Dual solutions in boundary layer flow of a moving fluid over a moving permeable surface in presence of prescribed surface temperature and thermal radiation

Swati Mukhopadhyay†

Department of Mathematics, The University of Burdwan, Burdwan-713104, W. B., India

(Received 3 April 2013; revised manuscript received 19 May 2013; published 3 December 2013)

An analysis of the heat transfer for a boundary layer forced convective flow past a moving permeable flat surface parallel to a moving fluid is presented. Prescribed surface temperature at the boundary is considered. A thermal radiation term in the energy equation is considered. The similarity solutions for the problem are obtained and the reduced ordinary differential equations are solved numerically. To support the validity of the numerical results, a comparison is made with the available results for some particular cases of this study. Dual solutions exist when the surface and the fluid move in the opposite directions.

Keywords: moving fluid, prescribed surface temperature, suction, thermal radiation

PACS: 47.15.Cb, 44.20.+b, 47.85.–g **DOI:** 10.1088/1674-1056/23/1/014702

1. Introduction

The flow and heat transfer of a viscous fluid over a moving surface have many important applications in modern industry, such as polymer industry, glass fiber drawing, crystal growing, plastic extrusion, continuous casting, etc. Many researchers[1–4] have investigated the boundary layer flow on a flat plate. They considered the boundary layer flow on a moving flat plate in a quiescent fluid. Afzal *et al.*[5] combined the Blasius and Sakiadis problems successfully using composite velocity. Later on, by studying the effects of radiation on the boundary layer, Batallar[6] and Cortell[7] extended respectively the Blasius and Sakiadis problems. Considering the suction/blowing effect on the boundary, a new dimension is added to the above mentioned study of Cortell[7] by Ishak.[8]

Moving wall boundary layer flow was studied by different researchers.[9,10] Recently, Cortell[11] discussed the effects of viscous dissipation on the thermal field for constant as well as prescribed power-law surface temperature. However, the existence of dual solutions was not discussed due to the smaller range of velocity ratio between the free stream and the composite velocity considered in that study. By investigating the effects of suction and injection for constant surface temperature, Ishak *et al.*[12] showed that dual solutions exist when the sheet moves in the opposite direction to the free stream. However, the prescribed surface temperature was disregarded in their study.[13] In this investigation, a quadratic power-law thermal boundary condition is considered. Continuous surface heat transfer problems have many practical applications in industrial manufacturing processes. Such processes are hot rolling, wire drawing and glass fiber production. In particular, the power-law form of the variable surface temperature can be used to calibrate the thermal boundary condition at the surface to experiments involving a heating element and a particular material. The radiative heat transfer effects on different flows are very important in space technology and high temperature processes.[14] However, the effects of radiation on the boundary layer are not known in detail. Recently, Mukhopadhyay *et al.*[14] reported the existence of dual solutions in the case of flow and heat transfer over a permeable moving plate in the presence of thermal radiation. However, in their study, a uniform wall temperature was assumed. In our investigation, the combined effects of suction at the boundary and thermal radiation are considered. The prescribed surface temperature is assumed. Dual solutions also exist in this case and the results are presented and explained through their graphical representations.

2. Formulation of the problem

We consider a forced convective, two-dimensional steady laminar boundary-layer flow of an incompressible, viscous fluid over a flat surface moving with constant velocity U_w in the same or opposite direction to the free stream U_∞ (directed towards the positive x direction). The x axis extends parallel to the surface, while the y axis extends upwards, normal to the surface. The governing equations for boundary layer flows and heat transfer are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\kappa}{\rho c_{\rm p}}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho c_{\rm p}}\frac{\partial q_{\rm r}}{\partial y}, \tag{3}$$

†Corresponding author. E-mail: swati_bumath@yahoo.co.in

© 2014 Chinese Physical Society and IOP Publishing Ltd

where u and υ are the components of velocity respectively in the x and y directions, μ is the coefficient of fluid viscosity, ρ is the fluid density, $\nu = \mu/\rho$ is the kinematic viscosity, T is the temperature, κ is the thermal conductivity of the fluid, q_r is the radiative heat flux, c_p is the specific heat at a constant pressure. Since the velocity of the fluid is low (laminar flow), the viscous dissipative heat is assumed to be negligible here. Using the Rosseland approximation for radiation,[15] we can write

$$q_r = -\frac{4\sigma}{3k^*}\frac{\partial T^4}{\partial y}, \tag{3a}$$

where σ is the Stefan–Boltzman constant and k^* is the absorption coefficient.

Assuming that the temperature difference within the flow is such that T^4 may be expanded in a Taylor series and expanding T^4 about T_∞ and neglecting higher orders, we obtain $T^4 \equiv 4T_\infty^3 T - 3T_\infty^4$. Therefore, equation (3) becomes

$$u\frac{\partial T}{\partial x} + \upsilon\frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{16\sigma T_\infty^3}{3\rho c_p k^*}\frac{\partial^2 T}{\partial y^2}. \tag{4}$$

The appropriate boundary conditions for the problem are given by

$$u = U_w, \quad \upsilon = V_w, \quad T = T_w, \quad y = 0, \tag{5a}$$

$$u \to U_\infty, \quad T \to T_\infty, \quad y \to \infty. \tag{5b}$$

Here, V_w is the velocity of suction ($V_w > 0$) or injection ($V_w < 0$), $T_w = T_\infty + Ax^2$ is the prescribed surface temperature, T_∞ is the free stream temperature assumed to be constant with $T_w > T_\infty$.

3. Similarity analysis and solution procedure

With the help of a composite velocity $U = U_w + U_\infty$, we now introduce the following dimensionless variables:

$$\eta = y\sqrt{\frac{U}{2\nu x}}, \; u = Uf'(\eta), \; \upsilon = U\frac{\eta f'(\eta) - f(\eta)}{\sqrt{2Re_x}}, \tag{6}$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \tag{7}$$

where $Re_x = Ux/\nu$ is the local Reynolds number.

Using the above equations in the boundary layer Eq. (2) and in the energy Eq. (4), we obtain the following equations:

$$f''' + ff'' = 0, \tag{8}$$

$$\left(1 + \frac{4}{3N}\right)\frac{\theta''}{Pr} + f\theta' - 4f'\theta = 0, \tag{9}$$

where $Pr = \mu c_p/\kappa$ is the Prandtl number and $N = kk^*/4\sigma T_\infty^3$ is the radiation parameter.

For the existence of similarity solutions V_w is taken as $V_w = -(\nu U/2x)^{1/2} v_0$, where v_0 is a quantity which determines the transpiration rate at the surface with $v_0 > 0$ for suction, $v_0 < 0$ for injection, and $v_0 = 0$ corresponding to an impermeable plate. Then, the boundary conditions (5a) and (5b) become

$$f' = 1 - r, \quad f = S, \quad \theta = 1, \quad \eta = 0, \tag{10}$$

$$f' = r, \quad \theta = 0, \quad \eta \to \infty, \tag{11}$$

where $r = U_\infty/U$ is the velocity ratio parameter and $S = v_0$ is the suction parameter (for details see Ref. [14]).

Equations (8) and (9) along with boundary conditions were solved numerically by the shooting method.[14]

4. Results and discussion

In order to obtain a clear insight of the physical problem, numerical computations have been carried out using the shooting method for various values of different parameters, such as velocity ratio parameter (r), suction parameter (S), radiation parameter (N), and Prandtl number (Pr) encountered in this problem. For illustrations of the results, numerical values are plotted in Figs. 1(a)–4(b).

In order to assess the accuracy of the method, the numerical results obtained in this study are compared with those available from the literatures for variable r and are presented in Table 1. The results agree well with those obtained by Cortell[11] and Ishak *et al.*[12]

It is noted that dual solutions exist for certain values of the chosen parameters. The two branches (upper and lower branches) merge with one another at a critical point (the critical value of r, say r_c). The boundary layer solutions beyond this point are impossible to be obtained due to boundary layer separation from the surface. Based on this investigation, $r_c = 1.54892$, 1.82134 for $S = 0$, 0.2 respectively. Therefore, it can be stated that there are dual solutions for $1 < r < 1.54892$ and no solution for $r > 1.54892$ for an impermeable surface. This agrees excellently with those reported by Afzal *et al.*[5] and Ishak *et al.*[12] for an impermeable sheet. Moreover, the suction widens the range of r, for which the solution exists.

Figures 1(a)–1(c) present the dual nature of stream function, velocity and shear stress profiles respectively for variable values of velocity ratio parameter r in the presence of suction, while figures 1(d)–1(e) display the dual nature of temperature profiles for variable r in the presence of both suction and thermal radiation.

For stream function profiles, a crossing over is noted for the upper branch solution [Fig. 1(a)]. Initially, $f(\eta)$ decreases, while for larger values of η, it increases, i.e., the region of reverse flow appears here.[14] This is quite obvious as the plate

and the free stream move in opposite directions. With the increase of the velocity ratio parameter, fluid velocity decreases near the wall but it increases after a fixed point ($\eta \approx 2$) in case of an upper branch solution [Fig. 1(b)]. Figure 1(c) exhibits increasing nature of shear stress with the increase of velocity ratio parameter for the upper branch solution. The temperature is found to increase with the increase of r for the upper branch solution but it decreases in the case of the lower branch [Fig. 1(d)]. The wall temperature gradient is negative for all cases considered here, which implies that heat is transferred from the plate surface to the fluid [Fig. 1(e)]; this is very significant and useful in technology, since cooling is one of the technical challenges faced in many industries.

The dual nature of velocity, shear stress, temperature and temperature gradient profiles for variable values of suction parameter S for $r = 1.5$ are exhibited respectively in Figs. 2(a)–2(d). Figure 2(a) shows that the fluid velocity increases with the increase of S for the upper branch solution but the opposite behavior is noted for the lower branch solution. The boundary layer thickness for the lower branch solution is larger than that of the upper branch solution. For the upper branch solution, though the shear stress increases initially, it decreases for larger η with the increase of S, as shown in Fig. 2(b). For the lower branch solution, the behavior is the opposite.

Table 1. Values of velocity gradient $f''(0)$ for non-porous flat surface.

r	$f''(0)$				
	Cortell[11]	Ishak *et al.*[12]		Present study	
		Upper branch	Lower branch	Upper branch	Lower branch
0	−0.627547	−0.627562		−0.627560	
0.1	−0.493711	−0.493760		−0.493757	
0.2	−0.363308	−0.363346		−0.363342	
0.25	−0.299593				
0.3	−0.237132	−0.237133		−0.237131	
0.4	−0.115777	−0.115810		−0.115807	
0.5	0.0	0.0		0.0	
0.6	0.109652	0.109638		0.109636	
0.7	0.212374	0.212373		0.212374	
0.75	0.261075				
0.8	0.307378	0.307355		0.307352	
0.9	0.393567	0.393563		0.393564	
1.0	0.469602	0.469601		0.469602	
1.1		0.533708	0.001493	0.533707	0.001491
1.2				0.583176	0.016172
1.3		0.583178	0.016171	0.613645	0.051940
1.4				0.616142	0.117885
1.5		0.613646	0.051941	0.565823	0.241874
		0.616140	0.117886		
		0.565821	0.241872		

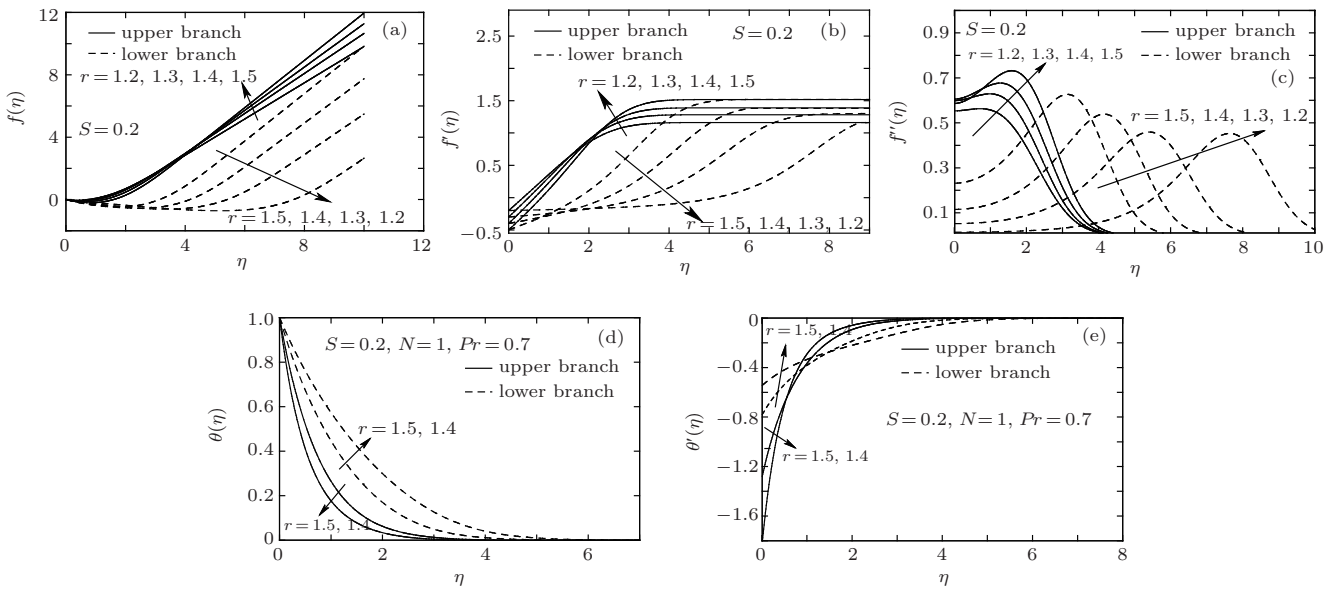


Fig. 1. Dual nature of (a) stream function, (b) velocity, (c) shear stress, (d) temperature and (e) temperature gradient profiles for several values of the velocity ratio parameter r.

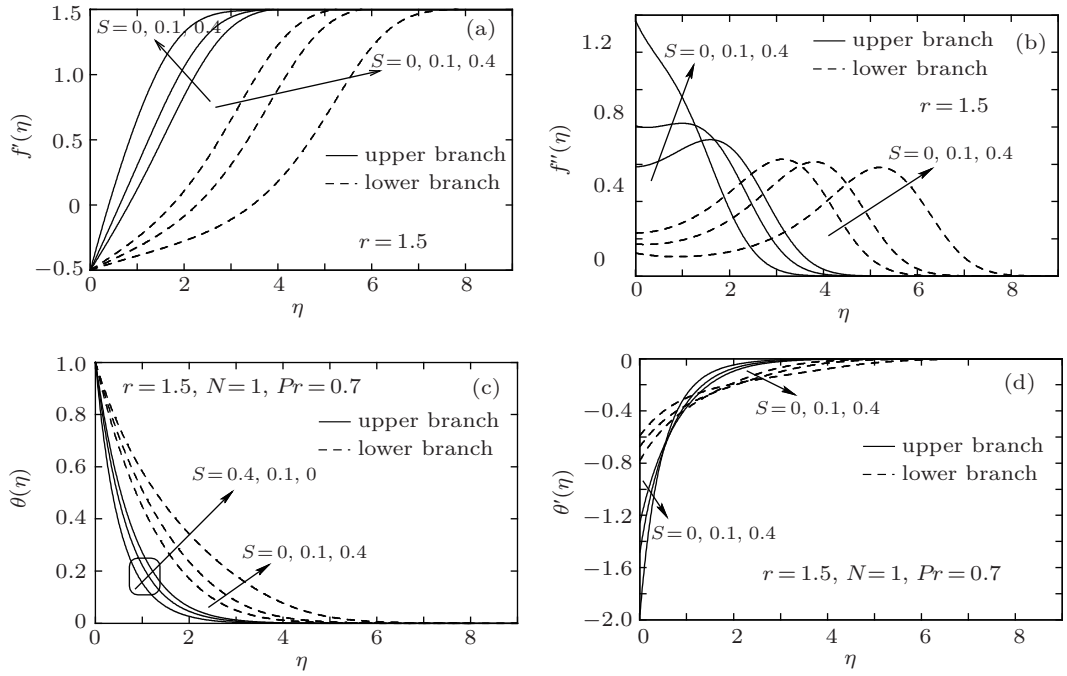


Fig. 2. Dual nature of (a) velocity (b) shear stress, (c) temperature and (d) temperature gradient profiles for several values of the suction parameter S.

In the case of the upper branch solution, the temperature decreases with S increasing; however, it increases for the lower branch solution [Fig. 2(c)]. This can be used in different experiments involving a heating/cooling element of a particular material. It is observed that the magnitude of the temperature gradient decreases initially with the increase of S for the upper branch solution but it increases with larger η [Fig. 2(d)]. The opposite nature is noted for the lower branch solution. The boundary layer thickness for the lower branch solution is higher compared to the upper branch solution.

Figures 3(a)–3(b) display the effects of thermal radiation on the temperature profile for $r = 1.3$, 1.5 respectively. A crossing over in the lower branch solution is noted in Fig. 3(a) but no crossing over is found in Fig. 3(b). Such a type of crossing over was not found in Ref. [14]. This is due to the combined effects of the velocity ratio parameter, prescribed surface temperature and thermal radiation. From both figures, it is very clear that the temperature decreases (for the upper branch solution) with the increasing values of the radiation parameter N. The increase in the radiation parameter means the release of heat energy from the flow region and so the fluid temperature decreases. This can be explained as follows. A decrease in the values of N for given k and T_∞ means a decrease in the Rosseland radiation absorptivity k^*. Since divergence of the radiative heat flux $\partial q_r/\partial y$ increases k^* decreases, which in turn causes it to increase the rate of radiative heat transfer to the fluid and hence the fluid temperature increases. For the upper branch solution, the magnitude of the temperature gradient decreases with the increase of the radiation parameter near the plate but it increases after a fixed point [Fig. 3(c)]. Thus, thermal radiation is very useful in high temperature processes as it plays an important role in controlling the heat transfer process in industry.

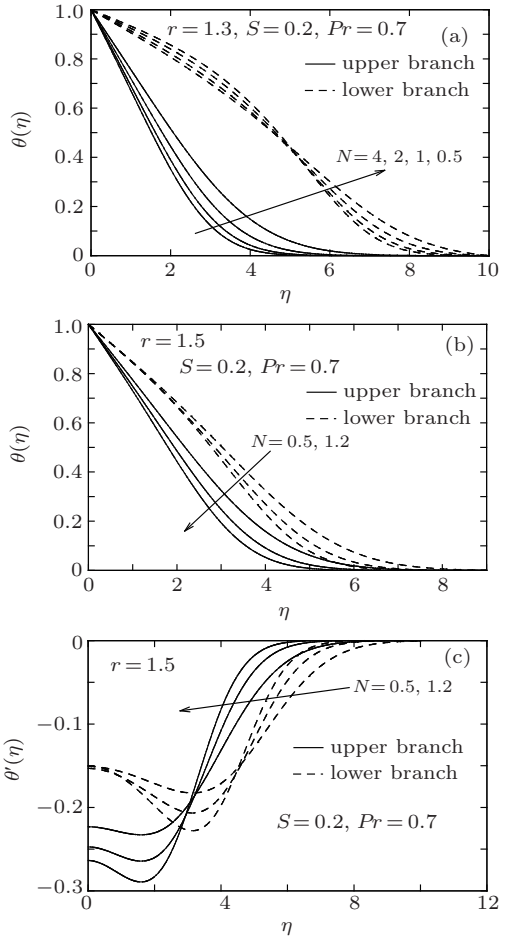


Fig. 3. Dual nature of (a) temperature (when $r = 1.3$), (b) temperature (when $r = 1.5$) and (c) temperature gradient profiles for several values of radiation parameter N.

Figures 4(a)–4(b) depict the dual nature of temperature and temperature gradient profiles for variable values of Pr for $r = 1.5$. The temperature decreases with Pr increasing for the upper branch solution. For the lower branch solution, initially, the temperature increases with Pr increasing, but away from the surface, the temperature is found to decrease [Fig. 4(a)]. Such a type of behavior was not observed in Ref. [13]. This behavior is due to the presence of a prescribed surface temperature and thermal radiation. The boundary layer thickness for the lower branch solution is higher than that of the upper branch solution. This supports the fact that the upper branch solutions are stable while the lower branch solutions are not. The increase in Pr results in the increase in the heat transfer rate [Fig. 4(b)]. Fluid with higher Pr has a relatively low thermal conductivity and thereby reduces the thermal boundary layer thickness and as a result the heat transfer rate at the surface increases. At present, the enhancement of heat transfer is an important topic. From an energy saving perspective, this result can be useful in electronic cooling, heat exchangers, double plane windows, etc.

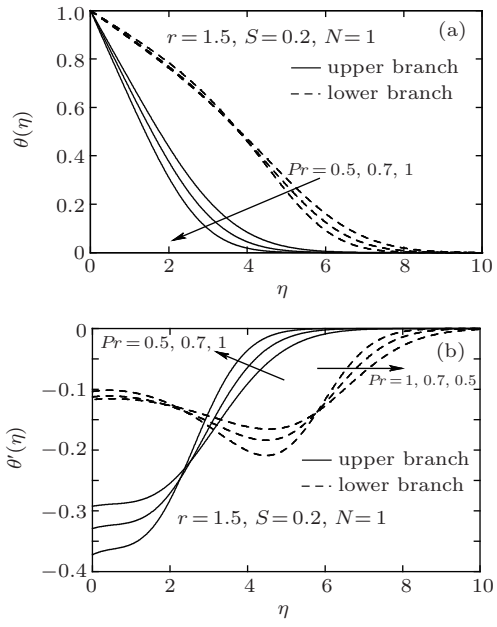


Fig. 4. Dual nature of (a) temperature and (b) temperature gradient profiles for several values of Pr.

Between these two solutions (upper and lower branch solutions), it is very important to know which solution is physically relevant.[16] It seems to suggest that the upper branch solution is physically relevant as the upper branch solution is the only solution valid for $0 \leq r - 1 \leq 1$. In the recent literature for similar problems, it is shown that the lower branch solution is unstable and hence unphysical. Since the stability of this problem is essentially an important issue, we refer the reader to Ref. [17]. Note that in similar studies where two solution branches were obtained, one solution is found to be stable, whereas other solution branches are unstable.[18–20]

5. Conclusions

Effects of thermal radiation on heat transfer characteristics for a moving non-isothermal surface parallel to a free stream with prescribed power-law surface temperature have been considered. Crossing over in the temperature profiles (for the lower branch solution) is noted due to presence of prescribed power law surface temperature and thermal radiation. Moreover, in this manuscript, a discussion about the multiplicity of solutions (numerically) for the momentum field and speculation on which is physically relevant have been provided. This brief discussion hints at a stable solution, which is very significant.

Acknowledgment

The author acknowledges the financial assistance received from UGC (New Delhi, India) through the Special Assistance Programme DSA Phase I.

References

[1] Blasius H 1908 *Zeitschrift für Mathematik und Physik* **56** 1
[2] Wang L 2004 *Appl. Math. Comp.* **157** 1
[3] Cortell R 2005 *Appl. Math. Comp.* **170** 706
[4] Sakiadis B C 1961 *AIChE J.* **7** 26
[5] Afzal N, Badaruddin A and Elgarvi A A 1993 *Int. J. Heat Mass Tranf.* **36** 3399
[6] Bataller R C 2008 *Appl. Math. Comp.* **198** 333
[7] Cortell R 2008 *Chin. Phys. Lett.* **25** 1340
[8] Ishak A 2009 *Chin. Phys. Lett.* **26** 034701
[9] Klemp J B and Acrivos A 1976 *J. Fluid Mech.* **76** 363
[10] Hussaini M Y, Lakin W D and Nachman A 1987 *SIAM J. Appl. Math.* **47** 699
[11] Cortell R 2007 *Theor. Comput. Fluid Dyn.* **21** 435
[12] Ishak A, Nazar R and Pop I 2009 *Chem. Engng. J.* **148** 63
[13] Mukhopadhyay S 2011 *Chin. Phys. Lett.* **28** 124706
[14] Mukhopadhyay S, Bhattacharyya K and Layek G C 2011 *Int. J. Heat Mass Transf.* **54** 2751
[15] Brewster M Q 1972 *Thermal Radiative Transfer Properties* (John Wiley and Sons)
[16] Zheng L C, Zhang X X and Li B T 2009 *Chin. Phys. Lett.* **26** 094101
[17] Bertolotti F P, Herbert T H and Spalart P R 1992 *J. Fluid Mech.* **242** 441
[18] Weidman P D, Kubitschek D G and Davis A M J 2006 *Int. J. Eng. Sci.* **44** 730
[19] Merkin J H 1994 *J. Appl. Math. Phys.* **45** 258
[20] Postelnicu A and Pop I 2011 *Appl. Math. Comp.* **217** 4359