# Growth rate of peeling mode in the near separatrix region of diverted tokamak plasma*

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An analytical expression of the peeling mode in the near separatrix region of diverted tokamak plasma is derived. It is shown that in diverted plasmas both with single and double $X$ points, though the perturbed potential energy of the unstable peeling mode tends to be large, its growth rate becomes very small due to the even larger kinetic energy. Compared to some recent studies that give qualitatively correct results about this growth rate, our result is directly related with the diverted equilibrium quantities suitable for application to realistic experiments.

**Keywords:** diverted plasma, tokamak pedestal, growth rate of peeling mode

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## 1. Introduction

The edge localized mode (ELM)[1] plays an important role in future tokamaks, such as the International Thermonuclear Experimental Reactor (ITER).[2] This mode, which appears in the pedestal region (an edge region with a steep pressure gradient just inside the separatrix), is thought to determine the width and height of the pedestal[3] and be triggered by ideal magnetohydrodynamic (MHD) modes, namely, the peeling[4,5] and ballooning modes.[5] These are localized modes with large toroidal mode number $n$ and mode width $\varepsilon \ll \psi_{\rm b}$ measured by the poloidal flux function, where $\psi_{\rm b}$ is the total poloidal flux within the plasma region. The peeling mode becomes unstable for axisymmetric toroidal equilibrium with non-zero edge current, and the criterion is as follows:[5]

$$2S/P - 1 > \sqrt{1 - 4D_{\rm M}}. \tag{1}$$

In Ref. [5], Connor, Hastie, Wilson, and Miller (CHWM) revisited the peeling mode problem that was first predicted in Ref. [4] by using the orthogonal curvilinear coordinates based upon the magnetic flux surfaces $(\psi, \chi, \zeta)$. They derived the same criterion as that in Ref. [4] and expressed the quantities in Eq. (1) as follows:

$$P = 2\pi (q')^2 \left( \oint \frac{JB^2}{R^2 B_\chi^2} {\rm d}\chi \right)^{-1}, \tag{2}$$

$$S = P + q' \oint \frac{\boldsymbol{j}\cdot\boldsymbol{B}}{R^2 B_\chi^2} J {\rm d}\chi \left( \oint \frac{JB^2 {\rm d}\chi}{R^2 B_\chi^2} \right)^{-1}, \tag{3}$$

and the Mercier coefficient $D_{\rm M}$ is defined by

$$D_{\rm M} = -Q/P, \tag{4}$$

where

$$\begin{aligned} Q &= p' \left[ \frac{1}{2\pi} \oint (\partial J/\partial \psi) {\rm d}\chi - I q' \oint \frac{J {\rm d}\chi}{R^2 B_\chi^2} \left( \oint \frac{JB^2 {\rm d}\chi}{R^2 B_\chi^2} \right)^{-1} \right] \\ &\quad + p'^2 \left[ \frac{I^2}{2\pi} \left( \oint \frac{J {\rm d}\chi}{R^2 B_\chi^2} \right)^2 \left( \oint \frac{JB^2 {\rm d}\chi}{R^2 B_\chi^2} \right)^{-1} - \frac{1}{2\pi} \oint \frac{J {\rm d}\chi}{B_\chi^2} \right]. \end{aligned} \tag{5}$$

The safety factor $q(\psi)$ is defined by $q = \oint \nu(\psi,\chi) {\rm d}\chi/2\pi$, $q'$ is its derivative with respect to $\psi$, $\nu = IJ/R^2$ represents its local value, while the Jacobian $J = 1/|\nabla\psi \times \nabla\chi \cdot \nabla\zeta|$. The poloidal magnetic field is $B_\chi = |\nabla\psi|/R$, with $R$ being the major radius of the plasma, and $I = RB_\zeta$ is also a flux function, representing the poloidal current flux.

Criterion (1) has been widely used by several authors to discuss the pedestal stability of the H-mode confinement. Because in most tokamak devices, the H-mode confinements were realized in diverted configurations, recently, the MHD stability at separatrix, especially, the $X$-point effect on such stability, was studied analytically[6–9] and numerically.[10–14] It was found that though at the magnetic surface very near the separatrix, the plasma can be unstable according to criterion (1), the growth rate of the peeling mode tends to zero due to a very large kinetic energy.[6–9] However, the previous analytical results are somewhat of qualitative character, the result in Ref. [6] is from a cylindrical model while that in Ref. [8] is obtained by using a circular cross-section equilibrium with an additional current filament just outside the boundary so that it does not give a complete formula about the growth rate that can be compared to a real equilibrium.

In this paper, we report an analytic result about the growth rate of the unstable peeling mode in the near separatrix region

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by using CHWM's formalism. We obtain the kinetic energy in a more accurate form so that the final result about the growth rate is analytically expressed. This can be compared to numerical results. It should be emphasized that these theoretical results are in conflict with the common experimental observations that do not show an $X$ point prohibiting the unstable ELM. Because the idealized separatrix is fragile and easily destroyed by error fields or MHD modes, one way to relax such a conflict is to assume that the realistic plasma–vacuum interface is replaced by an inward surface originally inside the separatrix. At this surface, the peeling mode is unstable.

## 2. Derivation of the growth rate of the peeling mode

According to the energy principle,[7,15] the growth rate of the unstable peeling mode can be calculated by[6–9]

$$\gamma^2 = \frac{-\delta W}{\int \mathrm{d}\tau \rho_0 |\boldsymbol{\xi}|^2}, \tag{6}$$

where $\boldsymbol{\xi}$ is the displacement function, and $\rho_0$ is the plasma mass density.

In Ref. [5], the perturbed potential energy $\delta W$ is derived through a series of minimization steps. Introduce new variables to replace three components of displacement $\boldsymbol{\xi}$

$$X = RB_\chi \xi_\psi, \;\; U = \xi_\zeta / R - I\xi_\chi / R^2 B_\chi, \;\; Z = \xi_\chi / B_\chi. \tag{7}$$

The first step is to minimize $\delta W$ by suitably choosing $Z$; this leads to a condition

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \frac{1}{J}\frac{\partial}{\partial\psi}(JX) + \mathrm{i}nU + \mathrm{i}Bk_{//}Z) = 0, \tag{8}$$

where

$$\mathrm{i}k_{//} = \frac{1}{JB}\left(\frac{\partial}{\partial\chi} + \mathrm{i}n\nu\right). \tag{9}$$

This makes $\delta W$ only contain terms about two variables $X$, $U$ and their derivatives. The peeling mode is a localized mode around the plasma–vacuum interface,[4,5] its radial width $\varepsilon \ll \psi_\mathrm{a}$. We use a new dimensionless radial scale $x = (\psi - \psi_0)/\varepsilon$, here the dimensionless variable $x$ is defined to represent the position of the rational magnetic surface with $q = m/n$ by $x = 0$ and the plasma–vacuum interface by $x = \Delta$.[5] To carry out further minimization, we expand $X$, $U$, $\delta W$ in $\varepsilon$, then minimize $\delta W$ order by order, and thus find $X_0 = 0$, $\delta W^{(-1)} = 0$, $\delta W^{(0)} = 0$. To minimize $\delta W^{(1)}$, the following conditions are obtained:[5]

$$\frac{\partial X_1}{\partial x} = -\mathrm{i}nU_0, \tag{10}$$

$$\boldsymbol{B}_0\cdot\nabla X_1 = 0. \tag{11}$$

This makes $\delta W^{(1)} = 0$. Subsequent minimizations lead to $\delta W^{(2)} = 0$ and an expression of $\delta W^{(3)}$ that contains $X_1$, $X_2$, $U_1$ and their derivatives. After the next minimization step, we obtain $\delta W^{(3)}$ (for simplicity, subscript now being omitted)

$$\delta W = \frac{\varepsilon^3}{2}\int \mathrm{d}x\left[Px^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2 + Q\xi^2 + S\frac{\mathrm{d}}{\mathrm{d}x}(x\xi^2)\right]. \tag{12}$$

A further minimization leads to the following Euler equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}\xi}{\mathrm{d}x}\right) - \frac{Q}{P}\xi = 0. \tag{13}$$

For the case of interest in which the rational surface is outside the plasma, we should select the following solution:[5]

$$\xi = RB_\chi\xi_\psi = \xi_0 x^\sigma, \tag{14}$$

where

$$\sigma = -\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{Q}{P}}\,. \tag{15}$$

We need to emphasize that this mode is a surface free mode so that it has its maximum at the boundary. However, in some numerical studies, boundary condition $\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\psi = 0$ is assumed[13,14] so that the resultant eigenfunction can be different. Thus we obtain the perturbed potential

$$\delta W = -\frac{\varepsilon^3}{2}\Delta^{2\sigma+1}[S + \sigma P]. \tag{16}$$

Now we can use the above results to calculate the kinetic energy $K = 2\pi\int J\mathrm{d}\psi\mathrm{d}\chi\rho_0|\boldsymbol{\xi}|^2$.

If we notice that the linear perturbed force vector is[13]

$$\begin{aligned}\boldsymbol{F} \;=\; & \boldsymbol{j}_0\times\boldsymbol{Q} + \boldsymbol{\nabla}\times\boldsymbol{Q}\times\boldsymbol{B}_0 \\ & - \boldsymbol{\nabla}[-\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0 - \gamma_\mathrm{s}p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}],\end{aligned} \tag{17}$$

with $\gamma_\mathrm{s}$ the ratio of specific heats, and consider the minimization condition, we find that this force is free from a change of type $\boldsymbol{\xi}_1 = \boldsymbol{\xi} + c\boldsymbol{\xi}\cdot\boldsymbol{B}/B$ with $c$ an arbitrary constant. Because for an unstable mode, a minimized kinetic energy corresponds to the largest growth rate, we put $c = 0$, or

$$B_\chi\xi_\chi + B_\zeta\xi_\zeta = 0. \tag{18}$$

From the tokamak ordering, we then obtain

$$\xi_\zeta \approx 0. \tag{19}$$

From condition (10), we obtain

$$U_0 = -\frac{I}{R^2B_\chi}\xi_\chi = \mathrm{i}\xi_0\frac{\sigma}{n}x^{\sigma-1}. \tag{20}$$

From Eqs. (14) and (20), we obtain

$$\begin{aligned}K \;=\; & \pi\varepsilon^3\rho_0\gamma^2\xi_0^2\left[\frac{1}{|2\sigma+1|}\left(\oint\frac{J\mathrm{d}\chi}{|\boldsymbol{\nabla}\psi|^2}\right)\Delta^{2\sigma+1}\right. \\ & \left. + \frac{1}{n^2}\left(\oint\frac{R^4B_\chi^2J\mathrm{d}\chi}{I^2}\right)\frac{\sigma^2}{|2\sigma-1|}\Delta^{2\sigma-1}\right].\end{aligned} \tag{21}$$

## 3. Approximate expressions near the $X$-point region

Now we turn to discuss how to calculate the integrals appearing in quantities $Q$, $P$, $S$. Clearly, the main contributions to these integrals are from the short arc near the $X$ point, and these integrals are all divergent when the magnetic surface considered approaches to the separatrix. For both single-null and double-null tokamak equilibriums, in the near the $X$-point region, we can expand the poloidal magnetic flux function into[16]

$$\psi(R,z)=\psi_X+\frac{1}{2}[\psi_{RR}(R-R_X)^2+\psi_{zz}(z-z_X)^2]+\cdots. \quad (22)$$

Without loss of generality, we assume $\alpha=\psi_{ZZ}/2>0$ and $\psi_{RR}=-\lambda\alpha$,

$$\psi=\psi_X-\lambda\alpha(R-R_X)^2+\alpha(z-z_X)^2. \quad (23)$$

Consider the case that the $X$ point is at the upper half part of the plasma, thus in the region near the $X$ point and within the plasma volume, the magnetic surface is approximately expressed by

$$z = z_x-\sqrt{\lambda(R-R_x)^2+\frac{(\psi-\psi_x)}{\alpha}}, \quad (24)$$

$$\begin{aligned}|\nabla\psi| &= 2\alpha[\lambda^2(R-R_X)^2+(z-z_X)^2]^{1/2}\\ &= 2\alpha\sqrt{\left(\frac{\psi-\psi_X}{\alpha}\right)+\lambda(1+\lambda)(R-R_X)^2}. \end{aligned} \quad (25)$$

When the toroidal current density at the $X$ point is zero, it can be seen from the Grad–Shafranov equation at the $X$ point that we have $\lambda=1$. In this case, two asymptotes through the $X$ point are orthogonal to each other, while for non-zero $j_\zeta(X)$ corresponding to the unstable peeling mode, $\lambda>1$, they are not orthogonal to each other.[16] Using these approximate results, we can write

$$\begin{aligned}\frac{J\mathrm{d}\chi}{|\nabla\psi|^2} &= \frac{\mathrm{d}l_\chi}{|\nabla\psi|^2B_\chi}\\ &= \frac{R}{4\alpha^2\Lambda^2}\frac{\mathrm{d}x}{(1+(\lambda+1)\lambda)x^2)\sqrt{1+x^2}}, \end{aligned} \quad (26)$$

where

$$\Lambda^2=(\psi-\psi_x)/\alpha,\ \ x=(R-R_x)/\Lambda. \quad (27)$$

We can write

$$\oint\frac{J\mathrm{d}\chi}{|\nabla\psi|^2}\approx\frac{R_x}{2\alpha^2\Lambda^2}\zeta(\lambda)=\frac{R_x}{2\alpha(\psi-\psi_x)}\zeta(\lambda), \quad (28)$$

$$\zeta(\lambda)=\int_0^\infty\frac{\mathrm{d}x}{(1+(\lambda+1)\lambda)x^2)\sqrt{1+x^2}}. \quad (29)$$

A useful result from this discussion is an approximate expression for quantity $q'(\psi)$

$$q'(\psi)=-\frac{I}{2\pi\alpha R_x(\psi-\psi_x)}. \quad (30)$$

In the same approximation, we can also obtain some relevant integrals

$$\oint\frac{B^2J\mathrm{d}\chi}{|\nabla\psi|^2}=\frac{I^2}{R_x\alpha^2(\psi-\psi_x)}\zeta(\lambda), \quad (31)$$

$$\oint\frac{\boldsymbol{j}\cdot\boldsymbol{B}J\mathrm{d}\chi}{|\nabla\psi|^2}\Big/\oint\frac{B^2J\mathrm{d}\chi}{|\nabla\psi|^2}=\left(\frac{j_\zeta}{B}\right)_x. \quad (32)$$

It is also found that in Eq. (5) for $Q$, the quantities in the two brackets cancel each other in the lowest order[6–8]

$$Q\approx0,\ \ \sigma=-1, \quad (33)$$

$$P=\frac{q'(\psi)\alpha}{I\zeta(\lambda)}, \quad (34)$$

$$S+\sigma P=q'\left(\frac{j_\zeta}{B}\right)_x. \quad (35)$$

This makes the change in potential energy be

$$\delta W=-\frac{\varepsilon^3}{2}\xi_0^2\Delta^{-1}q'(j_\zeta/B)_x. \quad (36)$$

Now consider the kinetic energy $K$, denote the distance between two neighboring rational magnetic surfaces as

$$\Delta_{m,m-1}=\frac{1}{\varepsilon nq'(\psi)}, \quad (37)$$

and

$$0<\tau=\Delta/\Delta_{m-1,m}=\frac{\psi_{m,n}-\psi_{\mathrm{a}}}{\psi_{m,n}-\psi_{m-1,n}}<1, \quad (38)$$

we obtain

$$\frac{1}{n\Delta}=\frac{\varepsilon q'}{\tau}. \quad (39)$$

The kinetic energy then becomes

$$\begin{aligned}K &= \frac{\varepsilon^3}{2}\xi_0^2\Delta^{2\sigma+1}2\pi\rho_0\gamma^2\left(\frac{1}{|2\sigma+1|}\oint\frac{J\mathrm{d}\chi}{|\nabla\psi|^2}\right.\\ &\quad\left.+\frac{\sigma^2}{|2\sigma-1|}\tau^2q'^2\oint\frac{R^4B_\chi^2J\mathrm{d}\chi}{I^2}\right). \end{aligned} \quad (40)$$

Now we find that unlike the normal perturbation component $\xi_\psi$ that is very large in the near $X$-point region, the poloidal perturbation component $\xi_\chi$ is very large in the whole magnetic surface due to the mode locality. It has its maximum in the middle plane and becomes very small in the near $X$-point region. It is seen that in the near separatrix region the second term is dominant. For the unstable peeling mode, the growth rate should be determined by

$$\frac{\gamma^2}{\omega_{\mathrm{A}}^2}=\frac{3}{2\pi}\frac{j_{//}/B}{q'}\frac{\tau^2}{\oint B_\chi\mathrm{d}l}, \quad (41)$$

where $\omega_{\mathrm{A}}=B/R\sqrt{\rho_0}$ is the Alfven frequency. Defining an average minor radius $\langle r\rangle$ and an average toroidal current density $\langle j_\zeta\rangle$, we have $\oint B_\chi\mathrm{d}l=\pi\langle r\rangle^2\langle j_\zeta\rangle$. As indicated by

some recent analyses, the presence of the $X$ point makes the growth rate of this unstable mode very small according to $\gamma \propto 1/\sqrt{q'}$.[6–9] We can rewrite Eq. (41) as

$$\frac{\gamma^2}{\omega_{\rm A}^2} = \frac{3\tau^2}{2\pi}\left(\frac{j_\zeta}{\langle j_\zeta\rangle}\right)\frac{1}{q'(\psi)\Phi(\psi)}, \tag{42}$$

with $\Phi(\psi) = \pi\langle r\rangle^2 B$ being the toroidal magnetic flux. For the largest growth rate, we let $\tau = 1$, i.e., $\Delta \approx \Delta_{m,n}$ and when the plasma–vacuum interface is close to a neighboring rational surface $q = (m-1)/n$, the mode becomes most unstable. It is clear that the instability is caused by the non-zero toroidal edge current and related with the total toroidal magnetic flux $\Phi(\psi)$; these are not indicated in the previous work.[6,8] The above result indicates that the growth rate is proportional to $\sqrt{(\psi - \psi_x)}$ so that the growth rate at the separatrix tends to zero as indicated already in the previous work.

## 4. Summary

We have derived an expression of the growth rate of the peeling mode in the near separatrix region of the diverted tokamak plasma, Eq. (42), which is related with real equilibrium quantities so that it can be directly compared to experimental or numerical results. It should be noted that the decrease of the growth rate to zero approaching the separatrix is not due to the decrease of the potential energy, but mainly due to the even faster increase in the poloidal perturbed velocity in the near separatrix region. This is in turn the property of the ideal incompressible MHD plasma combined with a special structure of the magnetic topology near the separatrix where the magnetic shear becomes divergently large. Obviously, in this region, non-ideal effects and the initial effect itself can affect the mode structure severely. Another important feature is the fragility of the $X$-point structure; it can be easily destructed by engineering effects (error field, ripple, etc.) and plasma perturbations. As indicated in a recent numerical study,[11] as an unstable ballooning or peeling mode grows, the plasma separatrix is perturbed and the magnetic surface splits. This unstable surface, approaching to the $X$ point, develops large oscillations around the original boundary. If such a destruction of the separatrix happens, our result implies that the last closed magnetic surface shifts inwardly and the growth rate of the peeling mode increases according to $\sqrt{(\psi - \psi_x)}$.

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