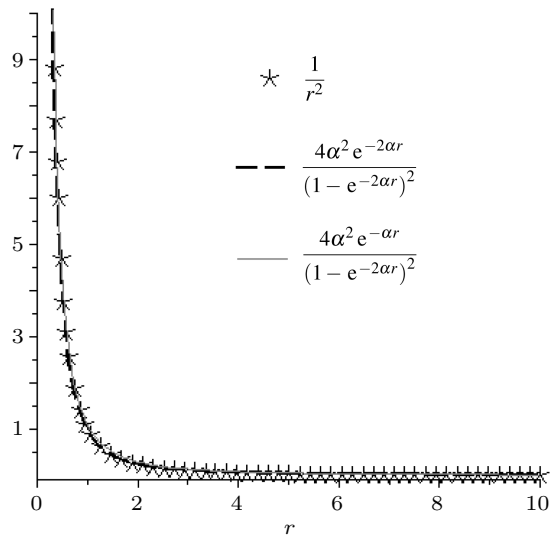


Fig. 1. $1/r^2$ and its approximation ($\alpha=0.1$).

5.2. Spin symmetry with Yukawa-like tensor interaction

In the spin symmetry limit case, we use the following scalar, vector, and tensor potentials:

$$\Delta(r)=C_{\rm s},$$
$$\Sigma(r)=D\left(1-\frac{b}{\mathrm{e}^{2\alpha r}-1}\right)^2-\frac{V_1\,\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}+\frac{V_2\,\mathrm{e}^{-2\alpha r}}{\left(1-\mathrm{e}^{-2\alpha r}\right)^2},$$
$$U(r)=-\frac{V_0\,\mathrm{e}^{-\alpha r}}{r}, \tag{63}$$

which corresponds to

$$\frac{\mathrm{d}^2F^{\rm s}_{n,\kappa}}{\mathrm{d}r^2}+\frac{(1-s)}{s(1-s)}\frac{\mathrm{d}F^{\rm s}_{n,\kappa}}{\mathrm{d}r}+\frac{1}{s^2(1-s)^2}$$
$$\times\left[-\delta^{\rm s}_1 s^2+\delta^{\rm s}_2 s-\delta^{\rm s}_3\right]F^{\rm s}_{n,\kappa}(s)=0, \tag{64}$$

with

$$\delta^{\rm s}_1=\left(\frac{\varepsilon^2_{\rm s}}{4\alpha^2}+\frac{\beta_{\rm s}D(1+b)^2}{4\alpha^2}+\frac{\beta_{\rm s}V_1}{4\alpha^2}+V_0\left(V_0-\frac{1}{2}\right)\right), \tag{65}$$

$$\delta^{\rm s}_2=\left(\frac{\varepsilon^2_{\rm s}}{2\alpha^2}+\frac{2\beta_{\rm s}D(1+b)}{4\alpha^2}+\frac{\beta_{\rm ss}V_1}{4\alpha^2}-\frac{\beta_{\rm s}V_2}{4\alpha^2}\right.$$
$$\left.-\kappa(\kappa+1)-\left(2\kappa+\frac{3}{2}\right)V_0\right), \tag{66}$$

$$\delta^{\rm s}_3=\left(\frac{\varepsilon^2_{\rm s}}{4\alpha^2}+\frac{\beta_{\rm s}D}{4\alpha^2}\right). \tag{67}$$

6. Pseudo-spin and spin symmetry approximate analytical solutions with Yukawa-like tensor interaction

In this section, we intend to investigate the solutions of Eqs. (59) and (64) for pseudo-spin and spin symmetries.

6.1. Pseudo-spin symmetry solution with Yukawa-like tensor interaction

Comparing Eq. (59) with Eq. (2), we obtain

$$c_1=1,\ \ c_2=1,\ \ c_3=1,$$
$$\xi_1=-\rho^{\rm ps}_1,\ \ \xi_2=\rho^{\rm ps}_2,\ \ \xi_3=-\rho^{\rm ps}_3, \tag{68}$$

and the other parameters are

$$c_4=0,\ \ c_5=-\frac{1}{2},\ \ c_6=\frac{1}{4}-\rho^{\rm ps}_1,\ \ c_7=-\rho^{\rm ps}_2,\ \ c_8=-\rho^{\rm ps}_3,$$
$$c_9=\frac{1}{4}-\rho^{\rm ps}_1-\rho^{\rm ps}_3-\rho^{\rm ps}_2,\ \ c_{10}=1+2\sqrt{-\rho^{\rm ps}_3},$$
$$c_{11}=2+2\left[\sqrt{\frac{1}{4}-\rho^{\rm ps}_1-\rho^{\rm ps}_3-\rho^{\rm ps}_2}+\sqrt{-\rho^{\rm ps}_3}\right],$$
$$c_{12}=\sqrt{-\rho^{\rm ps}_3},$$
$$c_{13}=-\frac{1}{2}-\left[\sqrt{\frac{1}{4}-\rho^{\rm ps}_1-\rho^{\rm ps}_3-\rho^{\rm ps}_2}+\sqrt{-\rho^{\rm ps}_3}\right]. \tag{69}$$

Substituting Eqs. (68) and (69) into Eq. (10) gives the energy relation as

$$n(n+1)+\frac{1}{2}+(2n+1)\left(\sqrt{\frac{1}{4}-\rho^{\rm ps}_1-\rho^{\rm ps}_3-\rho^{\rm ps}_2}+\sqrt{-\rho^{\rm ps}_3}\right)$$

$$-\rho_2^{\rm ps}-2\rho_3^{\rm ps}+2\sqrt{-\rho_3^{\rm ps}\left(\frac{1}{4}-\rho_1^{\rm ps}-\rho_3^{\rm ps}-\rho_2^{\rm ps}\right)}=0. \quad (70)$$

From Eq. (16) the lower and upper components of the wave function are obtained as follows:

$$G^{\rm ps}_{n,\kappa}(r)=N^{\rm ps\prime}_{n,\kappa}\,{\rm e}^{-2\alpha\sqrt{-\rho_3^{\rm ps}}r}\left(1-{\rm e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}-\rho_1^{\rm ps}-\rho_3^{\rm ps}-\rho_2^{\rm ps}}} \times P_n^{\left(2\sqrt{-\rho_3^{\rm ps}},2\sqrt{\frac{1}{4}-\rho_1^{\rm ps}-\rho_3^{\rm ps}-\rho_2^{\rm ps}}\right)}\left(1-{\rm e}^{-2\alpha r}\right), \quad (71)$$

$$F^{\rm ps}_{n,\kappa}(r)=\frac{1}{M-E^{\rm ps}_{n,\kappa}+C_{\rm ps}}\left(\frac{\rm d}{{\rm d}r}-\frac{\kappa}{r}+U(r)\right)G^{\rm ps}_{n,\kappa}(r), \quad (72)$$

where $E^{\rm ps}_{n,\kappa}\neq M+C_{\rm ps}$.

6.2. Spin symmetry solution with Yukawa-like tensor interaction

Comparing Eq. (64) with Eq. (2), we obtain

$$c_1=1,\ \ c_2=1,\ \ c_3=1,$$
$$\xi_1=\delta_1^{\rm s},\ \ \xi_2=\delta_2^{\rm s},\ \ \xi_3=\delta_3^{\rm s}, \quad (73)$$

and

$$c_4=0,\ \ c_5=-\frac{1}{2},\ \ c_6=\frac{1}{4}+\delta_1^{\rm s},\ \ c_7=-\delta_2^{\rm s},\ \ c_8=\delta_3^{\rm s},$$
$$c_9=\frac{1}{4}+\delta_1^{\rm s}+\delta_3^{\rm s}-\delta_2^{\rm s},\ \ c_{10}=1+2\sqrt{\delta_3^{\rm s}}, \quad (74)$$
$$c_{11}=2+2\left[\sqrt{\frac{1}{4}+\delta_1^{\rm s}+\delta_3^{\rm s}-\delta_2^{\rm s}}+\sqrt{\delta_3^{\rm s}}\right],\ \ c_{12}=\sqrt{\delta_3^{\rm s}},$$
$$c_{13}=-\frac{1}{2}-\left[\sqrt{\frac{1}{4}+\delta_1^{\rm s}+\delta_3^{\rm s}-\delta_2^{\rm s}}+\sqrt{\delta_3^{\rm s}}\right]. \quad (75)$$

Substituting Eqs. (75) and (74) into Eq. (10) gives the energy eigenvalues as

$$n(n+1)+\frac{1}{2}+(2n+1)\left(\sqrt{\frac{1}{4}+\delta_1^{\rm s}+\delta_3^{\rm s}-\delta_2^{\rm s}}+\sqrt{\delta_3^{\rm s}}\right)$$
$$-\delta_2^{\rm s}+2\delta_3^{\rm s}+2\sqrt{\delta_3^{\rm s}\left(\frac{1}{4}+\delta_1^{\rm s}+\delta_3^{\rm s}-\delta_2^{\rm s}\right)}=0. \quad (76)$$

Finally, we obtain the upper and lower wave functions as

$$F^{\rm s}_{n,\kappa}(r)=N^{\prime s}_{n,\kappa}\,{\rm e}^{-2\sqrt{\delta_3^{\rm s}}\alpha r}\left(1-{\rm e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\delta_1^{\rm s}+\delta_3^{\rm s}-\delta_2^{\rm s}}} \times P_n^{\left(2\sqrt{\delta_3^{\rm s}},2\sqrt{\frac{1}{4}+\delta_1^{\rm s}+\delta_3^{\rm s}-\delta_2^{\rm s}}\right)}(1-2\,{\rm e}^{-2\alpha r}), \quad (77)$$

$$G^{\rm s}_{n,\kappa}(r)=\frac{1}{M+E^{\rm s}_{n,\kappa}-C_{\rm s}}\left(\frac{\rm d}{{\rm d}r}+\frac{\kappa}{r}-U(r)\right)F^{\rm s}_{n,\kappa}(r). \quad (78)$$

7. Numerical results

Listed in Tables 1 and 2 are some numerical results for different states in the cases of pseudo-spin and spin symmetries for the Coulomb-like tensor interaction. It is clear from the tables that energy levels are degenerate for $H=0$, i.e. ($1s_{1/2}$,$0d_{3/2}$), ($1p_{3/2}$, $0f_{5/2}$), …, etc. in pseudo-spin symmetry and ($0p_{1/2}$, $0p_{3/2}$), ($0d_{3/2}$, $0d_{5/2}$), …, etc. in the spin symmetry limit. In the presence of the tensor interaction, these degeneracies are removed but new degeneracies are present. For example, we observe the degeneracy in ($1p_{3/2}$, $0d_{3/2}$), ($1d_{5/2}$, $0f_{5/2}$), …, etc. in the pseudo-spin symmetry and ($0p_{1/2}$, $0d_{5/2}$), ($0d_{3/2}$, $0f_{7/2}$), …, etc. in the spin symmetry. In the presence of the Yukawa-like tensor interaction, we report some numerical results in Tables 3 and 4. In these tables we can see the degeneracy pattern for $V_0=0$ which is changed by the tensor interaction. The effects of the Coulomb-like and Yukawa-like tensor interactions on the upper and lower components are shown in Figs. 2–5 for the spin and pseudo-spin symmetries. The behaviors of the energies of the system versus V_1 and V_2 for spin and pseudo-spin symmetry limits in the presence of Coulomb-like and Yukawa-like tensor interactions are given in Figs. 6–9. The sensitivities of the pseudo-spin doublets ($1p_{3/2}$, $0f_{5/2}$) and ($1d_{5/2}$, $0g_{7/2}$) and spin doublets ($0f_{5/2}$, $0f_{7/2}$) and ($0d_{3/2}$, $0d_{5/2}$) to the parameters H and V_0 are given in Figs. 10 and 11. It is seen that degeneracies are removed by tensor interactions. Furthermore, the magnitude of the energy difference between the two states in the doublets increases with increasing H and V_0.

Table 1. Energies of the pseudo-spin symmetry limit in the presence and absence of Coulomb-like tensor interaction $M=5$ fm^{-1}, $\alpha=0.1$ fm^{-1}, $V_1=0.3$ fm^{-1}, $V_2=-0.2$ fm^{-1}, $C_{\rm ps}=-5$ fm^{-1}, $d=0.5$ fm^{-1}, $b=0.4$, $\omega=\lambda=\alpha^2/2$. The units 1 fm $=10^{-15}$ m.

$\tilde{l}$	n,k	State	$E^{\rm ps}_{n,k(H=0)}$	$E^{\rm ps}_{n,k(H=0.5)}$	$n-1,k$	State	$E^{\rm ps}_{n-1,k(H=0)}$	$E^{\rm ps}_{n-1,k(H=0.5)}$
1	1, −1	$1s_{1/2}$	−3.539131397	−3.520208549	0, 2	$0d_{3/2}$	−3.539131397	−3.563916066
2	1, −2	$1p_{3/2}$	−3.593214178	−3.563916066	0, 3	$0f_{5/2}$	−3.593214178	−3.625669320
3	1, −3	$1d_{\frac{5}{2}}$	−3.660040675	−3.625669320	0, 4	$0g_{7/2}$	−3.660040675	−3.695271783
4	1, −4	$1f_{7/2}$	−3.730513890	−3.695271783	0, 5	$0h_{9/2}$	−3.730513890	−3.765119033

Table 2. Energies of the spin symmetry limit in the presence and absence of Coulomb-like tensor interaction $M=5$ fm^{-1}, $\alpha=0.1$ fm^{-1}, $V_1=0.3$ fm^{-1}, $V_2=0.2$ fm^{-1}, $C_{\rm s}=5$ fm^{-1}, $d=0.5$ fm^{-1}, $b=0.4$, $\omega=\lambda=\alpha^2/2$.

l	n,k	State	$E^{\rm s}_{n,k(H=0)}$	$E^{\rm s}_{n,k(H=0.5)}$	n,k	State	$E^{\rm s}_{n,k(H=0)}$	$E^{\rm s}_{n,k(H=0.5)}$
1	0, −2	$0p_{3/2}$	5.317481481	5.315039825	0, 1	$0p_{1/2}$	5.317481481	5.320826857
2	0, −3	$0d_{5/2}$	5.325007584	5.320826857	0, 2	$0d_{3/2}$	5.325007584	5.329941773
3	0, −4	$0f_{7/2}$	5.335537723	5.329941773	0, 3	$0f_{5/2}$	5.335537723	5.341697708
4	0, −5	$0g_{9/2}$	5.348321581	5.341697708	0, 4	$0g_{7/2}$	5.348321581	5.355310004

Table 3. Energies of the pseudo-spin symmetry limit in the presence and absence of Yukawa tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.6$ fm^{-1}, $V_2 = -0.7$ fm^{-1}, $C_{\rm ps} = -5$ fm^{-1}, $d = 0.8$ fm^{-1}, $b = 0.4$.

$\tilde{l}$	n,k	State	$E_{n,k(V_0=0)}$	$E_{n,k(V_0=0.5)}$	$n-1,k$	State	$E_{n-1,k(V_0=0)}$	$E_{n-1,k(V_0=0.5)}$
1	1, −1	$1s_{1/2}$	−1.113295446	−0.5882352666	0, 2	$0d_{3/2}$	−1.113295446	−1.584187124
2	1, −2	$1p_{3/2}$	−1.963481191	−1.584187124	0, 3	$0f_{5/2}$	−1.963481191	−2.295293090
3	1, −3	$1d_{5/2}$	−2.540637767	−2.295293090	0, 4	$0g_{7/2}$	−2.540637767	−2.752287475
4	1, −4	$1f_{7/2}$	−2.901919348	−2.752287475	0, 5	$0h_{9/2}$	−2.901919348	−3.034053680

Table 4. Energies of the spin symmetry limit in the presence and absence of Yukawa tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.6$ fm^{-1}, $V_2 = -0.7$ fm^{-1}, $C_{\rm s} = 5$ fm^{-1}, $d = 0.8$ fm^{-1}, $b = 0.4$.

l	n,k	State	$E^{\rm s}_{n,k(V_0=0)}$	$E^{\rm s}_{n,k(V_0=0.5)}$	n,k	State	$E^{\rm s}_{n,k(V_0=0)}$	$E^{\rm s}_{n,k(V_0=0.5)}$
1	0, −2	$0p_{3/2}$	3.031848201	2.634622578	0, 1	$0p_{1/2}$	3.031848201	3.415961005
2	0, −3	$0d_{5/2}$	3.764604633	3.415961005	0, 2	$0d_{\frac{3}{2}}$	3.764604633	4.072231660
3	0, −4	$0f_{7/2}$	4.339786943	4.072231660	0, 3	$0f_{5/2}$	4.339786943	4.570661003
4	0, −5	$0g_{9/2}$	4.768992074	4.570661003	0, 4	$0g_{7/2}$	4.768992074	4.938924901

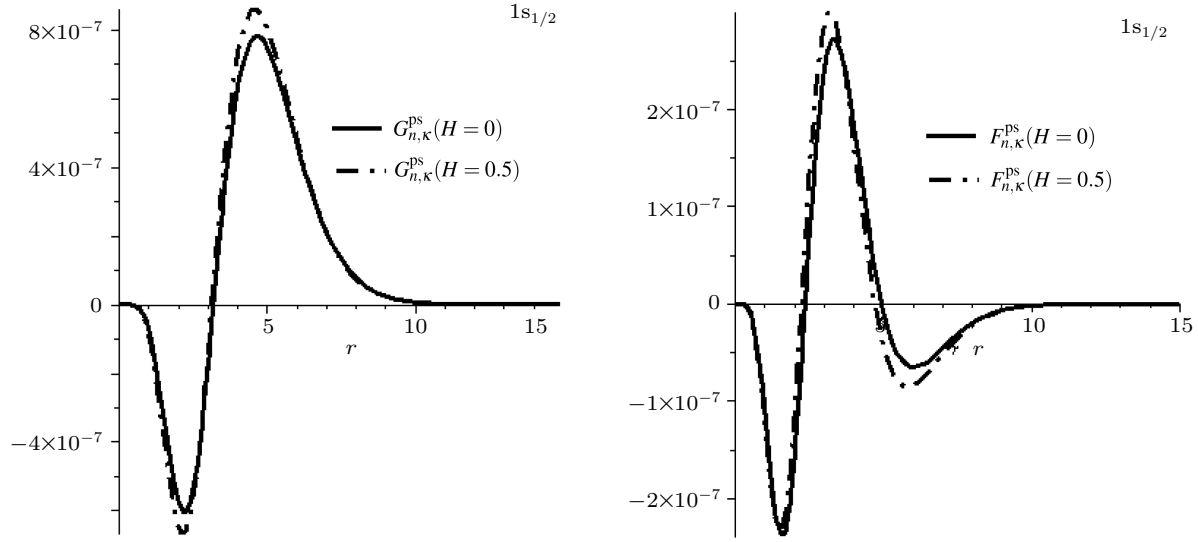


Fig. 2. Wave functions of $1s_{1/2}$ in the pseudo-spin symmetry in the presence and absence of the Coulomb tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.01$ fm^{-1}, $V_1 = 0.3$ fm^{-1}, $V_2 = 0.2$ fm^{-1}, $C_{\rm ps} = -5$ fm^{-1}, $d = 0.5$ fm^{-1}, $b = 0.4$, $\omega = \lambda = \alpha^2/2$.

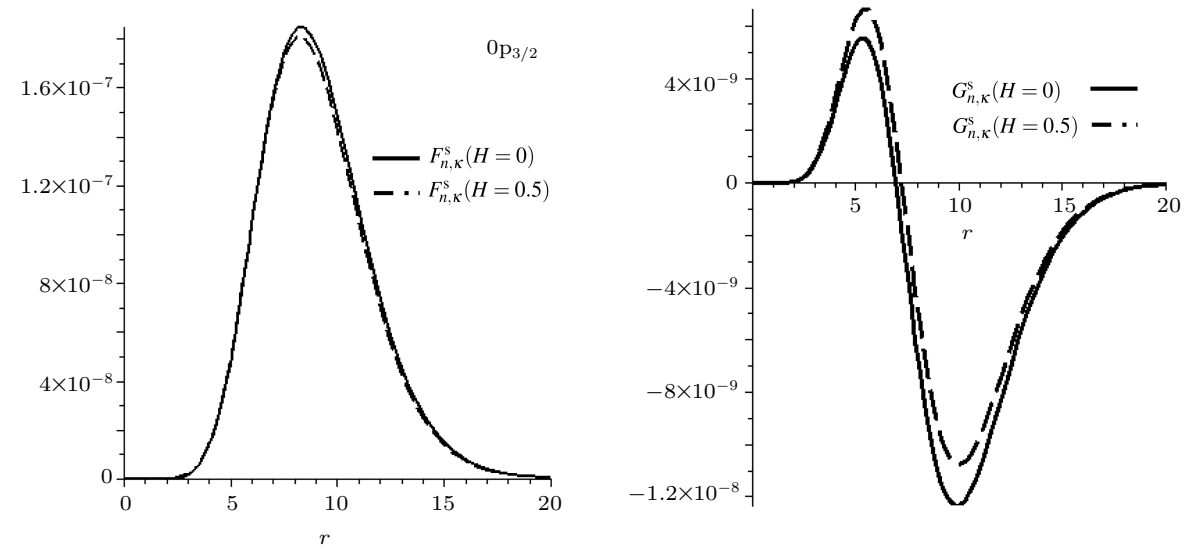


Fig. 3. Wave functions of $0p_{3/2}$ in the spin symmetry in the presence and absence of Coulomb-like tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.3$ fm^{-1}, $V_2 = 0.2$ fm^{-1}, $C_{\rm s} = 5$ fm^{-1}, $d = 0.5$ fm^{-1}, $b = 0.4$, $\omega = \lambda = \alpha^2/2$.

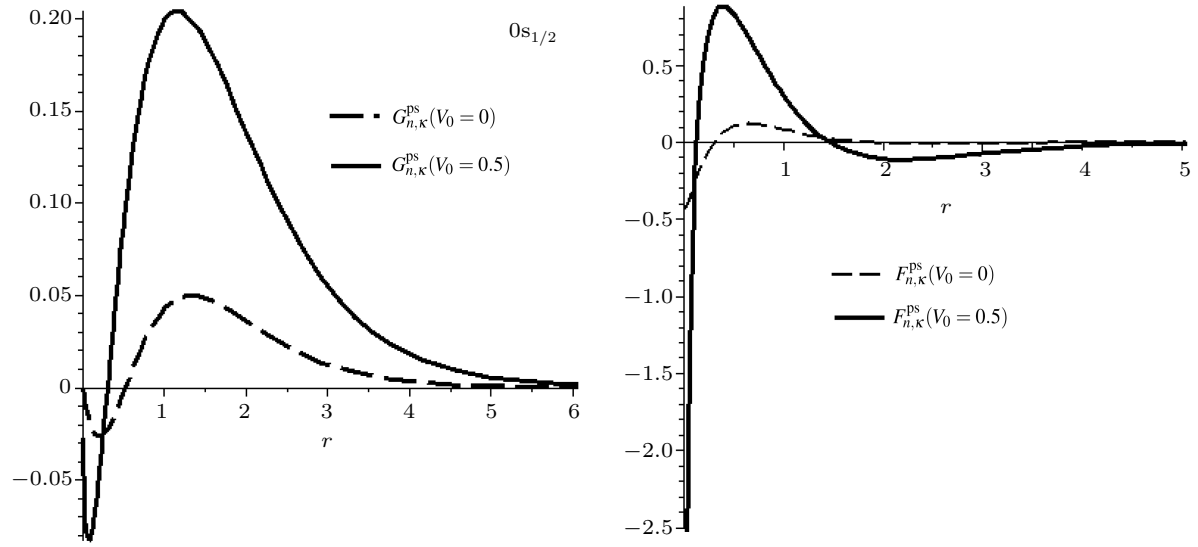


Fig. 4. Wave functions of $1s_{1/2}$ in the pseudo-spin symmetry in the presence and absence of Yukawa tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.6$ fm^{-1}, $V_2 = -0.7$ fm^{-1}, $C_{ps} = -5$ fm^{-1}, $d = 0.8$ fm^{-1}, $b = 0.4$.

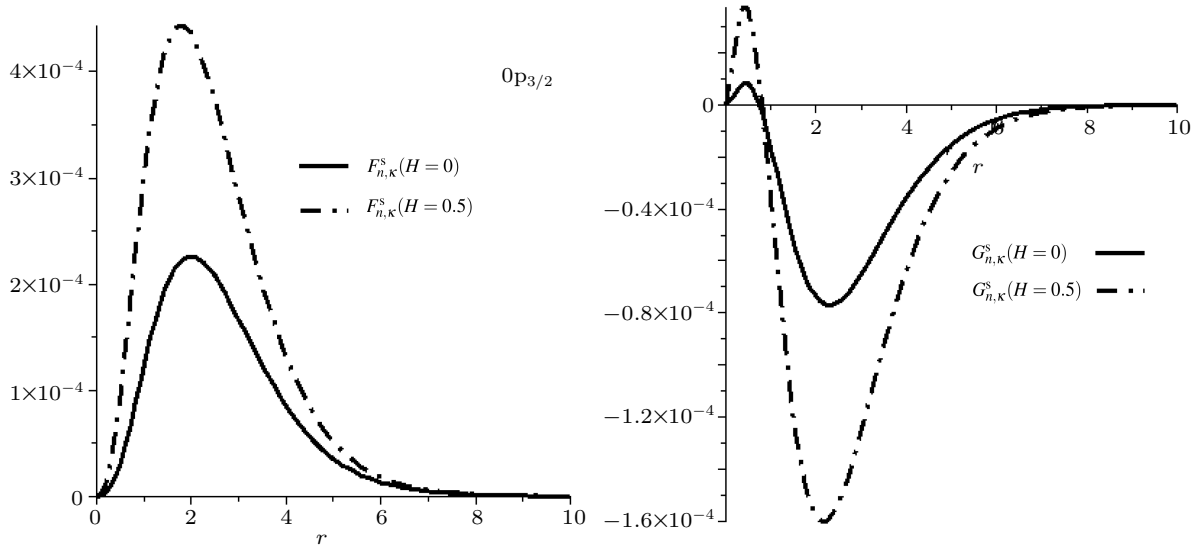


Fig. 5. Wave functions of $0p_{3/2}$ in the spin symmetry in the presence and absence of Yukawa tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.6$ fm^{-1}, $V_2 = -0.7$ fm^{-1}, $C_s = 5$ fm^{-1}, $d = 0.8$ fm^{-1}, $b = 0.4$.

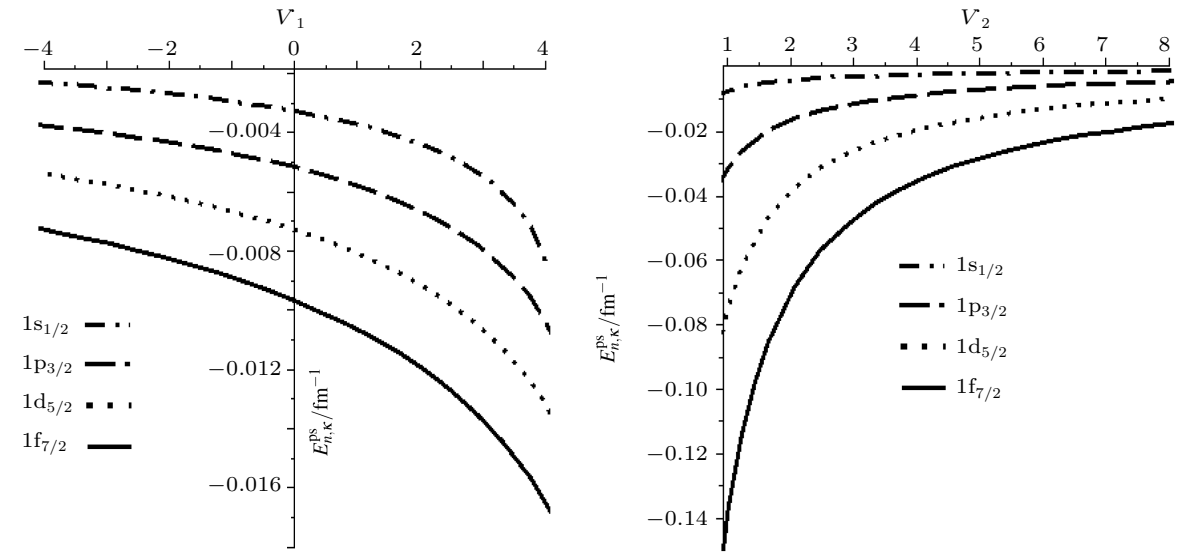


Fig. 6. Energy spectra in the pseudo-spin symmetry case versus V_1 and V_2 in the presence of Coulomb-tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.3$ fm^{-1}, $V_2 = -0.2$ fm^{-1}, $C_{ps} = -5$ fm^{-1}, $d = 0.5$ fm^{-1}, $b = 0.4$, $\omega = \lambda = \alpha^2/2$.

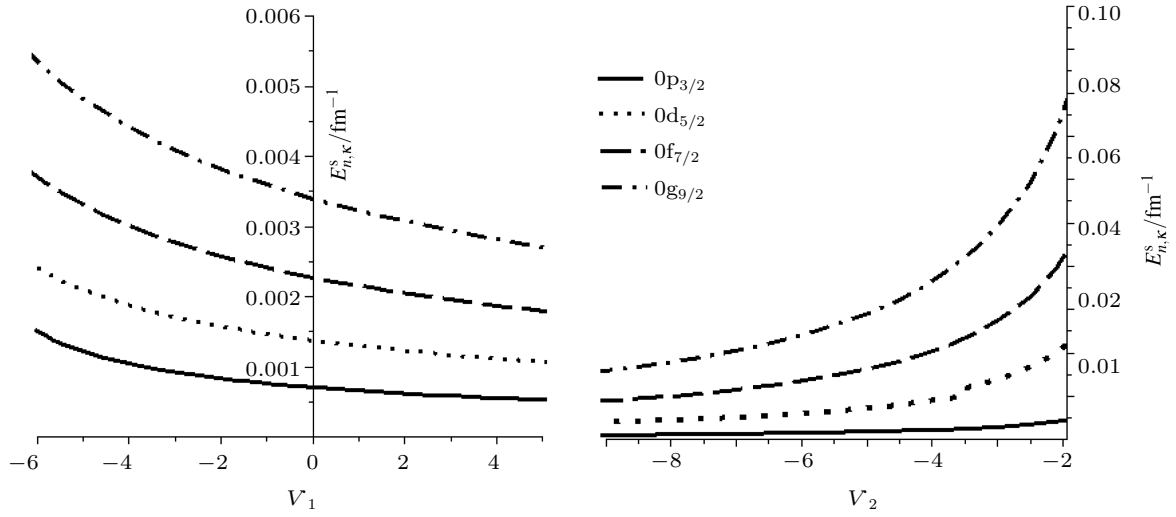


Fig. 7. Energy spectra in the spin symmetry case versus V_1 and V_2 in the presence of Coulomb-like tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.3$ fm^{-1}, $V_2 = 0.2$ fm^{-1}, $C_s = 5$ fm^{-1}, $d = 0.5$ fm^{-1}, $b = 0.4$, $\omega = \lambda = \alpha^2/2$.

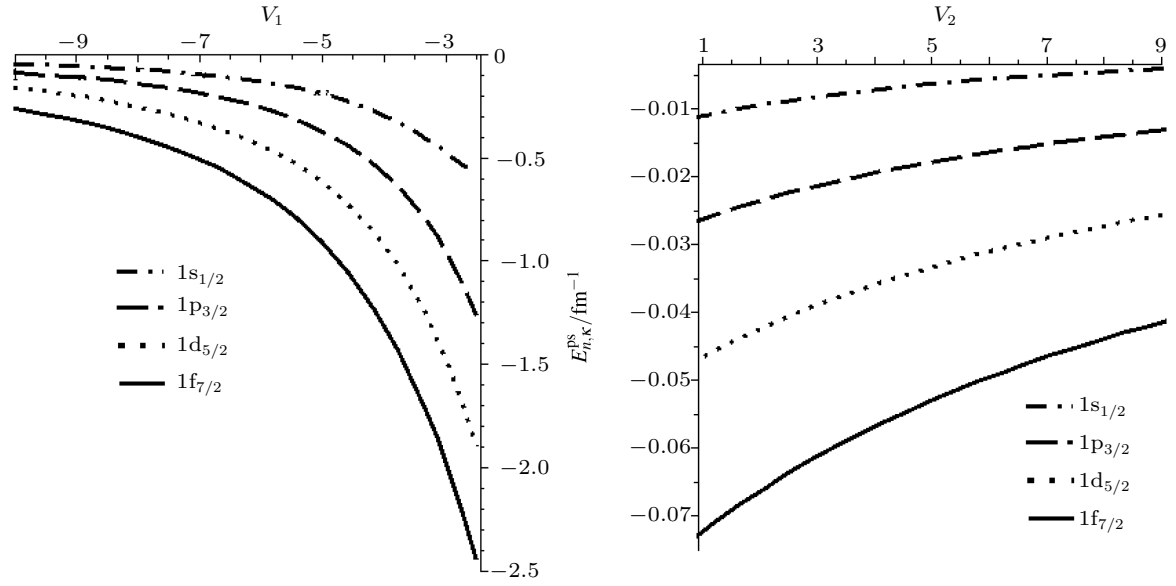


Fig. 8. Energy spectra in the pseudo-spin symmetry case versus V_1 and V_2 in the presence of Yukawa- like tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.6$ fm^{-1}, $V_2 = -0.7$ fm^{-1}, $C_{ps} = -5$ fm^{-1}, $d = 0.8$ fm^{-1}, $b = 0.4$.

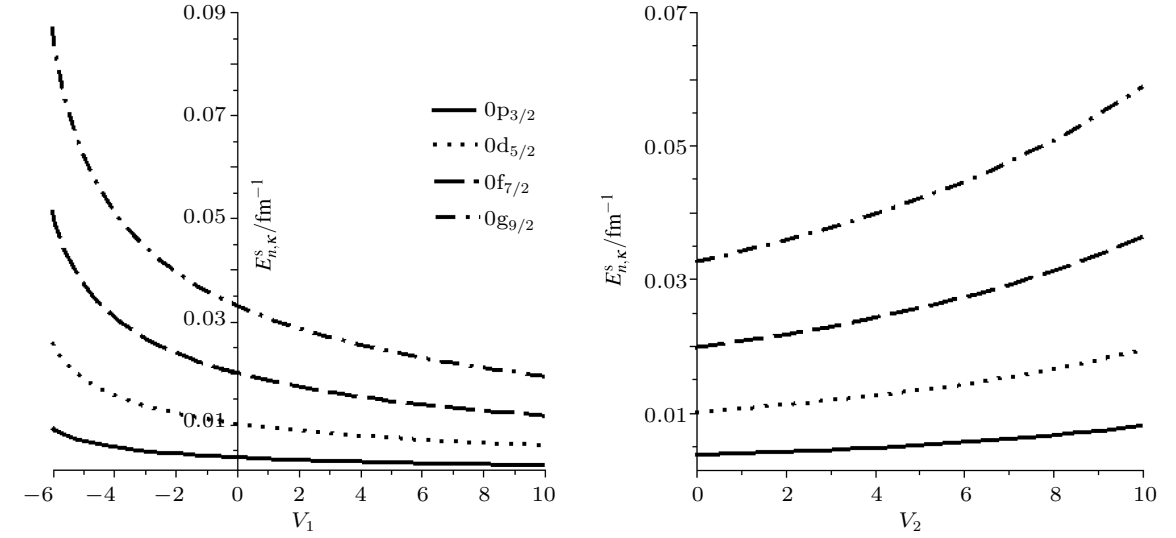


Fig. 9. Energy spectra in the spin symmetry case versus V_1 and V_2 in the presence of Yukawa-like tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.6$ fm^{-1}, $V_2 = -0.7$ fm^{-1}, $C_s = 5$ fm^{-1}, $d = 0.8$ fm^{-1}, $b = 0.4$.

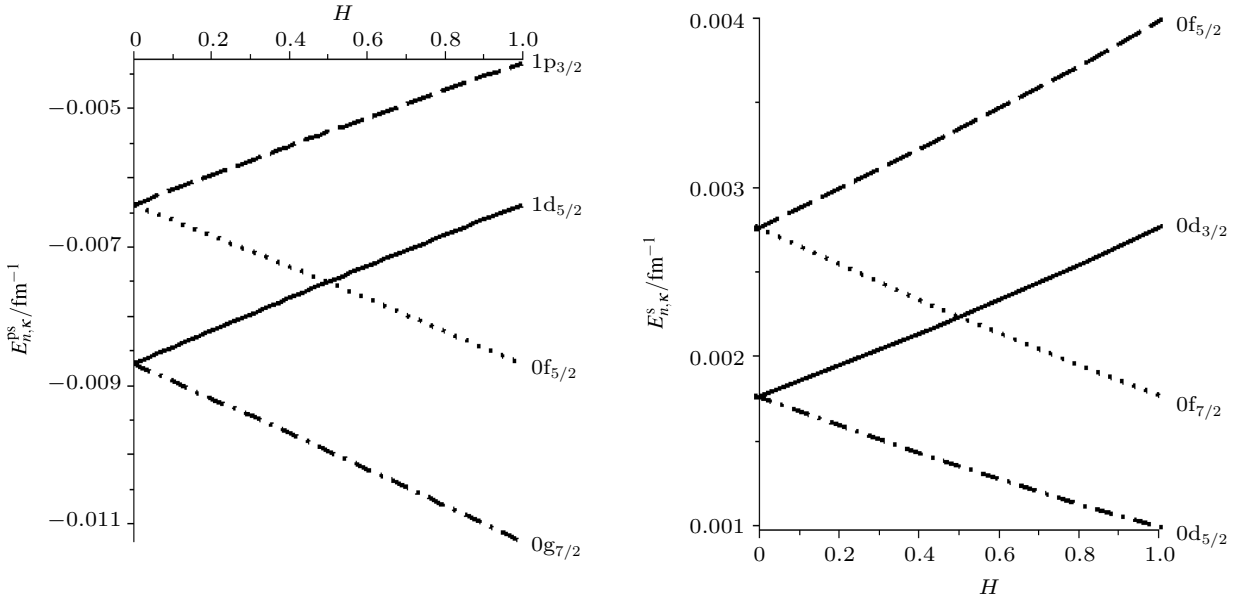


Fig. 10. Energy spectra in the pseudo-spin and spin symmetries versus H for Coulomb-tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.3$ fm^{-1}, $V_2 = 0.2$ fm^{-1}, $C_s = 5$ fm^{-1}, $d = 0.5$ fm^{-1}, $b = 0.4$, $\omega = \lambda = \alpha^2/2$.

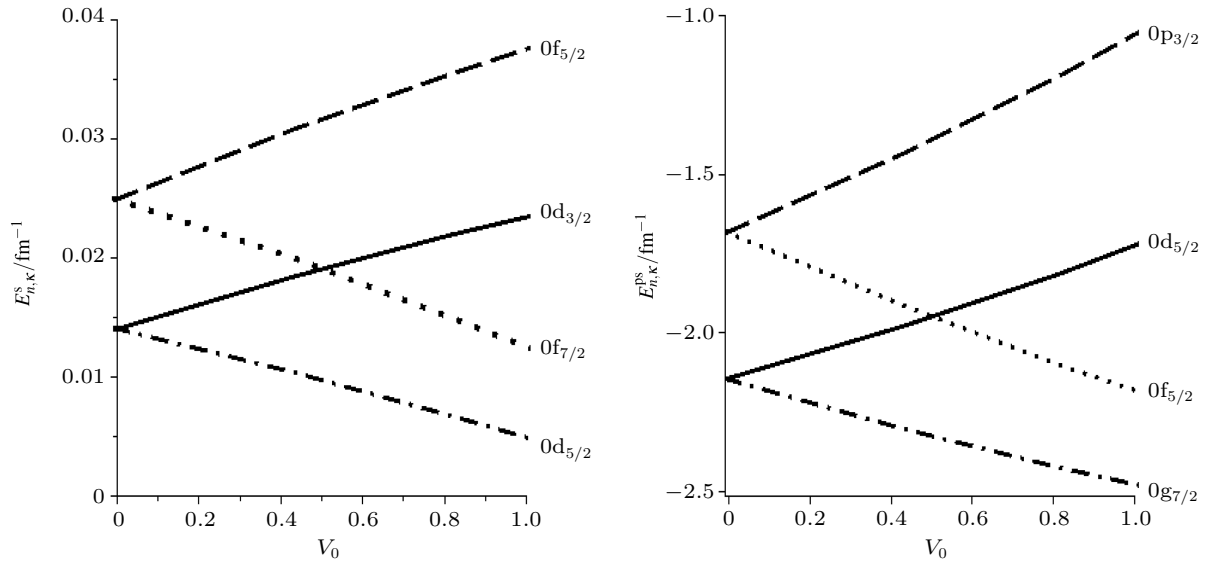


Fig. 11. Energy spectra in the pseudo-spin and spin symmetry cases versus V_0 for Yukawa-like tensor interaction for $M = 5$ fm^{-1}, $\alpha = 0.1$ fm^{-1}, $V_1 = 0.6$ fm^{-1}, $V_2 = -0.7$ fm^{-1}, $C_{ps} = -5$ fm^{-1}, $C_s = 5$ fm^{-1}, $d = 0.8$ fm^{-1}, $b = 0.4$.

8. Conclusions

The motivation of the present work is to investigate the spin and pseudo-spin symmetry limits of the Dirac equation which play significant roles in hadron and nuclear spectroscopy. We consider the Deng–Fan and Eckart potentials which are rather general forms of the exponential potentials. In the case of the tensor interaction, we consider physically motivated Coulomb-like and Yukawa-like tensor interactions. In our calculations, we use the analytical NU method to obtain the arbitrary-state solutions, and we introduce an exponential counterpart into the inverse square centrifugal term. In our obtained results, the influence of the tensor term on the degeneracy of the system is well observed. As the higher states and the general form of the energy eigenvalues are obtained, the results can be used to study the decay modes, bound-states, scattering parameters including the transition and reflection factors, the charge radius, and other static properties.

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