Continuity and momentum equations for moist atmospheres*

Ran Ling-Kun(冉令坤)†, Gao Shou-Ting(高守亭), and Cao Jie(曹 洁)

Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029, *China*

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The moist atmosphere with occurring precipitation is considered to be a multiphase fluid composed of dry air, water vapor and hydrometeors. These compositions move with different velocities: they take a macroscopic motion with a reference velocity and a relative motion with a velocity deviated from the reference velocity. The reference velocity can be chosen as the velocities of dry air, a gas mixture and the total air mixture. The budget equations of continuity and momentum are formulated in the three reference-velocity frames. It is shown that the resulting equations are dependent on the chosen reference velocity. The diffusive flux due to compositions moving with velocities deviated from the reference velocity and the internal sources due to the phase transitions of water substances result in additional source terms in continuity and momentum equations. A continuity equation of the total mass is conserved and free of diffusive flux divergence if the reference velocity is referred to the velocity of the total air mixture. However, continuity equations in the dry-air and gas-mixture frames are not conserved due to the mass diffusive flux divergence. The diffusive flux introduces additional source terms in the momentum equation. In the dry-air frame, the diffusive flux of water substances and the phase transitions of water substances contribute to the change of the total momentum. The additional sources of total momentum in the frame of a gas mixture are associated with the diffusive flux of hydrometeors, the phase transitions of hydrometeors and the gas-mixture diffusive flux. In the frame of total air mixture, the contribution to the total momentum comes from the diffusive flux of all atmospheric compositions instead of the phase transitions. The continuity and momentum equations derived here are more complicated than the traditional model equations. With increasing computing power, it becomes possible to simulate atmospheric processes with these sophisticated equations. It is helpful to the improvement of precipitation forecast.

Keywords: diffusive flux, phase transition, reference velocity

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1. Introduction

With the advance of computing power, numerical simulations of sophisticated atmospheric phenomena with more realistic physical processes and higher spatial and temporal resolutions become available. This makes it possible to reproduce the details of atmospheric events in numerical forecasting models with less error.[1]

Although atmospheric models have accomplished great achievements, the prediction of heavy precipitation still does not show significant progress as expected. No matter how high the resolution and how sophisticated the physical parameterizations are designed in numerical simulations, the precipitation forecast is still lagging behind other forecasts, such as temperature and pressure forecasting.[2–4] Although a lot of effort has been ingested into data assimilation to improve the precipitation forecast, the results are still not promising. Therefore, predictions of heavy precipitation cannot be improved by only increasing the resolution of the model or the accuracy of the initial conditions on the current dynamic basis. More attention needs to be paid to the dynamical frame inside the model.

A remarkable character of the precipitating atmosphere is atmospheric mass loss, which can typically be explained as below. Convergence and convection of water vapor typically cause condensation to occur and form cloud water droplets. These droplets continue to grow via various microphysical processes, such as collision and coalescence.[5] When the cloud droplets are transformed to precipitable hydrometeors and acquire large fall velocities, a precipitation takes place.[6,7] Precipitable hydrometeors fall to the ground with a prescribed terminal velocity relative to their ambient air and leave the atmosphere. Thus, the precipitation takes away a large amount of water substance from the atmospheric system, which results in atmospheric mass loss; this mass loss has come to the attention of meteorologists. Schubert *et al.* budget this mass loss in a mass continuity equation.[8] A matching forcing of mass loss in modified potential vorticity equations was explored.[9–11] Qiu *et al.* demonstrated that in heavily precipitating systems, the mass loss exerts a significant impact on the numerical results.[12] Gary *et al.* concluded that the mass sink contributes to the lower pressure, stronger wind speeds and heavier precipitation in Hurricane Lili (2002).[13] How-

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†Corresponding author. E-mail: rlk@mail.iap.ac.cn

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ever, the mass loss of the precipitating atmosphere is not taken into account in many sophisticated meso-scale models.[14–16]

The mass loss due to precipitation will modify the microphysical and dynamic fields. Therefore, the mass loss should be involved not only in the continuity equation of mass, but also in the momentum equation. The influence of precipitation on momentum is more complicated.[17,18] Satoh drafted a conservative scheme for a compressible non-hydrostatic model involving the transport of momentum due to rain and tested it for experiments of a squall line.[19]

The mass loss is usually introduced into model equations in the following way. The real atmosphere can be regarded as a mixture of dry air, water vapor, cloud water, rain water, ice, snow and graupel.[20] These compositions move with different velocities. Therefore, their movement can be decomposed to a macroscopic motion with a reference velocity and a relative motion, with a velocity deviated from the reference velocity. The choice of reference velocity is a key step to determining the influence of precipitation on the momentum in a mixing atmosphere. The reference velocity may be assumed to be the barycentric velocity of the mixing atmosphere, or merely the velocity of the mixture of dry air and water vapor, or even the velocity of dry air. Wacker and Herbert formulated and discussed the continuity equation of mass in the three cases of reference velocity.[21] Wacker *et al*. argued that choosing the barycentric velocity as the reference motion can provide the most convenient form for the total mass continuity equation.[22] Ooyama interpreted the velocity of cloudy air, i.e., dry air, water vapor, cloud droplets and cloud ice, as an atmospheric wind field to derive a set of equations for precipitating atmosphere.[23] A similar set of equations was also derived by Bannon in a multiphase and multicomponent fluids perspective;[24] however, the momentum equation contains terms that are different from those of Ooyama.[23] Bott presented a derivation of a model equation system for an open precipitating atmosphere consisting of cloudy air, rain water and snow.[25] The rain water and snow were assumed to move with the same horizontal velocity as the cloud air and that they fell with a constant terminal velocity relative to the vertical velocity of cloud air.

Although there has been some research on model equations for multicomponents in the frame of a specific reference velocity,[26] the literature on the comparison of momentum equations for different reference velocities is unavailable. The mass loss in continuity and momentum equations has also been little addressed. Therefore, this compilation will contribute to deriving mass and momentum equations for the three different reference velocities and discuss the influence of mass loss due to precipitation on atmospheric mass and momentum. Their distinctions in the three reference velocities are focused on further next.

2. Continuity and momentum equations in three reference velocity frames

An arbitrary control volume $V(t)$ in space with surface S is considered. Let A describe the flow property per unit volume. The material derivative of the property in volume $V(t)$ is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V(t)} A\,\mathrm{d}\tau &= \int\limits_{V(t)} \frac{\partial A}{\partial t}\,\mathrm{d}\tau + \int\limits_{S} A\boldsymbol{v}_A\cdot\boldsymbol{n}\,\mathrm{d}s \\ &= \int\limits_{V(t)} \left[\frac{\partial A}{\partial t} + \nabla\cdot(A\boldsymbol{v}_A)\right]\mathrm{d}\tau = \int\limits_{V(t)} \dot{A}\,\mathrm{d}\tau, \quad (1)\end{aligned}$$

where $\boldsymbol{v}_A$ is the velocity with which A is transported, $\dot{A}$ denotes the source or sink of A and $\boldsymbol{n}$ denotes the unit vector normal to the volume surface S. Since the control volume $V(t)$ is arbitrary, we have

$$\frac{\partial A}{\partial t} + \nabla\cdot(A\boldsymbol{v}_A) = \dot{A}. \quad (2)$$

Here, A may be an arbitrary quantity of the scalar function or vector,[27–29] such as density ρ and wind vector $\boldsymbol{v} = (u, v, w)$.

The atmospheric volume we consider is made of dry air and a water substance. The latter is recognized in three phases: vapor, liquid and solid condensates. The condensates are further classified into two modes: airborne and precipitating. Airborne includes cloud water and ice particles; the precipitating condensates involve rain water, snow and graupel. For convenience, the dry air, water vapor, cloud water, rain water, ice, snow and graupel in order are denoted by $k = 0, 1, 2, \ldots, 6$. For partial density ρ_k in volume $V(t)$, the partial mass is $M_k = \int_{V(t)} \rho_k\,\mathrm{d}\tau$. The total density and the total mass in volume $V(t)$ can be given by summing the matching partial quantities, namely,

$$\rho = \sum_{k=0}^{6} \rho_k, \quad M = \sum_{k=0}^{6} M_k = \int\limits_{V(t)} \rho\,\mathrm{d}\tau. \quad (3)$$

According to Eq. (1), the partial mass M_k in volume V with velocity $\boldsymbol{v}_k$ satisfies the following equation:

$$\begin{aligned}\frac{\mathrm{d}M_k}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V(t)} \rho_k\,\mathrm{d}\tau \\ &= \int\limits_{V(t)} \left[\frac{\partial \rho_k}{\partial t} + \nabla\cdot(\rho_k\boldsymbol{v}_k)\right]\mathrm{d}\tau = \int\limits_{V(t)} \dot{\rho}_k\,\mathrm{d}\tau, \quad (4)\end{aligned}$$

where $\dot{\rho}_k$ describes the internal source or sink of partial density due to the phase transition of water substances. Because of the arbitrariness of V, one may obtain from Eq. (4)

$$\frac{\partial \rho_k}{\partial t} + \nabla\cdot(\rho_k\boldsymbol{v}_k) = \dot{\rho}_k. \quad (5)$$

The dry air is assumed to be source-free, namely, $\dot{\rho}_0 = 0$. The phase transitions of water substances are often treated in a parameterization way. For example, Lin *et al.* parameterized them as follows:[30]

$$\dot{\rho}_1 = e_c + e_r - c - d_i + s_i - d_s + s_s - d_g + s_g \quad \text{(for water vapor)}, \tag{6}$$

$$\dot{\rho}_2 = c - e_c - T_c \quad \text{(for cloud water)}, \tag{7}$$

$$\dot{\rho}_3 = m - e_r - f - T_r \quad \text{(for rain)}, \tag{8}$$

$$\dot{\rho}_4 = d_i - s_i - T_i \quad \text{(for ice)}, \tag{9}$$

$$\dot{\rho}_5 = d_s - s_s - m_s + f_s - T_s \quad \text{(for snow)}, \tag{10}$$

$$\dot{\rho}_6 = d_g - s_g - m_g + f_g - T_g \quad \text{(for graupel)}, \tag{11}$$

where $m = m_s + m_g$ and $f = f_s + f_g$. The symbols c, e, f, m, d and s stand for the rates of condensation, evaporation, freezing, melting, deposition and sublimation, respectively. The subscripts c, r, i, s and g denote cloud water, rain, ice, snow and graupel, respectively. The terms T_c, T_r, T_i, T_s and T_g are microphysical transfer rates between hydrometeor species; their sum is zero. The parameterization scheme has been widely used in numerical models, such as weather research and forecast (WRF) and advanced regional prediction system (ARPS) models. The interested reader may refer to Ref. [31]. It is hypothesized in the parameterization that the phase transitions do not change the total mass. The hypothesis is often adopted in numerical models. It means the sum of internal source rate $\dot{\rho}_k$ caused by the phase transition vanishing

$$\sum_{k=1}^{6} \dot{\rho}_k = 0. \tag{12}$$

It is noted that a partial mass of atmosphere in the control volume moves with a different velocity. The movement involves two motions: a macroscopic motion of the control volume with a reference velocity $\boldsymbol{v}$ and a relative motion with a deviated velocity, i.e., $\boldsymbol{v}_k - \boldsymbol{v}$. Then, Eq. (5) may be rewritten as

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \boldsymbol{v}) = -\nabla \cdot [\rho_k (\boldsymbol{v}_k - \boldsymbol{v})] + \dot{\rho}_k. \tag{13}$$

The partial mass k moves with $\boldsymbol{v}_k$, which is different from $\boldsymbol{v}$. It can penetrate the surface of the control volume moving with $\boldsymbol{v}$. Therefore, the term $\rho_k (\boldsymbol{v}_k - \boldsymbol{v})$ in Eq. (13) denotes the diffusive flux of the partial mass through the surface. In the frame of reference velocity, the partial mass is not conserved and is changed by the partial mass diffusive flux as well as the phase transitions.

The total momentum per unit volume is taken as $A = \sum_{k=0}^{6} \rho_k \boldsymbol{v}_k$. The external forces acting on any volume of the atmosphere are the total ambient pressure gradient force, the Coriolis force due to Earth's rotation, the gravity force and frictional forces. Using Eq. (2), the budget equation for the total momentum in Cartesian coordinates rotating with Earth's angular velocity $\boldsymbol{\Omega}$ is

$$\frac{\partial}{\partial t}\left(\sum_{k=0}^{6} \rho_k \boldsymbol{v}_k\right) + \nabla \cdot \left(\sum_{k=0}^{6} \rho_k \boldsymbol{v}_k \boldsymbol{v}_k\right) = -\nabla p - \rho \nabla \varphi - f\boldsymbol{k} \times \sum_{k=0}^{6} \rho_k \boldsymbol{v}_{kh} + \boldsymbol{F}, \tag{14}$$

where p is the pressure, φ is the geopotential height, f is the Coriolis constant and $\boldsymbol{F}$ is the friction force. The first term on the right-hand side of Eq. (14) implies the assumption that the pressure gradient force does not act on the hydrometeors of rain water, snow and graupel, but on the dry air, water vapor, cloud water and ice particles. This treatment is justified by Ooyama.[23] He pointed out that since the pressure gradient force acts on the control volume, it does not directly accelerate the condensate whose volume is negligibly small. Bott took the treatment based on the fact that the surfaces of rain and snow are much smaller than those of cloud air (dry air, water vapor, cloud water and cloud ice).[25] The second term on the right-hand side of Eq. (14) represents the change in total momentum due to gravity. As usual, the Coriolis term $-2\boldsymbol{\Omega} \times \sum_{k=0}^{6} \rho_k \boldsymbol{v}_{kh}$ is approximated by $-f\boldsymbol{k} \times \sum_{k=0}^{6} \rho_k \boldsymbol{v}_{kh}$ with f the Coriolis parameter and $\boldsymbol{v}_{kh}$ the horizontal component of $\boldsymbol{v}_k$. Additionally, the term $\boldsymbol{F}$ stands for the viscous stress tensor. The equation has a similar form in the literature of Cotton *et al.*[32] It should be noted that the second term on the left-hand side can be expressed as

$$\nabla \cdot \left(\sum_{k=0}^{6} \rho_k \boldsymbol{v}_k \boldsymbol{v}_k\right) = \frac{\partial}{\partial x}\left(\sum_{k=0}^{6} \rho_k \boldsymbol{v}_k u_k\right) + \frac{\partial}{\partial y}\left(\sum_{k=0}^{6} \rho_k \boldsymbol{v}_k v_k\right) + \frac{\partial}{\partial z}\left(\sum_{k=0}^{6} \rho_k \boldsymbol{v}_k w_k\right), \tag{15}$$

where $\boldsymbol{v}_k = (u_k, v_k, w_k)$ and means the divergence of the momentum flux. In the following section, there are many expressions analogous to the left-hand side of Eq. (15) and they can be expanded like Eq. (15).

For continuity and momentum equations in a numerical model, the reference velocity must be specified. In the following subsections, we choose the velocities of dry air, gas mixture and total air mixture as the reference velocity to complete the derivation of these equations respectively.

2.1. Velocity of dry air as reference

In this subsection, the reference velocity refers to the velocity of dry air $\boldsymbol{v}_0$, namely,

$$\boldsymbol{v} = \boldsymbol{v}_0. \tag{16}$$

In this frame, the partial mass continuity equations can be deduced from Eq. (5) as

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v}) = 0, \tag{17}$$

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \boldsymbol{v}) = -\nabla \cdot \left(\rho_k \boldsymbol{J}_{k,0}\right) + \dot{\rho}_k, \quad k = 1,2,\ldots,6, \tag{18}$$

where $\boldsymbol{J}_{k,0} = \boldsymbol{v}_k - \boldsymbol{v}$ is the diffusion flux of water substance k through the surface of the volume and $\rho_0 \boldsymbol{v}$ and $\rho_k \boldsymbol{v}$ are the dry-air and water-substance partial mass fluxes transferred with the dry-air reference velocity, respectively.

The continuity equation of the total mass is yielded by summing up Eqs. (17) and (18) over $k = 0,1,\ldots,6$ on conditions (3) and (12)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = -\nabla \cdot \sum_{k=1}^{6} \left(\rho_k \boldsymbol{J}_{k,0}\right). \tag{19}$$

The divergence of the mass diffusive flux on the right-hand side of Eq. (19) is non-zero in the dry-air frame. Therefore, the diffusion term serves as a source to make the total mass non-conserved.

With the conditions $\boldsymbol{v}_k = \boldsymbol{v} + \boldsymbol{J}_{k,0}$ and $\boldsymbol{v}_{k\mathrm{h}} = \boldsymbol{v}_\mathrm{h} + \boldsymbol{J}_{k,0\mathrm{h}}$, Eq. (18) may be rewritten as

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v}) \\
&= -\nabla p - \rho \nabla \varphi - f\boldsymbol{k} \times \rho \boldsymbol{v}_\mathrm{h} + \boldsymbol{F} \\
&\quad - \frac{\partial}{\partial t}\left(\sum_{k=1}^{6} \rho_k \boldsymbol{J}_{k,0}\right) - \nabla \cdot \left(\sum_{k=1}^{6} \rho_k \boldsymbol{J}_{k,0}\boldsymbol{v}\right) \\
&\quad - \sum_{k=1}^{6} \nabla \cdot \left(\rho_k \boldsymbol{v}_k \boldsymbol{J}_{k,0}\right) - f\boldsymbol{k} \times \sum_{k=1}^{6} \rho_k \boldsymbol{J}_{k,0\mathrm{h}}.
\end{aligned} \tag{20}$$

Using Eq. (18), the equation for the partial mass diffusive flux is given by

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho_k \boldsymbol{J}_{k,0}\right) + \nabla \cdot \left(\rho_k \boldsymbol{J}_{k,0}\boldsymbol{v}\right) \\
&= \frac{\partial}{\partial t}\left(\rho_k \boldsymbol{J}_{k,0}\right) + \nabla \cdot \left(\rho_k \boldsymbol{J}_{k,0}\boldsymbol{v}_k\right) - \nabla \cdot \left(\rho_k \boldsymbol{J}_{k,0}\boldsymbol{J}_{k,0}\right) \\
&= \boldsymbol{J}_{k,0}\left[\frac{\partial}{\partial t}\rho_k + \nabla \cdot (\rho_k \boldsymbol{v}_k)\right] \\
&\quad + \rho_k \left(\frac{\partial \boldsymbol{J}_{k,0}}{\partial t} + (\boldsymbol{v}_k \cdot \nabla)\boldsymbol{J}_{k,0}\right) - \nabla \cdot \left(\rho_k \boldsymbol{J}_{k,0}\boldsymbol{J}_{k,0}\right) \\
&= \boldsymbol{J}_{k,0}\dot{\rho}_k - \nabla \cdot \left(\rho_k \boldsymbol{J}_{k,0}\boldsymbol{J}_{k,0}\right) \\
&\quad + \rho_k \left(\frac{\partial \boldsymbol{J}_{k,0}}{\partial t} + (\boldsymbol{v}_k \cdot \nabla)\boldsymbol{J}_{k,0}\right).
\end{aligned} \tag{21}$$

Combining Eq. (20) with Eq. (21), one obtains the prognostic equation of atmospheric wind field in flux form

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v}) \\
&= -\nabla p - \rho \nabla \varphi - f\boldsymbol{k} \times \rho \boldsymbol{v}_\mathrm{h} + \boldsymbol{F} - \sum_{k=1}^{6} \nabla \cdot \left(\rho_k \boldsymbol{v}\boldsymbol{J}_{k,0}\right) \\
&\quad - \sum_{k=1}^{6} \boldsymbol{J}_{k,0}\dot{\rho}_k - \sum_{k=1}^{6}\left[\rho_k \left(\frac{\partial \boldsymbol{J}_{k,0}}{\partial t} + (\boldsymbol{v}_k \cdot \nabla)\boldsymbol{J}_{k,0}\right)\right] \\
&\quad - f\boldsymbol{k} \times \sum_{k=1}^{6} \rho_k \boldsymbol{J}_{k,0\mathrm{h}}.
\end{aligned} \tag{22}$$

The last four terms on the right-hand side of Eq. (22) serves as additional sources to the change of momentum in the reference velocity frame. These sources are related to the diffusion of water substances. They occur due to the diffusive flux divergence of the reference velocity momentum (the fifth term), the diffusive flux of the phase transition (the sixth term), the material derivative of the diffusive flux (the seventh term) and the Corlirios force resulting from the diffusive flux (the eighth term). If all compositions of atmosphere move at the same velocity as the dry air, $\boldsymbol{J}_{k,0} = 0$ and Eq. (22) reduces to the traditional momentum equation.

It should be emphasized that the occurrence of additional source terms in the above equations is a feature of atmospheric models dealing with the assumption that water substances move with their own velocities different from the reference velocity. The above equations are rigorously derived from fundamental physical laws and axiomatic statements, such as mass conservation principles and Newton's second law. In this form, however, the numerical evaluation of these equations would be rather complicated. It is significant to find possibilities for simplifications in numerical investigations, e.g., the justified neglect of certain terms. However, Qiu *et al.* argued that there is no need to simplify the model equations, particularly in view of the low additional computer cost that may be saved.[12]

2.2. Velocity of gas mixture as reference

In this subsection, the velocity of the gas mixture (dry air and water vapor) is employed as the reference velocity, namely,

$$\boldsymbol{v} = \frac{\rho_0}{\rho_\mathrm{m}}\boldsymbol{v}_0 + \frac{\rho_1}{\rho_\mathrm{m}}\boldsymbol{v}_1. \tag{23}$$

Note that this concept involves the weighting factors ρ_0/ρ_m and ρ_1/ρ_m, with $\rho_\mathrm{m} = \sum_{k=0}^{1}\rho_k$ denoting the density of the gas mixture. The diffusive flux of composition k relative to the gas mixture can be written as

$$\boldsymbol{J}_{k,1} = \boldsymbol{v}_k - \boldsymbol{v}. \tag{24}$$

Then, one can deduce

$$\sum_{k=0}^{k=1} \rho_k \boldsymbol{J}_{k,1} = 0. \tag{25}$$

Using Eqs. (5) and (24), the continuity equation of the partial mass in the gas-mixture frame is expressed as

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \boldsymbol{v}) = -\nabla \cdot \left(\rho_k \boldsymbol{J}_{k,1}\right) + \dot{\rho}_k, \quad k = 0,1,\ldots,6, \tag{26}$$

which is the same as Eq. (18) in the dry-air frame, except $\boldsymbol{J}_{k,1}$. Summing Eq. (26) over $k=0,1$ may give the continuity equation of the gas mixture

$$\frac{\partial \rho_{\rm m}}{\partial t}+\nabla\cdot(\rho_{\rm m}\boldsymbol{v})=\dot{\rho}_1, \tag{27}$$

where Eq. (25) is used. It is clear from Eq. (27) that the density of the gas mixture depends on internal source $\dot{\rho}_1$, i.e., the rate of phase changes between water vapor and hydrometeors.

The continuity equation of the total mass is given by summing Eq. (26) over $k=0,1,\ldots,6$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=-\nabla\cdot\sum_{k=2}^{6}\left(\rho_k\boldsymbol{J}_{k,1}\right), \tag{28}$$

where conditions (12) and (25) are made use of. Evidently, the total mass is not conserved in the gas-mixture frame because of the hydrometeor diffusive flux.

By utilizing Eqs. (21) and (26), the prognostic equation of the total momentum in the gas-mixture frame can be given by

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho\boldsymbol{v})+\nabla\cdot(\rho\boldsymbol{v}\boldsymbol{v})\\ =&-\nabla p-\rho\nabla\varphi-f\boldsymbol{k}\times\rho\boldsymbol{v}_{\rm h}+\boldsymbol{F}-\sum_{k=2}^{6}\nabla\cdot\left(\rho_k\boldsymbol{v}\boldsymbol{J}_{k,1}\right)\\ &-\sum_{k=2}^{6}\boldsymbol{J}_{k,1}\dot{\rho}_k-\sum_{k=2}^{6}\rho_k\left(\frac{\partial \boldsymbol{J}_{k,1}}{\partial t}+\boldsymbol{v}_k\cdot\nabla\boldsymbol{J}_{k,1}\right)-f\boldsymbol{k}\\ &\times\sum_{k=2}^{6}\rho_k\boldsymbol{J}_{k,1\rm h}-\nabla\cdot\sum_{k=0}^{1}\left(\rho_k\boldsymbol{J}_{k,1}\boldsymbol{J}_{k,1}\right).\end{aligned} \tag{29}$$

Although the second four terms on the right-hand side of Eq. (29) appear in Eq. (22), they are the summations taken for hydrometeors. Besides these terms, there is an additional source term (the last term on the right-hand side of Eq. (29)), which indicates the contribution from the diffusive flux divergence of the gas-mixture mass diffusion to the reference-velocity momentum.

2.3. Velocity of the total air mixture as reference

In this subsection, the reference velocity is chosen as the velocity of the total air mixture, namely, the mean velocity of the dry air, water vapor and hydrometeors

$$\boldsymbol{v}=\sum_{k=0}^{6}\frac{\rho_k}{\rho}\boldsymbol{v}_k, \tag{30}$$

where the weighting factor ρ_k/ρ is involved. In this case, the diffusion flux of composition k against the total mixture of atmosphere is

$$\boldsymbol{J}_k=\boldsymbol{v}_k-\boldsymbol{v}. \tag{31}$$

Hence, one has

$$\sum_{k=0}^{6}\rho_k\boldsymbol{J}_k=0. \tag{32}$$

The partial-mass continuity equation becomes

$$\frac{\partial \rho_k}{\partial t}+\nabla\cdot(\rho_k\boldsymbol{v})=-\nabla\cdot(\rho_k\boldsymbol{J}_k)+\dot{\rho}_k,\quad k=0,1,\ldots,6. \tag{33}$$

Summing Eq. (33) over $k=0,1,\ldots,6$ on conditions (11), (31) and (32) may yield the total-mass continuity equation

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=0, \tag{34}$$

which is free of diffusive flux. Therefore, the total mass is conserved in the total air-mixture frame. It is different from the cases with dry-air and gas-mixture reference velocities in which the non-conservation of the total mass results from the partial-mass diffusive flux.

The total momentum equation can be derived from Eq. (14) by applying Eq. (32)

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho\boldsymbol{v})+\nabla\cdot(\rho\boldsymbol{v}\boldsymbol{v})\\ =&-\nabla p-\rho\nabla\varphi-f\boldsymbol{k}\times\rho\boldsymbol{v}_{\rm h}+\boldsymbol{F}-\nabla\cdot\sum_{k=0}^{6}(\rho_k\boldsymbol{J}_k\boldsymbol{J}_k).\end{aligned} \tag{35}$$

Clearly, Eq. (35) is significantly different from Eq. (22) in the dry-air frame and Eq. (29) in the gas-mixture frame. There is only one additional source (the last term on the right-hand side of Eq. (35)), which is associated with the diffusion of dry air and water substances. The additional source is caused by the summation of the diffusive flux divergence of the partial mass diffusion.

The distinctions among the continuity and momentum equations in the three reference frames are clear. First, the continuity equations of the total mass in dry-air and gas-mixture frames are not conserved due to the diffusions of water substances and hydrometeors, respectively. The total-mass continuity equation in the total-air mixture frame is conserved. Second, the total-mass momentum equation contains four additional sources due to the diffusions of water substances in the dry-air frame, five additional sources due to the diffusions of hydrometeors in the gas-mixture frame and one additional source due to the diffusions of dry air and water substances in the total air-mixture frame.

2.4. Simplicity of diffusive flux

In the three cases of reference velocity, additional source terms come from the diffusive fluxes due to the compositions of the atmosphere moving with different velocities and the internal source due to the phase transitions of the water substances. The diffusion fluxes may be parameterized by means of the theory of irreversible processes in atmospheric thermodynamics.[33] For a cloudy atmosphere without precipitation, scale analytical considerations yield the conclusion that $\boldsymbol{J}_{k,0}$, $\boldsymbol{J}_{k,1}$ and $\boldsymbol{J}_k$ are several orders of magnitude smaller than

the corresponding turbulent fluxes.[34] Because of this, in atmospheric numerical models, the diffusive fluxes of dry air, water vapor and airborne (cloud water and ice) may safely be neglected. In other words, in the cloudy atmosphere, it is justified to assume that all compositions move with the same reference velocity $\boldsymbol{v}$.

However, for precipitating atmosphere, the velocity $\boldsymbol{v}_k$ of falling precipitation particles may distinctly deviate from $\boldsymbol{v}$. This becomes obvious in heavily precipitating systems. For example, in tropical cyclones, a huge amount of precipitable particles fall down and provide a large contribution to the vertical component w of $\boldsymbol{v}$. On the contrary, the vertical velocity of composition w_k may be small. Therefore, in the precipitating atmosphere, it seems to be no longer justified to neglect diffusive fluxes like in the cloudy system.

The continuity equations of some numerical models are treated in the following way. The diffusion fluxes for dry air, water vapor and airborne particles are neglected. The diffusion fluxes of precipitable liquid and solid particles, such as rain, snow and graupel, are taken as the sedimentation fluxes of precipitation.

According to the treatment, for the dry-air reference velocity, one has

$$\boldsymbol{v}_k = \boldsymbol{v}, \quad \boldsymbol{J}_{k,0} = 0, \quad k = 1,2,4, \tag{36}$$

$$\boldsymbol{v}_k = \boldsymbol{v} + W_k \boldsymbol{k}, \quad \boldsymbol{J}_{k,0} = W_k \boldsymbol{k}, \quad k = 3,5,6, \tag{37}$$

where $\boldsymbol{k}$ is the unit vertical vector pointing upwards and W_k are the terminal velocities of precipitable particles against reference vertical velocity w. The sign conventions for w and W_k are the same. So W_k is always negative (downward). This treatment is also included in the continuity equations (3.1) and (3.2) of Ooyama.[23] Thus, the continuity equation in the dry-air frame (Eq. (19)) becomes

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = -\frac{\partial}{\partial z} (\rho_3 W_3 + \rho_5 W_5 + \rho_6 W_6). \tag{38}$$

The term on the right-hand side indicates the mass flux divergence of precipitable particles due to terminal velocities. If Eq. (38) is vertically integrated, it becomes

$$\int_{z_\mathrm{b}}^{z_\mathrm{t}} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \right] \mathrm{d}z = (\rho_3 W_3 + \rho_5 W_5 + \rho_6 W_6)|_{z_\mathrm{b}}, \tag{39}$$

where z_b is the model bottom and z_t is the model top. Since W_k does not vanish on the ground, the right-hand term is the mass flux of precipitable particles by the terminal velocity on the ground, namely, the surface precipitation. Thus, the term stands for the influence of precipitation (or a loss of precipitating water to the ground) on the evolution of the total atmospheric mass.

Ooyama postulated that the material derivative of W_k along the path precipitation is ignored,[23] namely,

$$\frac{\partial \boldsymbol{J}_{k,0}}{\partial t} + \boldsymbol{v}_k \cdot \boldsymbol{\nabla} \boldsymbol{J}_{k,0} = 0. \tag{40}$$

Using Eqs. (36)–(40), the total momentum equation in the dry-air frame (Eq. (22)) is then reduced to

$$\begin{aligned} &\frac{\partial}{\partial t} (\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \boldsymbol{v}) \\ &= -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \varphi - f \boldsymbol{k} \times \rho \boldsymbol{v}_\mathrm{h} + \boldsymbol{F} \\ &\quad - \frac{\partial}{\partial z} [\boldsymbol{v} (\rho_3 W_3 + \rho_5 W_5 + \rho_6 W_6)] \\ &\quad - (\dot{\rho}_3 W_3 + \dot{\rho}_5 W_5 + \dot{\rho}_6 W_6) \boldsymbol{k}, \end{aligned} \tag{41}$$

where the last two terms on the right-hand side account for the vertical flux divergence of the precipitable-particle momentum by the terminal velocity and the flux of precipitable-particle phase transitions by the terminal velocity, respectively. The equation involves terminal fall speeds of rain, snow and graupel, which is different from Ooyama.[23] The vertical integration of Eq. (41) becomes

$$\begin{aligned} &\int_{z_\mathrm{b}}^{z_\mathrm{t}} \left[\frac{\partial}{\partial t} (\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \boldsymbol{v}) \right] \mathrm{d}z \\ &= \int_{z_\mathrm{b}}^{z_\mathrm{t}} \left[-\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \varphi - f \boldsymbol{k} \times \rho \boldsymbol{v}_\mathrm{h} + \boldsymbol{F} \right. \\ &\quad \left. - (\dot{\rho}_3 W_3 + \dot{\rho}_5 W_5 + \dot{\rho}_6 W_6) \boldsymbol{k} \right] \mathrm{d}z \\ &\quad + [\boldsymbol{v} (\rho_3 W_3 + \rho_5 W_5 + \rho_6 W_6)]|_{z_\mathrm{b}}. \end{aligned} \tag{42}$$

The last term on the right-hand side is the terminal-velocity flux of the precipitable-particle momentum on the ground, indicating the forcing of the surface precipitation to the total atmospheric momentum.

For the gas-mixture reference velocity, the continuity and momentum Eqs. (28) and (29) can be parameterized in the way mentioned above. The resulting equations are the same as Eqs. (38) and (41) in the dry-air frame and omitted here.

For the reference velocity of the total air mixture, under conditions (36) and (37), the total momentum equation (35) becomes

$$\begin{aligned} &\frac{\partial}{\partial t} (\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \boldsymbol{v}) \\ &= -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \varphi - f \boldsymbol{k} \times \rho \boldsymbol{v}_\mathrm{h} + \boldsymbol{F} \\ &\quad - \frac{\partial}{\partial z} \left[(\rho_3 W_3^2 + \rho_5 W_5^2 + \rho_6 W_6^2) \boldsymbol{k} \right], \end{aligned} \tag{43}$$

which excludes the two additional source terms in Eq. (41). The only additional source term on the right-hand side comes from the vertical gradient of the terminal-velocity kinetic energy due to precipitation. It is noted that the additional source

term does not appear in Ooyama's momentum equation.[23] The vertical integration of Eq. (43) may yield

$$\int_{z_b}^{z_t}\left[\frac{\partial}{\partial t}(\rho \boldsymbol{v})+\boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}\boldsymbol{v})\right]\mathrm{d}z$$
$$=\int_{z_b}^{z_t}\left[-\boldsymbol{\nabla}p-\rho\boldsymbol{\nabla}\varphi-f\boldsymbol{k}\times\rho\boldsymbol{v}_h+\boldsymbol{F}\right]\mathrm{d}z$$
$$+\left[\left(\rho_3 W_3^2+\rho_5 W_5^2+\rho_6 W_6^2\right)\boldsymbol{k}\right]\Big|_{z_b}. \quad (44)$$

It is clear that the last right-hand term implies the terminal-velocity kinetic energy at the ground. The term is associated with the surface precipitation and contributes to the total momentum.

3. Conclusions

This paper focuses on the theoretical fundamentals of moist atmospheric dynamics. The moist atmosphere considered here is made of dry air, water vapor, cloud water, rain water, ice, snow and graupel. Since the atmospheric compositions move with different velocities, diffusive fluxes inevitably appear for a specific reference velocity. The continuity and momentum equations are formulated in the reference-velocity frames of dry air, gas mixture and total-air mixture. The mass conservation might not be maintainable in a mixing atmosphere due to the diffusive flux of the partial mass and the phase transitions of water substances. The diffusive flux and phase transitions also introduce additional source terms in the momentum equation.

The total mass and the total momentum equations differ in their structures and physical processes for the three reference velocities. For example, the terms of diffusion flux divergence show up in the continuity equations of water substances if the dry-air velocity is introduced as the reference velocity. However, the additional divergence terms do not occur in the continuity equation of reference density, which is related to the reference velocity, such as, ρ_0 coupled with $\boldsymbol{v}_0$ (Eq. (17)), ρ_m with $\boldsymbol{v}=\frac{\rho_0}{\rho_m}\boldsymbol{v}_0+\frac{\rho_1}{\rho_m}\boldsymbol{v}_1$ (Eq. (27)), and $\rho=\sum_{k=0}^{6}\rho_k$ with $\boldsymbol{v}=\sum_{k=0}^{6}\frac{\rho_k}{\rho}\boldsymbol{v}_k$ (Eq. (34)). The terms of diffusive flux still appear as additional sources or sinks in the total momentum equations for the three reference velocities.

In the dry-air frame, the change of the total momentum caused by the additional sources involves the diffusive flux of water substances and the phase transitions of water substances. In the gas-mixture frame, the additional sources of the total momentum are associated with the diffusive flux of hydrometeors, the phase transitions of hydrometeors and the gas-mixture diffusive flux. In the frame of total air mixture, the diffusive flux of all atmospheric compositions instead of the phase transitions makes contribution to the total momentum.

Furthermore, for simplicity, the diffusive flux is treated in the same way as in the work by Ooyama.[23] It is shown that the continuity and momentum equations in dry-air and gas-mixture frames are the same in form, different from those in the frame of total air mixture. The terminal velocity in the two former frames results in the loss of the total mass, while the total mass is conserved in the latter frame. The vertical flux of the momentum and the phase transition by the terminal velocity exerts an impact on the total momentum in dry-air and gas-mixture frames, while the vertical gradient of the terminal-velocity kinetic energy due to precipitation serve as the only additional source of momentum in the frame of total air mixture.

The three versions of mass and momentum equations derived here contain additional source terms associated with diffusions and phase transitions and are more complicated than the traditional model equations. Heavy precipitation is usually triggered by the interaction of dynamic, thermodynamical and microphysical processes. Owing to computer limitations, at the beginning, the governing model equations had to be drastically simplified. Many traditional models involved the complicated parameterization of cloud microphysical processes. Nevertheless, not all of the sources or sinks were taken into account in the model momentum equations. On the other hand, with increasing computing power, it becomes possible to simulate atmospheric processes with more sophisticated equations, such as the momentum equations obtained here. On the basis of an accurate description of dynamic processes, the numerical model can reproduce more real precipitation events. It sheds some light on ways of improving heavy precipitation numerical forecasting.

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