New interaction solutions of the Kadomtsev–Petviashvili equation*

Liu Xi-Zhong(刘希忠)[a], Yu Jun(俞 军)[a]†, Ren Bo(任 博)[a], and Yang Jian-Rong(杨建荣)[b]

[a] *Institute of Nonlinear Science, Shaoxing University, Shaoxing* 312000, *China*

[b] *Department of Physics and Electronics, Shangrao Normal University, Shangrao* 334001, *China*

(Received 24 January 2014; revised manuscript received 8 April 2013; published online 10 August 2014)

The residual symmetry relating to the truncated Painlevé expansion of the Kadomtsev–Petviashvili (KP) equation is nonlocal, which is localized in this paper by introducing multiple new dependent variables. By using the standard Lie group approach, new symmetry reduction solutions for the KP equation are obtained based on the general form of Lie point symmetry for the prolonged system. In this way, the interaction solutions between solitons and background waves are obtained, which are hard to find by other traditional methods.

Keywords: Kadomtsev–Petviashvili equation, localization procedure, residual symmetry, Bäcklund transformation, symmetry reduction solution

PACS: 02.30.Jr, 02.30.Ik, 05.45.Yv, 47.35.Fg **DOI:** 10.1088/1674-1056/23/10/100201

1. Introduction

In nonlinear science, many effective ways exist to derive exact solutions, including soliton solutions and different wave solutions for integrable systems. Among which, the symmetry analysis plays an important role. The application of the symmetry group in dealing with differential equations can be dated back to Sophus Lie,[1] and various symmetry methods have been developed to find exact solutions for different partial differential equations since then. As we know, once the Lie point symmetry group for a nonlinear system is obtained, one can reduce the system by using both the classical and nonclassical Lie group methods[2–7] and the symmetry reduction solutions are consequently derived. The Darboux transformations (DT) and the Bäcklund transformations (BT) are two other effective methods,[8–10] from which, in principle, one can obtain new solutions from the known ones. However, one can only obtain multiple soliton solutions by taking the seed solution as constant. For the seed solutions taken as nonconstant nonlinear waves, it is difficult to find new solutions by using BT or DT.

The integrability of a nonlinear system depends on different senses, such as possessing a Lax pair, having infinitely many symmetries, etc.[11] As for testing the Painlevé integrability of a nonlinear system, the Painlevé analysis plays an essential role. For any Painlevé integrable system, a truncated expansion exists which relates one solution to another, so it is a particular form of BT. In many cases, the BT for an integrable system has been obtained by various ways, while the corresponding symmetry is rarely studied. To study the BT-related symmetry and use it to construct exact solutions of a nonlinear system by a symmetry reduction method would open a new window for studying the properties of integrable systems in terms of symmetry theory. However, the BT-related symmetries are nonlocal,[12–17] and thus difficult to use directly in constructing exact solutions of nonlinear systems. To overcome that obstacle, one direct way is to localize it by introducing new dependent variables so that the BT-related symmetry becomes localized in the new system.[18–20] Recently, it was found that the symmetry related to the Painlevé truncated expansion is just the residue with respect to the singular manifold, which is called residual symmetry.[21] In fact, hierarchies of symmetries, including nonlocal symmetries to the Kadomtsev–Petviashvili (KP) equation, are pretty systematically generated from the Lax formulation.[22] Here, in this paper, we just consider the residual symmetry of the KP equation and use it to construct interacting exact solutions between solitons and different nonlinear waves, which are difficult to obtain by other approaches.

The KP equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \tag{1}$$

was first derived to study the evolution of long ion–acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations.[23] The KP equation was widely considered as a natural extension of the classical Korteweg–de Vries equation to two spatial dimensions, and has wide applications in almost all physical fields, such as nonlinear optics,[24] ion–acoustic waves in plasmas, ferromagnetics, Bose–Einstein condensation, and string theory.

The paper is organized as follows. In Section 2, we first obtain the residual symmetry of the KP equation by using the Painlevé truncated expansion method, then localize it by introducing multiple new dependent variables. For the enlarged

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11347183, 11275129, 11305106, 11405110, and 11365017) and the Natural Science Foundation of Zhejiang Province of China (Grant Nos. Y7080455 and LQ13A050001).

†Corresponding author. E-mail: junyu@usx.edu.cn

© 2014 Chinese Physical Society and IOP Publishing Ltd

KP system, the finite group transformation theorem is consequently obtained by using Lie's first principle. In Section 3, the general form of the Lie point symmetry group, which includes the localized residual symmetry as a special case for the enlarged KP system, is obtained. The similarity reductions for the system are given according to the standard Lie point symmetry approach with interacting solutions between solitons and nonlinear waves. The last section is devoted to a short summary and discussion.

2. Localization of residual symmetry and new Bäcklund transformation

It is well known that the KP equation is Painlevé-integrable, which means a truncated expansion exists

$$u=\sum_{i=0}^{\alpha}u_i\phi^{i-\alpha}, \tag{2}$$

where ϕ is the singular manifold and α is a positive integer. Substituting Eq. (2) into Eq. (1) and balancing the nonlinear and dispersion terms, we have the truncated Painlevé expansion in the form

$$u=\frac{u_0}{\phi^2}+\frac{u_1}{\phi}+u_2. \tag{3}$$

Now, substituting Eq. (3) into the KP Eq. (1), we have

$$\begin{aligned}
&3u_{2yy}-6u_{2x}^2-6u_2u_{2xx}+u_{2xxxx}+u_{2xt}\\
&\quad+\phi^{-1}(3u_{1yy}-6u_1u_{2xx}\quad+u_{1xt}-6u_2u_{1xx}\\
&\quad-12u_{1x}u_{2x}+u_{1xxxx})+\phi^{-2}(-4u_{1x}\phi_{xxx}-u_1\phi_{xt}\\
&\quad-u_1\phi_{xxxx}-6u_{1xx}\phi_{xx}-6u_{1y}\phi_y+6u_2u_1\phi_{xx}\\
&\quad-u_{1x}\phi_t+12u_1\phi_xu_{2x}-3u_1\phi_{yy}+3u_{0yy}+12u_2u_{1x}\phi_x\\
&\quad-6u_0u_{2xx}+u_{0xt}-6u_2u_{0xx}-6u_1u_{1xx}-4u_{1xxx}\phi_x\\
&\quad-12u_{0x}u_{2x}+u_{0xxxx}-u_{1t}\phi_x-6u_{1x}^2)+\phi^{-3}(-12u_{0xx}\phi_{xx}\\
&\quad-12u_2u_1\phi_x^2+6u_1\phi_y^2+8u_1\phi_x\phi_{xxx}+12u_{1xx}\phi_x^2\\
&\quad+24u_2u_{0x}\phi_x+24u_{1x}\phi_x\phi_{xx}-8u_{0xxx}\phi_x+24u_1u_{1x}\phi_x\\
&\quad-2u_{0t}\phi_x-2u_{0x}\phi_t+6u_1^2\phi_{xx}-6u_0\phi_{yy}-6u_0u_{1xx}\\
&\quad+24u_0\phi_xu_{2x}-8u_{0x}\phi_{xxx}+6u_1\phi_{xx}^2-2u_0\phi_{xxxx}-12u_{0x}u_{1x}\\
&\quad+12u_2u_0\phi_{xx}+2u_1\phi_t\phi_x-2u_0\phi_{xt}-12u_{0y}\phi_y\\
&\quad-6u_1u_{0xx})+\phi^{-4}(-6u_{0x}^2+72u_{0x}\phi_x\phi_{xx}+24u_0\phi_x\phi_{xxx}\\
&\quad+36u_{0xx}\phi_x^2-36u_1\phi_x^2\phi_{xx}-18u_1^2\phi_x^2+36u_0u_{1x}\phi_x\\
&\quad+18u_0\phi_{xx}^2-36u_2u_0\phi_x^2+6u_0\phi_t\phi_x+36u_1u_{0x}\phi_x\\
&\quad-24u_{1x}\phi_x^3+18u_0u_1\phi_{xx}+18u_0\phi_y^2-6u_0u_{0xx})\\
&\quad+\phi^{-5}(12u_0^2\phi_{xx}-72u_0u_1\phi_x^2-96u_{0x}\phi_x^3\\
&\quad-144u_0\phi_x^2\phi_{xx}+48u_0u_{0x}\phi_x+24u_1\phi_x^4)\\
&\quad+\phi^{-6}\left[6u_0\phi_x^2(2\phi_x^2-u_0)\right]=0.
\end{aligned} \tag{4}$$

For the arbitrariness of ϕ, the coefficients of different powers of ϕ should equal zero. By vanishing the coefficients of ϕ^0 and ϕ^{-1} in Eq. (4), it is interesting to find that u_1 satisfies the linearized KP equation with u_2 as a solution. So, u_1 is a symmetry of the KP equation, which is called residual symmetry.

Vanishing the coefficients of ϕ^{-6}, ϕ^{-5}, and ϕ^{-4} in Eq. (4), we have

$$u_0=2\phi_x^2,\quad u_1=-2\phi_{xx} \tag{5}$$

and

$$u_2=\frac{1}{6}\phi_x^{-2}(-3\phi_{xx}^2+4\phi_x\phi_{xxx}+\phi_x\phi_t+3\phi_y^2). \tag{6}$$

Vanishing the coefficient of ϕ^{-3} or ϕ^{-2} in Eq. (4) and using Eqs. (5), (6), we get the Schwarz form of the KP equation

$$\left[\frac{\phi_t}{\phi_x}+\{\phi;x\}+\frac{3}{2}\frac{\phi_y^2}{\phi_x^2}\right]_x+3\left(\frac{\phi_y}{\phi_x}\right)_y=0, \tag{7}$$

where

$$\{\phi;x\}=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}.$$

The Schwarz form (7) is invariant under the Möbious transformation

$$\phi\to\frac{a_1\phi+b_1}{a_2\phi+b_2},\quad a_1a_2\neq b_1b_2, \tag{8}$$

which means equation (7) possesses three symmetries $\sigma_\phi=d_1$, $\sigma_\phi=d_2\phi$, and

$$\sigma_\phi=d_3\phi^2 \tag{9}$$

with arbitrary constants d_1, d_2, and d_3. It is obvious that the residual symmetry $\sigma^u=u_1$ is related to the Möbious transformation symmetry (9) by the linearized equation (6).

Clearly, the residual symmetry

$$\sigma_u=-2\phi_{xx} \tag{10}$$

with ϕ satisfying Eq. (7) is nonlocal, which is just the generator of the BT (3).

The residual symmetry of the KP equation is localized as a Lie point symmetry

$$\sigma_u=h, \tag{11a}$$

$$\sigma_g=\phi g, \tag{11b}$$

$$\sigma_h=g^2+\phi h, \tag{11c}$$

$$\sigma_\phi=\frac{\phi^2}{2}, \tag{11d}$$

for the prolonged system

$$(u_t-6uu_x+u_{xxx})_x+3u_{yy}=0, \tag{12a}$$

$$u=\frac{1}{6}\phi_x^{-2}(-3\phi_{xx}^2+4\phi_x\phi_{xxx}+\phi_x\phi_t+3\phi_y^2), \tag{12b}$$

$$g=\phi_x, \tag{12c}$$

$$h=g_x, \tag{12d}$$

which is a special case of integrable couplings in Ref. [25]. These results can be easily verified by substituting Eq. (11) into the linearized equations (12) as

$$\sigma_{\phi,x}-\sigma_g=0, \tag{13a}$$

$$\sigma_{g,x}-\sigma_h=0, \tag{13b}$$

$$\begin{aligned}&\phi_x\sigma_{\phi,x}\phi_t+6\sigma_u\phi_x^3-(\sigma_{\phi,t}+4\sigma_{\phi,xxx})\phi_x^2\\&\quad+2(3\phi_{xx}\sigma_{\phi,xx}+2\sigma_{\phi,x}\phi_{xxx}-3\phi_y\sigma_{\phi,y})\phi_x\\&\quad-6\sigma_{\phi,x}\phi_{xx}^2+6\sigma_{\phi,x}\phi_y^2=0,\end{aligned} \tag{13c}$$

$$\begin{aligned}&(\sigma_{\phi,x}\phi_{xx}+\phi_x\sigma_{\phi,xx})\phi_t-(\sigma_{\phi,xxxx}+\sigma_{\phi,xt}+3\sigma_{\phi,yy})\phi_x^2\\&\quad+\big[-2\phi_{xt}\sigma_{\phi,x}+(\sigma_{\phi,t}+4\sigma_{\phi,xxx})\phi_{xx}+4\sigma_{\phi,xx}\phi_{xxx}\\&\quad-2\phi_{xxxx}\sigma_{\phi,x}-6\phi_{yy}\sigma_{\phi,x}\big]\phi_x-9\phi_{xx}^2\sigma_{\phi,xx}\\&\quad+4\sigma_{\phi,x}\phi_{xx}\phi_{xxx}+3\sigma_{\phi,xx}\phi_y^2+6\phi_{xx}\phi_y\sigma_{\phi,y}=0.\end{aligned} \tag{13d}$$

In other words, the prolonged system (12) has the Lie point symmetry vector

$$V=h\partial_u+g\phi\partial_g+(g^2+h\phi)\partial_h+\frac{\phi^2}{2}\partial_\phi, \tag{14}$$

which localizes the residual symmetry.

Now, let us proceed to study the finite transformation form of Eq. (14), which can be stated in the following theorem.

Theorem 1 If $\{u,g,h,\phi\}$ is a solution of the prolonged system (12), then so is $\{\hat{u},\hat{g},\hat{h},\hat{\phi}\}$ with

$$\hat{u}=\frac{(2g^2-2h\phi+u\phi^2)\varepsilon^2+(4h-4u\phi)\varepsilon+4u}{(\varepsilon\phi-2)^2}, \tag{15a}$$

$$\hat{g}=\frac{4g}{(\varepsilon\phi-2)^2}, \tag{15b}$$

$$\hat{h}=\frac{4h}{(\varepsilon\phi-2)^2}-\frac{8\varepsilon g^2}{(\varepsilon\phi-2)^3}, \tag{15c}$$

$$\hat{\phi}=-\frac{2\phi}{\varepsilon\phi-2}, \tag{15d}$$

with arbitrary group parameter ε.

Proof Using Lie's first theorem on vector (14) with the corresponding initial condition, i.e.,

$$\frac{\mathrm{d}\hat{u}(\varepsilon)}{\mathrm{d}\varepsilon}=\hat{h}(\varepsilon),\quad \hat{u}(0)=u, \tag{16}$$

$$\frac{\mathrm{d}\hat{g}(\varepsilon)}{\mathrm{d}\varepsilon}=\hat{\phi}(\varepsilon)\hat{g}(\varepsilon),\quad \hat{g}(0)=g, \tag{17}$$

$$\frac{\mathrm{d}\hat{h}(\varepsilon)}{\mathrm{d}\varepsilon}=\hat{g}(\varepsilon)^2+\hat{\phi}(\varepsilon)\hat{h}(\varepsilon),\quad \hat{h}(0)=h, \tag{18}$$

$$\frac{\mathrm{d}\hat{\phi}(\varepsilon)}{\mathrm{d}\varepsilon}=\frac{\hat{\phi}(\varepsilon)^2}{2},\quad \hat{\phi}(0)=\phi, \tag{19}$$

we find the solutions (15). Thus the theorem is proved.

3. New symmetry reductions of the KP equation

The Lie point symmetry of the prolonged KP system (12) has the general form

$$V=X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+T\frac{\partial}{\partial t}+U\frac{\partial}{\partial u}+G\frac{\partial}{\partial g}+H\frac{\partial}{\partial h}+\Phi\frac{\partial}{\partial\phi}, \tag{20}$$

which means that equation (12) is invariant under the following transformation

$$\begin{aligned}\{x,y,t,u,g,h,\phi\}\ \to\ &\{x+\varepsilon X,y+\varepsilon Y,t+\varepsilon T,u\\&+\varepsilon U,g+\varepsilon G,h+\varepsilon H,\phi+\varepsilon\Phi\}\end{aligned} \tag{21}$$

with an infinitesimal parameter ε. Equivalently, the symmetry in the form (21) can be written as a function form

$$\sigma_u=Xu_x+Yu_y+Tu_t-U, \tag{22a}$$

$$\sigma_g=Xg_x+Yg_y+Tg_t-G, \tag{22b}$$

$$\sigma_h=Xh_x+Yh_y+Th_t-H, \tag{22c}$$

$$\sigma_\phi=X\phi_x+Y\phi_y+T\phi_t-\Phi. \tag{22d}$$

Substituting Eq. (22) into Eq. (13) and eliminating u_x, g_x, g_t, h_t, ϕ_x, ϕ_t, and u_{yy} in terms of the prolonged system (12), we get more than 900 determining equations for the functions X, Y, T, U, G, H, and Φ. Calculated by computer algebra, the solutions in general forms are

$$\begin{aligned}X&=\frac{1}{2}f_{1t}x-\frac{1}{12}f_{1tt}y^2-\frac{1}{6}f_{2t}y+f_3,\\Y&=f_{1t}y+f_2,T=\frac{3}{2}f_1+c_1,\\U&=-\frac{1}{12}f_{1tt}x+\frac{1}{72}f_{1ttt}y^2\\&\quad+\frac{1}{36}f_{2tt}y-\frac{1}{6}f_{3t}-uf_{1t}-hc_2,\\G&=-\frac{1}{2}gf_{1t}-c_2g\phi+c_3g,\\H&=-c_2g^2-hf_{1t}-c_2h\phi+c_3h,\\\Phi&=-\frac{1}{2}c_2\phi^2+c_3\phi+c_4,\end{aligned} \tag{23}$$

where f_1, f_2, f_3 are arbitrary functions of t and c_1, c_2, c_3, c_4 are arbitrary constants.

It is noted that, if we take $f_1=f_2=f_3=c_1=c_3=c_4=0$ and $c_2=1$ in Eq. (23), the obtained general form of the symmetry is degenerated into the special form in Eq. (14). The time-related x translation invariance and t translation invariance symmetries can also be obtained by setting $f_1=f_2=c_1=c_2=c_3=c_4=0$, $f_3\neq0$, and $f_1=f_2=f_3=c_2=c_3=c_4=0$, $c_1=1$, respectively.

Consequently, the symmetries in Eqs. (22) can be written as

$$\begin{aligned}\sigma_u\ =\ &\left(\frac{1}{2}f_{1t}x-\frac{1}{12}f_{1tt}y^2-\frac{1}{6}f_{2t}y+f_3\right)u_x+(f_{1t}y+f_2)u_y\\&+\left(\frac{3}{2}f_1+c_1\right)u_t+\frac{1}{12}f_{1tt}x\\&-\frac{1}{72}f_{1ttt}y^2-\frac{1}{36}f_{2tt}y+\frac{1}{6}f_{3t}+uf_{1t}+hc_2,\end{aligned}$$

$$\begin{aligned}
\sigma_g &= \left(\frac{1}{2}f_{1t}x-\frac{1}{12}f_{1tt}y^2-\frac{1}{6}f_{2t}y+f_3\right)g_x+(f_{1t}y+f_2)g_y\\
&\quad+\left(\frac{3}{2}f_1+c_1\right)g_t+\frac{1}{2}gf_{1t}+c_2g\phi-c_3g,\\
\sigma_h &= \left(\frac{1}{2}f_{1t}x-\frac{1}{12}f_{1tt}y^2-\frac{1}{6}f_{2t}y+f_3\right)h_x+(f_{1t}y+f_2)h_y\\
&\quad+\left(\frac{3}{2}f_1+c_1\right)h_t+c_2g^2+hf_{1t}+c_2h\phi-c_3h,\\
\sigma_\phi &= \left(\frac{1}{2}f_{1t}x-\frac{1}{12}f_{1tt}y^2-\frac{1}{6}f_{2t}y+f_3\right)\phi_x+(f_{1t}y+f_2)\phi_y\\
&\quad+\left(\frac{3}{2}f_1+c_1\right)\phi_t+\frac{1}{2}c_2\phi^2-c_3\phi-c_4. \qquad (24)
\end{aligned}$$

Similarity reduction solutions of the enlarged system can be found by setting $\sigma_u=\sigma_g=\sigma_h=\sigma_\phi=0$ in Eq. (24), which is equivalent to solving the corresponding characteristic equations

$$\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}y}{Y}=\frac{\mathrm{d}t}{T}=\frac{\mathrm{d}u}{U}=\frac{\mathrm{d}g}{G}=\frac{\mathrm{d}h}{H}=\frac{\mathrm{d}\phi}{\Phi}. \qquad (25)$$

In the following part of the paper, three nontrivial cases are considered regarding the symmetry reduction.

Case 1 $f_1\neq 0$ and $c_1\neq 0$.

In this case, without loss of generality, we rewrite the arbitrary functions f_2 and f_3 as

$$f_2=(3f_1+2c_1)^{5/3}m_{1t}, \qquad (26)$$

$$f_3=-\frac{1}{3}(3f_1+2c_1)^{4/3}\left[(3f_1+2c_1)m_{1t}^2-m_{2t}\right] \qquad (27)$$

with m_1 and m_2 being arbitrary functions of t.

Now, by solving Eq. (25), we get

$$\phi=\frac{\Delta\tanh\left[\Delta(m_3+\Phi)\right]+c_3}{c_2},\quad \Delta=\sqrt{2c_2c_4+c_3^2}, \qquad (28)$$

$$g=\frac{G}{(3f_1+2c_1)^{1/3}\cosh^2\left[\Delta(m_3+\Phi)\right]}, \qquad (29)$$

$$h=-\frac{2c_2G^2\sinh\left[\Delta(m_3+\Phi)\right]-H\Delta\cosh\left[\Delta(m_3+\Phi)\right]}{(3f_1+2c_1)^{2/3}\Delta\cosh^3\left[\Delta(m_3+\Phi)\right]}, \qquad (30)$$

$$\begin{aligned}
u &= \frac{1}{36(3f_1+2c_1)^{2/3}\Delta^2\left[1+\tanh^2\left[\frac{1}{2}\Delta(m_3+\Phi)\right]\right]^2}\\
&\quad\times\left\{\Delta^2(\Omega+36U)\tanh^4\left[\frac{1}{2}\Delta(m_3+\Phi)\right]\right.\\
&\quad-144c_2H\Delta\tanh^3\left[\frac{1}{2}\Delta(m_3+\Phi)\right]\\
&\quad+\left[2\Delta^2(\Omega+36U)+288c_2^2G^2\right]\tanh^2\left[\frac{1}{2}\Delta(m_3+\Phi)\right]\\
&\quad\left.-144c_2H\Delta\tanh\left[\frac{1}{2}\Delta(m_3+\Phi)\right]+\Delta^2(\Omega+36U)\right\}, \qquad (31)
\end{aligned}$$

where Φ, U, G, and H are group invariants functions of similarity variables ξ and η, which can be expressed as

$$\begin{aligned}
\xi &= \frac{f_{1t}}{6(3f_1+2c_1)^{4/3}}y^2+\frac{1}{3}(3f_1+2c_1)^{1/3}m_{1t}y\\
&\quad+\frac{x}{(3f_1+2c_1)^{1/3}}-\frac{2}{3}m_2, \qquad (32)\\
\eta &= \frac{y}{(3f_1+2c_1)^{2/3}}-2m_1. \qquad (33)
\end{aligned}$$

In Eq. (32), we have made

$$m_3=\int\frac{1}{3f_1+2c_1}\mathrm{d}t \qquad (34)$$

and

$$\begin{aligned}
\Omega &= -\frac{6f_{1t}}{(3f_1+2c_1)^{1/3}}x\\
&\quad+\left[\frac{f_{1tt}}{(3f_1+2c_1)^{1/3}}-\frac{2f_{1t}^2}{(3f_1+2c_1)^{4/3}}\right]y^2\\
&\quad+2[3(3f_1+2c_1)^{1/3}m_{1t}f_{1t}+(3f_1+2c_1)^{4/3}m_{1tt}]y\\
&\quad+2(3f_1+2c_1)^2m_{1t}^2-4(3f_1+2c_1)m_{2t} \qquad (35)
\end{aligned}$$

for simplicity.

Substituting Eqs. (28), (29), (30), and (31) into the prolonged KP system, we get the corresponding symmetry reduction equations

$$\Delta^2\Phi_\xi-c_2G=0, \qquad (36)$$

$$\Delta^2\Phi_{\xi\xi}-c_2H=0, \qquad (37)$$

$$-3\Phi_\eta^2+8\Delta^2\Phi_\xi^4+6\Phi_\xi^2U-(1+4\Phi_{\xi\xi\xi})\Phi_\xi+3\Phi_{\xi\xi}^2=0, \qquad (38)$$

and

$$\begin{aligned}
&-3\Phi_{\xi\xi}\Phi_\eta^2-4\Delta^2\Phi_{\xi\xi}\Phi_\xi^4+(3\Phi_{\eta\eta}+\Phi_{\xi\xi\xi\xi})\Phi_\xi^2\\
&\quad-(1+4\Phi_{\xi\xi\xi})\Phi_{\xi\xi}\Phi_\xi+3\Phi_{\xi\xi}^3=0. \qquad (39)
\end{aligned}$$

By solving out G, H, and U from Eqs. (36), (37), (38) and then substituting them into Eq. (31), we have the following theorem.

Theorem 2 If Φ is a solution of the reduction equation (39), then the u given by

$$\begin{aligned}
u &= \frac{1}{36(3f_1+2c_1)^{2/3}\left[1+\tanh^2\left[\frac{1}{2}\Delta(m_3+\Phi)\right]\right]^2\Phi_\xi^2}\\
&\quad\times\left[(18\Phi_\eta^2-48\Delta^2\Phi_\xi^4+\Omega\Phi_\xi^2\right.\\
&\quad+6(4\Phi_{\xi\xi\xi}+1)\Phi_\xi-18\Phi_{\xi\xi}^2)\tanh^4\left[\frac{1}{2}\Delta(m_3+\Phi)\right]\\
&\quad-144\Phi_\xi^2\Phi_{\xi\xi}\Delta\tanh^3\left[\frac{1}{2}\Delta(m_3+\Phi)\right]\\
&\quad+2(18\Phi_\eta^2+96\Delta^2\Phi_\xi^4+\Omega\Phi_\xi^2+6(4\Phi_{\xi\xi\xi}+1)\Phi_\xi\\
&\quad-18\Phi_{\xi\xi}^2)\tanh^2\left[\frac{1}{2}\Delta(m_3+\Phi)\right]\\
&\quad-144\Phi_\xi^2\Phi_{\xi\xi}\Delta\tanh\left[\frac{1}{2}\Delta(m_3+\Phi)\right]+18\Phi_\eta^2\\
&\quad\left.-48\Delta^2\Phi_\xi^4+\Omega\Phi_\xi^2+(24\Phi_{\xi\xi\xi}+6)\Phi_\xi-18\Phi_{\xi\xi}^2\right] \qquad (40)
\end{aligned}$$

is a solution of the KP equation.

From Eq. (40), we can obtain some interesting solutions. Even Φ is a trivial solution of Eq. (39). As an illustration, we take the solution of Eq. (39) as $\Phi = \xi + \eta$ and then we find that

$$\begin{aligned} u = & \frac{1}{36(3f_1+2c_1)^{2/3}\left[1+\tanh^2\left[\frac{1}{2}\Delta(m_3+\xi+\eta)\right]\right]^2} \\ & \times\left[(24-48\Delta^2+\Omega)\tanh^4\left[\frac{1}{2}\Delta(m_3+\xi+\eta)\right]\right. \\ & +2(24+96\Delta^2+\Omega)\tanh^2 \\ & \left.\times\left[\frac{1}{2}\Delta(m_3+\xi+\eta)\right]+24-48\Delta^2+\Omega\right] \end{aligned} \tag{41}$$

with ξ, η expressed by Eqs. (32) and (33), is a nontrivial solution of the KP equation.

Case 2 $f_1 = c_1 = 0$ and $f_2 \neq 0$.

Similar to Case 1, we get the second-type symmetry reduction solutions for the enlarged KP system

$$\phi = \frac{1}{c_2}\left\{c_3 - \Delta\tanh\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right]\right\}, \tag{42}$$

$$g = \frac{g'}{\cosh^2\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right]}, \tag{43}$$

$$\begin{aligned} h = & \frac{1}{\Delta\cosh^3\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right]} \\ & \times\left\{2c_2g'^2\sinh\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right]\right. \\ & \left.+\Delta h'\cosh\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right]\right\}, \end{aligned} \tag{44}$$

$$\begin{aligned} u = & \frac{2c_2^2}{\Delta^2}g'^2\tanh^2\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right] \\ & +\frac{2c_2}{\Delta}h'\tanh\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right] \\ & -\frac{1}{72f_2f_{2t}^2}\left\{\left[(12y+36\phi'f_2)f_{3t}\right.\right. \\ & +(3\phi'f_2-y)(y+3\phi'f_2)f_{2tt}-72u'f_2\left.\right]f_{2t}^2 \\ & \left.-36(2f_{3t}+f_{2tt}f_2\phi')f_{2t}f_3+36f_{2tt}f_3^2\right\}, \end{aligned} \tag{45}$$

where $u' \equiv u'(x',t), g' \equiv g'(x',t), h' \equiv h'(x',t), \phi' \equiv \phi'(x',t)$ with

$$x' = -x - \frac{f_{2t}}{12f_2}y^2 + \frac{f_3}{f_2}y. \tag{46}$$

The corresponding symmetry reduction equations can be obtained by substituting Eqs. (42), (43), (44), and (45), into a prolonged KP system (12). The results are

$$3\Delta^2\phi'_{x'} + 2c_2g' = 0, \tag{47}$$

$$3\Delta^2\phi'_{x'x'} - 2c_2h' = 0, \tag{48}$$

$$\begin{aligned} & 12\phi'_{x'}f_{2t}^2\phi'_tf_2^2 + 216f_2^2\Delta^2f_{2t}^2\phi_{x'}'^4 \\ & -9\left[(4f_2^2f_{3t}\phi' + f_2^3f_{2tt}\phi'^2 - 8f_2^2u' + 4f_3^2)f_{2t}^2\right. \\ & -4(2f_2f_{3t}f_3 + f_2^2f_{2tt}\phi'f_3)f_{2t} + 4f_2f_{2tt}f_3^2\left.\right]\phi_{x'}'^2 \\ & -24(f_{2t}f_2f_{3t} - f_{2tt}f_2f_3 + 2f_{2t}^2f_2^2\phi'_{x'x'x'})\phi'_{x'} \\ & -4(1-9f_2^2\phi_{x'x'}'^2)f_{2t}^2 = 0, \end{aligned} \tag{49}$$

and

$$\begin{aligned} & 6\phi'_{x'}\phi'_{x'x'}f_{2t}^2\phi'_tf_2^2 - 54f_2^2\Delta^2\phi'_{x'x'}f_{2t}^2\phi_{x'}'^4 \\ & -3\phi_{x'}'^3f_{2t}^3f_2 - 6(\phi'_{x't}f_2^2 - f_2^2\phi'_{x'x'x'x'})f_{2t}^2\phi_{x'}'^2 \\ & -12(\phi'_{x'x'}f_{2t}f_2f_{3t} - \phi'_{x'x'}f_{2tt}f_2f_3 + 2f_{2t}^2f_2^2\phi'_{x'x'}\phi'_{x'x'x'})\phi'_{x'} \\ & +2(9f_2^2\phi_{x'x'}'^3 - \phi'_{x'x'})f_{2t}^2 = 0. \end{aligned} \tag{50}$$

Similar to theorem 2, solving out g', h', and ϕ' from Eqs. (47), (48), (49) and substituting them into Eq. (45), we have the following theorem.

Theorem 3 If ϕ' is a solution of the symmetry reduction equation (50), then the u given by

$$\begin{aligned} u = & -\frac{9}{2}\Delta^2\phi_{x'}'^2\tanh^2\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right] \\ & +3\Delta\phi'_{x'x'}\tanh\left[\frac{\Delta(6f_3 - f_{2t}y - 3f_{2t}f_2\phi')}{2f_{2t}f_2}\right] \\ & -\frac{1}{72f_2^2f_{2t}^2\phi_{x'}'^2}\left[12\phi'_{x'}f_{2t}^2\phi'_tf_2^2 + 216f_2^2\Delta^2f_{2t}^2\phi_{x'}'^4\right. \\ & +(12f_2f_{3t}y - 36f_3^2 - f_2f_{2tt}y^2)f_{2t}^2\phi_{x'}'^2 \\ & -24(f_{2t}f_2f_{3t} - f_{2tt}f_2f_3 + 2f_{2t}^2f_2^2\phi'_{x'x'x'})\phi'_{x'} \\ & \left.-4(1-9f_2^2\phi_{x'x'}'^2)f_{2t}^2\right] \end{aligned} \tag{51}$$

is a solution of the KP equation.

Case 3 $f_1 = f_2 = c_1 = 0$.

In this case, executing a similar procedure as in Cases 1 and 2, we can find the third type of group-invariant solutions

$$g = -\frac{g''}{\cosh^2\left[\frac{\Delta(x+\phi'')}{2f_3}\right]}, \tag{52}$$

$$\begin{aligned} h = & \frac{1}{\Delta\cosh^3\left[\frac{\Delta(x+\phi'')}{2f_3}\right]\left[\sinh\left[\frac{\Delta(x+\phi'')}{2f_3}\right]+\cosh\left[\frac{\Delta(x+\phi'')}{2f_3}\right]\right]} \\ & \times\left\{2c_2g''^2 - h''\Delta\cosh^2\left[\frac{\Delta(x+\phi'')}{2f_3}\right] - h''\Delta\sinh\left[\frac{\Delta(x+\phi'')}{2f_3}\right]\cosh\left[\frac{\Delta(x+\phi'')}{2f_3}\right]\right\}, \end{aligned} \tag{53}$$

$$\phi = \frac{1}{c_2}\left[c_3+\tanh\left[\frac{\Delta(x+\phi'')}{2f_3}\right]\Delta\right], \tag{54}$$

$$\begin{aligned} u = {} & \frac{2c_2^2}{\Delta^2}g''^2\tanh^2\left[\frac{\Delta(x+\phi'')}{2f_3}\right] \\ & -\frac{2c_2}{\Delta^2}(2c_2g''^2-\Delta h'')\tanh\left[\frac{\Delta(x+\phi'')}{2f_3}\right] \\ & -\frac{f_{3t}}{6f_3}x+u'' \end{aligned} \tag{55}$$

with group-invariant functions $u''\equiv u''(y,t)$, $g''\equiv g''(y,t)$, $h''\equiv h''(y,t)$, $\phi''\equiv\phi''(y,t)$.

The corresponding symmetry reduction equations are

$$\Delta^2+2c_2f_3g''=0, \tag{56}$$

$$-\Delta^3+2c_2f_3^2h''=0, \tag{57}$$

$$-2\Delta^2+3f_3^2\phi_y''^2-6u''f_3^2-f_3f_{3t}\phi''+f_3^2\phi_t=0, \tag{58}$$

$$-3f_3\phi_{yy}''+f_{3t}=0. \tag{59}$$

Again, after solving out g'', h'', and u'' by Eqs. (56), (57), (58) and substituting them into Eq. (55), then we have the following theorem

Theroem 4 If ϕ'' is a solution of the symmetry reduction equation (59), then the u given by

$$\begin{aligned} u = {} & \frac{\Delta^2}{2f_3^2}\tanh^2\left[\frac{\Delta(x+\phi'')}{2f_3}\right]+\frac{1}{2}\phi_y''^2 \\ & +\frac{1}{6}\phi_t''-\frac{f_{3t}}{6f_3}(x+\phi'')-\frac{\Delta^2}{3f_3^2} \end{aligned} \tag{60}$$

is a solution of the KP equation.

It is obvious that equations (40) and (51) can be considered as interaction solutions of a soliton with the background wave, while equation (60) is corresponding to the single solitary wave with nonhomogeneous background for f_3 not being a constant.

4. Conclusion and discussion

In summary, the residual symmetry coming from the Painlevé truncated expansion of the KP equation is localized by introducing multiple new dependent variables. For the prolonged KP system, the new Bäcklund transformation is derived by using Lie's first principle. The general form of Lie point symmetry for the prolonged system is found, from which it can be found that the residual symmetry is included as a special case. The Lie point symmetry group for the prolonged KP system covers the corresponding one for the original KP equation. So many more rich interaction solutions can be found by using the classical Lie group approach. From the form of symmetry reduction solutions for the KP equation, we see that the residual symmetry is to add an additional soliton to the background wave. In other words, the interaction solutions between soliton and background wave are successfully obtained. Some special explicit solutions under certain constraints are also given.

In this paper, we have shown that the residual symmetry can be used to construct interaction solutions for nonlinear equations. It is meaningful to try to use other forms of nonlocal symmetries, which can be found from Bäcklund transformation, the bilinear forms, and negative hierarchies, the nonlinearizations,[26,27] and self-consistent source,[28] etc., to construct new exact solutions. Despite the bright prospect, some open questions still exist which deserve to be probed. First, it still remains unclear what kind of nonlocal symmetries must have close prolongations and can be applied to construct exact solutions. Secondly, different nonlocal symmetries, for example the residual symmetry and the DT-related symmetry, lead to different symmetry reduction solutions, so what is the link between them? Can the nonlocal symmetries be classified so that symmetries in one kind leads to essentially the same kind of interaction solutions? All in all, applying nonlocal symmetries to construct exact solutions for nonlinear systems is adventurous but meaningful.

References

[1] Lie S 1891 *Vorlesungen ber Differentialgleichungen mit Bekannten Infinitesimalen Transformationen* (Leipzig: Teuber) (reprinted by Chelsea: New York; 1967)

[2] Olver P J 1993 *Application of Lie Groups to Differential Equation, Graduate Texts in Mathematics*, 2nd edn. (NewYork: Springer-Verlag)

[3] Bluman G W and Kumei S 1989 *Symmetries and Differential Equation* (Berlin: Springer-Verlag)

[4] Wu J L and Lou S Y 2012 *Chin. Phys. B* **21** 120204

[5] Hu X R, Chen Y and Huang F 2010 *Chin. Phys. B* **19** 080203

[6] Liu X Z 2010 *Chin. Phys. B* **19** 080202

[7] Jing J C and Li B 2013 *Chin. Phys. B* **22** 010303

[8] Gu C H, Hu H and Zhou Z X 2005 *Darboux Transformations in Integrable Systems Theory and Their Applications to Geometry* (Dordrecht: Springer)

[9] Rogers C and Schief W K 2002 *Bäcklund and Darboux Transformations Geometry and Modern Applications in Soliton Theory* (Cambridge: Cambridge University Press)

[10] Jin Y, Jia M and Lou S Y 2013 *Chin. Phys. Lett.* **30** 020203

[11] Scott A 2005 *Encyclopedia of Nonlinear Science* (Taylor and Francis)

[12] Lou S Y and Hu X B 1993 *Chin. Phys. Lett.* **10** 577

[13] Lou S Y and Hu X B 1997 *J. Math. Phys.* **38** 6401

[14] Lou S Y and Hu X B 1997 *J. Phys. A: Math. Gen.* **30** L95

[15] Lou S Y 1998 *Physica Scripta* **57** 481

[16] Hu X B, Lou S Y and Qian X M 2009 *Stud. Appl. Math.* **122** 305

[17] Li Y Q, Chen J C, Chen Y and Lou S Y 2014 *Chin. Phys. Lett.* **31** 010201

[18] Lou S Y, Hu X R and Chen Y 2012 "Nonlocal Symmetries Related to Bäcklund Transformation and Their Applications", arXiv: 1201.3409

[19] Hu X R, Lou S Y and Chen Y 2012 *Phys. Rev. E* **85** 056607

[20] Cheng X P, Chen C L and Lou S Y 2012 "Interactions Among Different Types of Nonlinear Waves Described by the Kadomtsev–Petviashvili Equation", arXiv: 1208.3259

[21] Gao X N, Lou S Y and Tang X Y 2013 *JHEP* **05** 029

[22] Ma W X 1992 *J. Phys. A: Math. Gen.* **25** 5329

[23] Kadomtsev B B and Petviashvili V I 1970 *Sov. Phys. Dokl.* **15** 539

[24] Pelinovsky D E, Stepanyants Yu A and Kivshar Yu A 1995 *Phys. Rev. E* **51** 5016

[25] Ma W X and Kaup D 2013 *Nonlinear and Modern Mathematical Physics* (American Institute of Physics, Tampa, Florida, USA)

[26] Cao C W and Geng X G 1990 *J. Phys. A: Math. Gen.* **23** 4117

[27] Cheng Y and Li Y S 1991 *Phys. Lett. A* **157** 22

[28] Zeng Y B, Ma W X and Lin R L 2000 *J. Math. Phys.* **41** 5453