Control schemes for synchronizing two subnetworks with weak couplings*

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In this paper, we propose a well-designed network model with a parameter and study full and partial synchronization of the network model based on the stability analysis. The network model is composed of a star-coupled subnetwork and a globally coupled subnetwork. By analyzing the special coupling configuration, three control schemes are obtained for synchronizing the network model. Further analysis indicates that even if the inner couplings in each subnetwork are very weak, two of the control schemes are still valid. In particular, if the outer coupling weight parameter θ is larger than $(n^2-2n)/4$, or the subnetwork size n is larger than θ^2, the two subnetworks with weak inner couplings can achieve synchronization. In addition, the synchronizability is independent of the network size in case of $0<\theta<n/(n+1)$. Finally, we carry out some numerical simulations to confirm the validity of the obtained control schemes. It is worth noting that the main idea of this paper also applies to any network consisting of a dense subnetwork and a sparse network.

Keywords: synchronization, weak coupling, star-global network, control scheme

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1. Introduction

Synchronization of coupled chaotic oscillators was firstly reported in Ref. [1] and then it becomes a problem of significant importance. In view of its complexity and variousness, many different types of synchronization have been investigated such as full synchronization,[2,3] partial synchronization,[3] projective synchronization,[4] and burst synchronization.[5] Up to now, the researches on synchronization have been applied to various disciplines, ranging from physics, engineering, biology, mathematics, to social sciences.

Full synchronization requires that all oscillators in the network behave in same fashion eventually. Theoretically, full synchronization occurs in a network if and only if the full synchronization manifold is globally attractive for the network. The established theoretical frameworks for full synchronization mainly consist of the local linearization method and Lyapunov function method. One of the prominent results of the former is the master stability function method.[2] Through this method, it is shown that the local asymptotical stability of a full synchronization manifold is determined by the maximum Lyapunov exponent of the node dynamics and the eigenvalues of the topology matrix. One of the prominent results of the latter is the global asymptotical stability of the full synchronization manifold. Similar to the master stability function method, it is also shown that the eigenvalues of the topology matrix are key factors contributing to full synchronization.[3] In addition, another crucial requirement for the method is that the node dynamics satisfies the QUAD condition.[3] However, Liu *et al.*[6] has obtained a different condition for full synchronization to replace the QUAD condition.

Partial synchronization requires that the oscillators spit into clusters, and all the oscillators in the same cluster behave in the same fashion. Many previous studies have shown that partial synchronization can be induced by cluster structures of networks. As we know, cluster structures exist widely in real world networks, such as friendship circles and colleagueship circles in social networks. In those clustered networks, partial synchronization usually occurs instead of full synchronization.[7,8] Therefore, the investigation into partial synchronization of a clustered network has attracted increasing attention. For example, Feng *et al.*[9] investigated the synchronization in a clustered network composed of two BA scale-free subnetworks (clusters). Wu *et al.*[10] proposed a simple control scheme to achieve generalized outer synchronization in a network composed of two identical subnetworks. In a linearly and symmetrically coupled network, Wu and Chen[3] presented a sufficient and necessary condition for a partial synchronization manifold being an invariant manifold. Then by decomposing the phase space into a direct sum of the partial synchronization manifold and the transverse space, they obtained a criterion for the global attractiveness of the partial synchronization manifold.[3] Based on this criterion, Ma *et al.*[11] analyzed the partial synchronization bifurcations in a globally coupled network with a parameter.

In this paper, we investigate full and partial synchronization in a network composed of a dense subnetwork and a sparse subnetwork. It should be a meaningful and interesting topic to discuss that under what conditions a dense subnetwork can drive a sparse subnetwork to realize synchronization. As

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a special case, we firstly propose a model named a star-global network, which is composed of a star-coupled network and a globally coupled network. Based on the criterion for the global attractiveness of the synchronization manifold,[3] we propose three different control schemes to synchronize the network, and find that two of the control schemes are still effective even if the inner couplings in the subnetworks are very weak. As we know, the synchronizability of a star-coupled network is independent of network size. However, the obtained results indicate that the synchronizability of a star-global network is independent of network size if the outer coupling weight parameter $\theta < n/(n+1)$. Otherwise, the synchronizability increases with the network size. Therefore, the increase in the network size can lead to synchronization, and a brief threshold of the network size for synchronization is that $n \geq \theta^2$. We also show that synchronization can be induced by a sufficiently large outer coupling weight parameter θ, no matter how weak the inner couplings are. As a special case, a brief threshold for full synchronization is that $\theta \geq (n^2-2n)/4$, where $n \geq 3$. The obtained control schemes are also valid for any network composed of a dense subnetwork and a sparse subnetwork. In brief, both the network size and the outer coupling weight parameter can lead to full synchronization in such type of networks.

The rest of the paper is organized as follows. In Section 2, we introduce some relevant concepts and a lemma for the global attractiveness of the synchronization manifold. In Section 3, we analyze the eigenvalues of the topology matrix for the star-global network, and propose three different control schemes for the synchronization. In Section 4, some numerical examples are demonstrated to verify the validity of the control schemes. The paper is concluded in Section 5.

2. Preliminaries

During the past years, many studies of synchronization phenomena focused on the oscillator networks coupled linearly and symmetrically.[11,12] In this paper, we study the following modified network with a coupling weight parameter:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i,t) + \varepsilon \sum_{j=1}^{N} a_{ij}(\theta)\boldsymbol{\Gamma}\boldsymbol{x}_j, \quad i = 1,\ldots,N, \tag{1}$$

where $\boldsymbol{x}_i = (x_i^1,\ldots,x_i^m)^{\mathrm{T}}$ is the state variable of the i-th oscillator, $t \in [0,+\infty)$ is a continuous time, $\boldsymbol{f}: \boldsymbol{R}^m \times [0,+\infty) \to \boldsymbol{R}^m$ is a continuous map, $\varepsilon > 0$ is the coupling strength, $\boldsymbol{A}(\theta) = (a_{ij}(\theta)) \in \boldsymbol{R}^{N\times N}$ is the topology matrix with $a_{ij}(\theta) = a_{ji}(\theta)$, θ is a parameter, and $\boldsymbol{\Gamma} = \mathrm{diag}\{\gamma_1,\ldots,\gamma_m\}$ is a nonnegative matrix determining the interaction of variables.

In the following, we introduce some basic concepts and a necessary Lemma for full and partial synchronization of the network (1).

2.1. Synchronization manifold

Denote $\mathbb{M}$ as the manifold of codimension p defined by a vector equation $\boldsymbol{H}(x) = 0$, where $\boldsymbol{H} = (h_1,\ldots,h_p)$, $1 \leq p \leq K-1$. The manifold $\mathbb{M}$ is called an invariant manifold of the system

$$\dot{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x},t), \ \ \boldsymbol{x} \in \boldsymbol{R}^K, \ \ \boldsymbol{X}: \boldsymbol{R}^K \times [0,+\infty) \to \boldsymbol{R}^K, \tag{2}$$

if the vector field (2) is tangent to $\mathbb{M}$, i.e., $(\mathrm{grad}\boldsymbol{H}\cdot\boldsymbol{X})\big|_{\boldsymbol{H}=0} \equiv 0$. For the existence of invariant manifolds, one can refer to the papers by Golubitsky and his co-workers.[13]

The set

$$\begin{aligned}\boldsymbol{M} = \{&\boldsymbol{x} = (\boldsymbol{x}_1^{\mathrm{T}},\ldots,\boldsymbol{x}_N^{\mathrm{T}})^{\mathrm{T}} : \boldsymbol{x}_i \in \boldsymbol{R}^m, \ i = 1,\ldots,N, \\ &\text{and} \ \ \boldsymbol{x}_1 = \cdots = \boldsymbol{x}_N\}\end{aligned} \tag{3}$$

is called a full synchronization manifold. In the case of $m = 1$, denote

$$\begin{aligned}\tilde{\boldsymbol{M}} = \{&(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_N) : \boldsymbol{z}_i \in \boldsymbol{R}, \ \ i = 1,\ldots,N, \\ &\text{and} \ \ \boldsymbol{z}_1 = \cdots = \boldsymbol{z}_N\}.\end{aligned}$$

We also call

$$\begin{aligned}\boldsymbol{L} = \Bigg\{&\boldsymbol{x} = (\boldsymbol{x}_1^{\mathrm{T}},\ldots,\boldsymbol{x}_N^{\mathrm{T}})^{\mathrm{T}} : \boldsymbol{x}_i \in \boldsymbol{R}^m, \ \ i = 1,\ldots,N, \\ &\text{and} \ \ \sum_{i=1}^{N} \boldsymbol{x}_i = 0\Bigg\}\end{aligned}$$

the transverse space for the full synchronization manifold M. In the case of $m = 1$, denote

$$\tilde{\boldsymbol{L}} = \left\{(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_N) : \boldsymbol{z}_i \in \boldsymbol{R}, \ \ i = 1,\ldots,N, \ \ \text{and} \ \ \sum_{i=1}^{N} \boldsymbol{z}_i = 0\right\}.$$

Definition[3] The manifold M is globally attractive for the system (1). If for any initial condition $(\boldsymbol{x}_1^{\mathrm{T}}(0),\ldots,\boldsymbol{x}_N^{\mathrm{T}}(0))^{\mathrm{T}}$, there holds

$$\lim_{t\to+\infty} \sum_{i=1}^{N} ||\boldsymbol{x}_i(t) - \boldsymbol{x}_1(t)|| = 0,$$

where $||\cdot||$ denotes the 2-norm of vectors.

2.2. Lemma for the global attractiveness of M

It is well known that every vector $\boldsymbol{x} = (\boldsymbol{x}_1^{\mathrm{T}},\ldots,\boldsymbol{x}_N^{\mathrm{T}})^{\mathrm{T}} \in \boldsymbol{R}^{mN}$ can be decomposed into

$$\boldsymbol{x} = \boldsymbol{\alpha} \oplus \boldsymbol{\beta}, \ \ \boldsymbol{\alpha} \in \boldsymbol{M}, \ \ \boldsymbol{\beta} \in \boldsymbol{L}.$$

Therefore, the manifold $\boldsymbol{M}$ is globally attractive for the system (1) if and only if the dynamical flow in the transverse subspace $\boldsymbol{L}$ converges to zero, i.e., $\lim\limits_{t\to+\infty} \boldsymbol{\beta} = 0$.

Lemma 1[3] Let $\boldsymbol{P} = \mathrm{diag}(p_1,\ldots,p_m)$ be a positive definite diagonal matrix, $\boldsymbol{\Delta} = \mathrm{diag}(\delta_1,\ldots,\delta_m)$ be a diagonal matrix, and $\boldsymbol{E}_N \in \boldsymbol{R}^{N\times N}$ be the identity matrix. Under the following three conditions:

(i) the manifold $\boldsymbol{M}$ is an invariant manifold of the system (1);

(ii) $\boldsymbol{f}(\boldsymbol{u},t) \in \mathrm{QUAD}(\boldsymbol{\Delta},\boldsymbol{P},\boldsymbol{R}^m)$, i.e., there exists a constant $\epsilon > 0$ such that for any $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{R}^m$ and $t \geq 0$,

$$\begin{aligned}&(\boldsymbol{u}-\boldsymbol{v})^{\mathrm{T}}\boldsymbol{P}\{[\boldsymbol{f}(\boldsymbol{u},t) - \boldsymbol{f}(\boldsymbol{v},t)] - \boldsymbol{\Delta}(\boldsymbol{u}-\boldsymbol{v})\} \\ \leq\ &-\epsilon(\boldsymbol{u}-\boldsymbol{v})^{\mathrm{T}}(\boldsymbol{u}-\boldsymbol{v});\end{aligned}$$

(iii) the matrices $\varepsilon\gamma_j \boldsymbol{A}(\theta)+\delta_j \boldsymbol{E}_N$, $j=1,2,\ldots,m$, are negative semi-definite in the transverse space $\tilde{\boldsymbol{L}}$, i.e.,

$$z^{\mathrm{T}}(\varepsilon\gamma_j \boldsymbol{A}(\theta)+\delta_j \boldsymbol{E}_N)z \le 0,$$

for all $z \in \tilde{\boldsymbol{L}}$, and $j=1,2,\ldots,m$, or in particular,

$$\varepsilon\gamma_j \lambda_{\max}^{\tilde{M}}(\boldsymbol{A}(\theta))+\delta_j \le 0, \quad j=1,2,\ldots,m, \tag{4}$$

where

$$\lambda_{\max}^{\tilde{M}}(\boldsymbol{A}(\theta)) = \max\{\lambda \in \sigma(\boldsymbol{A}(\theta)) : \Xi_{\boldsymbol{A}(\theta)}(\lambda) \not\subseteq \tilde{\boldsymbol{M}}\},$$

$\sigma(\boldsymbol{A}(\theta))$ is the set of all the eigenvalues of $\boldsymbol{A}(\theta)$, and $\Xi_{\boldsymbol{A}(\theta)}(\lambda)$ is the eigenspace of $\boldsymbol{A}(\theta)$ corresponding to the eigenvalue λ.

Then, the manifold $\boldsymbol{M}$ is globally attractive for the system (1).

We remark that the condition (ii) is a crucial requirement for Lemma 1. It implies that the oscillators of the network (1) can synchronize with each other when the coupling is made sufficiently large. In fact, many chaotic oscillators have been proved to satisfy the condition (ii) such as Chua circuits,[14] and standard Hopfield neural networks.[15] In addition, it has been proved that if the trajectories of the uncoupled systems $\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i,t)$ are eventually dissipative and will be in the absorbing domain $\mathscr{B} \subset \boldsymbol{R}^m$, then the trajectory of the coupled system (1) is also eventually dissipative and will be in the absorbing domain $\mathscr{B}\times\cdots\times\mathscr{B} \subset \boldsymbol{R}^{mN}$.[16] Therefore, the condition (ii) can also be confined to the absorbing domain $\mathscr{B}$. For instance, x-coupled or y-coupled Lorenz systems have been proved to satisfy the condition $\boldsymbol{f}(\boldsymbol{u},t) \in \mathrm{QUAD}(\boldsymbol{\Delta},\boldsymbol{P},\mathscr{B})$.[17,18] However, some systems are not the case such as a lattice of x-coupled Rössler systems, in which the stability of synchronization regime is lost with the increase in couplings.[19]

3. Control schemes for synchronization

In this section, we take the topology matrix of the network (1) as follows:

$$\boldsymbol{A}(\theta) = \begin{pmatrix} \boldsymbol{G}^{\dagger}(\theta) & \theta\boldsymbol{E}_n \\ \theta\boldsymbol{E}_n & \boldsymbol{G}^{*}(\theta) \end{pmatrix}_{2n\times 2n}, \tag{5}$$

where $\theta \ge 0$ is the outer coupling weight parameter between the two subnetworks,

$$\boldsymbol{G}^{\dagger}(\theta) = \begin{pmatrix} 1-n-\theta & 1 & \cdots & 1 \\ 1 & 1-n-\theta & \cdots & 1 \\ & \cdots & \cdots & \\ 1 & 1 & \cdots & 1-n-\theta \end{pmatrix},$$

$$\boldsymbol{G}^{*}(\theta) = \begin{pmatrix} 1-n-\theta & 1 & 1 & \cdots & 1 \\ 1 & -1-\theta & 0 & \cdots & 0 \\ & \cdots & \cdots & & \\ 1 & 0 & 0 & \cdots & -1-\theta \end{pmatrix}.$$

As a special case, the topology structure of a star-global network with $n=5$ is shown in Fig. 1. As can be seen, $\boldsymbol{G}^{\dagger}(\theta)|_{\theta=0}$ is the topology matrix of a globally coupled network, $\boldsymbol{G}^{*}(\theta)|_{\theta=0}$ is the topology matrix of a star-coupled network, while the identity matrix $\boldsymbol{E}_n$ implies that the i-th oscillator in $\boldsymbol{G}^{\dagger}(\theta)$ is coupled with the $(n+i)$-th oscillator in $\boldsymbol{G}^{*}(\theta)$. In this paper, a network corresponding to the topology matrix $\boldsymbol{A}(\theta)$ is called a star-global network, which is described by

$$\begin{cases} \dot{x}_i = f(x_i,t)+\varepsilon\sum\limits_{j=1}^{n}\Gamma(x_j-x_i) \\ \qquad +\varepsilon\theta\Gamma(x_{n+i}-x_i), \quad i=1,\ldots,n, \\ \dot{x}_{n+1} = f(x_{n+1},t)+\varepsilon\sum\limits_{j=n+2}^{2n}\Gamma(x_j-x_{n+1}) \\ \qquad +\varepsilon\theta\Gamma(x_1-x_{n+1}), \\ \dot{x}_i = f(x_i,t)+\varepsilon\Gamma(x_{n+1}-x_i) \\ \qquad +\varepsilon\theta\Gamma(x_{i-n}-x_i), \quad i=n+2,\ldots,2n. \end{cases} \tag{6}$$

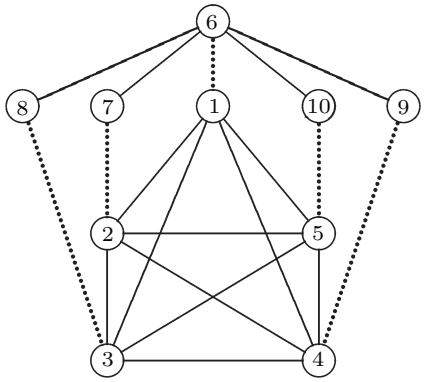


Fig. 1. The topology structure of a star-global network composed of 10 nodes. The solid lines are the inner edges of the two subnetworks, the dotted lines are the outer edges between the two subnetworks. The weights of the former are 1, and the weights of the latter are θ.

Since each row sum of the matrix $\boldsymbol{A}(\theta)$ is equal to zero, $\boldsymbol{A}(\theta)$ is a diffusive matrix and has a zero eigenvalue. The eigenspace corresponding to the zero eigenvalue is $\tilde{\boldsymbol{M}}$. Further analyzing the eigenvalues of $\boldsymbol{A}(\theta)$, one can obtain

$$\sigma(\boldsymbol{A}(\theta)) = \Big\{0,-2\theta,-n,-n-2\theta, \frac{1}{2}[-2\theta-n-1\pm\sqrt{(n-1)^2+4\theta^2}]\Big\}.$$

Then, based on the following inequality:

$$n+2\theta \ge n \ge \frac{1}{2}[2\theta+n+1-\sqrt{(n-1)^2+4\theta^2}],$$

we have

$$|\lambda_{\max}^{\tilde{M}}(A(\theta))| = \min\Big\{2\theta, \frac{1}{2}[2\theta+n+1-\sqrt{(n-1)^2+4\theta^2}]\Big\}. \tag{7}$$

Consequently, we obtain the following theorem with the help of Lemma 1.

Theorem 1 Assume that the trajectory of each uncoupled system $\dot{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{u},t)$ will eventually stay in the absorbing domain $\mathscr{B}$, and there exist a positive definite diagonal matrix $\boldsymbol{P} = \mathrm{diag}\{p_1,\ldots,p_m\}$ and a nonnegative diagonal matrix

$\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1, \ldots, \delta_m\}$ such that $\boldsymbol{f}(\boldsymbol{u}, t) \in \mathrm{QUAD}(\boldsymbol{\Delta}, \boldsymbol{P}, \mathscr{B})$, the following claims are true.

(i) If $0 < \theta < n/(n+1)$ and $\varepsilon \geq \delta_j/2\theta\gamma_j$ for all $j \in \{j : \gamma_j > 0\}$, then the manifold $\boldsymbol{M}$ is globally attractive for the star-global network (6).

(ii) If $\theta \geq n/(n+1)$ and

$$\varepsilon \geq \frac{\delta_j\left(2\theta + n + 1 + \sqrt{(n-1)^2 + 4\theta^2}\right)}{2\gamma_j(n + \theta + n\theta)}$$

for all $j \in \{j : \gamma_j > 0\}$, then the manifold $\boldsymbol{M}$ is globally attractive for the star-global network (6).

Proof Obviously, the conditions (i) and (ii) of Lemma 1 are satisfied. Noticing Eqs. (4) and (7), we can deduce the following.

(i) If $\theta < n/(n+1)$ holds, we have $2\theta < [2\theta + n + 1 - \sqrt{(n-1)^2 + 4\theta^2}]/2$, and then $|\lambda_{\max}^{\tilde{\boldsymbol{M}}}(\boldsymbol{G})| = 2\theta$. Hence, $\varepsilon \geq \delta_j/2\theta\gamma_j$ is sufficient for the condition (iii) of Lemma 1.

(ii) If $\theta \geq n/(n+1)$ holds, we have $|\lambda_{\max}^{\tilde{\boldsymbol{M}}}(\boldsymbol{G})| = [2\theta + n + 1 - \sqrt{(n-1)^2 + 4\theta^2}]/2$. The condition (iii) of Lemma 1 is equivalent to

$$\varepsilon \geq \frac{\delta_j\left(2\theta + n + 1 + \sqrt{(n-1)^2 + 4\theta^2}\right)}{2\gamma_j(n + \theta + n\theta)}.$$

According to Lemma 1, the proof is completed.

In the following, we fix the coupling strength ε as a constant and study synchronization induced by the network size and the parameter θ. For simplicity of notations, we denote $\zeta_j = \delta_j/\varepsilon\gamma_j$. Based on Theorem 1, we obtain two corollaries as follows.

Corollary 1 Assume that the trajectory of each uncoupled system $\dot{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{u}, t)$ will be eventually stay in the absorbing domain $\mathscr{B}$, and there exist a positive definite diagonal matrix $\boldsymbol{P} = \mathrm{diag}\{p_1, \ldots, p_m\}$ and a nonnegative diagonal matrix $\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1, \ldots, \delta_m\}$ such that $\boldsymbol{f}(\boldsymbol{u}, t) \in \mathrm{QUAD}(\boldsymbol{\Delta}, \boldsymbol{P}, \mathscr{B})$, the following claims are true.

(i) If $0 < \theta < n/(n+1)$, then the synchronizability of the star-global network (6) is independent of the network size.

(ii) If $\theta \geq \max\{n/(n+1), \zeta_j - 1\}$, and the network size

$$n \geq \zeta_j - \frac{\theta(1-\zeta_j)}{1 + \theta - \zeta_j}, \tag{8}$$

then the manifold $\boldsymbol{M}$ is globally attractive for the star-global network (6).

(iii) As a special case, if $\theta \geq \max\{1, \zeta_j\}$, and $n \geq \theta^2$, then the manifold $\boldsymbol{M}$ is globally attractive for the star-global network (6).

Proof (i) As a direct result of Theorem 1, the first claim is true.

(ii) The condition (8) can be rewritten as follows:

$$n \geq \frac{-\zeta_j^2 + (1+2\theta)\zeta_j - \theta}{1 + \theta - \zeta_j}.$$

Noticing $\theta \geq \zeta_j - 1$, we have

$$(1 + \theta - \zeta_j)n \geq -\zeta_j^2 + (1+2\theta)\zeta_j - \theta, \tag{9}$$

which leads to

$$(2\theta + n + 1 - 2\zeta_j)^2 \geq (n-1)^2 + 4\theta^2.$$

From the condition $\theta \geq \zeta_j - 1$, we obtain that $2\theta \geq 2\zeta_j - 1 - n$, and then

$$2\zeta_j < 2\theta + n + 1 - \sqrt{(n-1)^2 + 4\theta^2}.$$

Consequently, we have

$$\varepsilon \geq \frac{\delta_j\left(2\theta + n + 1 + \sqrt{(n-1)^2 + 4\theta^2}\right)}{2\gamma_j(n + \theta + n\theta)},$$

which meets the conditions of Theorem 1. Therefore, the manifold $\boldsymbol{M}$ is globally attractive for the star-global network (6).

(iii) In case of $\zeta_j < 1$, we have $\theta \geq \max\{1, \zeta_j\} = 1$, and the right hand of Eq. (8) is less than ζ_j. Then the condition (8) is satisfied. In case of $\zeta_j \geq 1$, we have $\theta \geq \zeta_j$, and the condition $n \geq \theta^2$ is sufficient for inequality (9). Then the condition (8) is satisfied. According to Corollary 1(ii), the manifold $\boldsymbol{M}$ is globally attractive for the star-global network (6).

Similarly, we obtain another corollary.

Corollary 2 Assume that the trajectory of each uncoupled system $\dot{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{u}, t)$ will eventually stay in the absorbing domain $\mathscr{B}$, and there exist a positive definite diagonal matrix $\boldsymbol{P} = \mathrm{diag}\{p_1, \ldots, p_m\}$ and a nonnegative diagonal matrix $\boldsymbol{\Delta} = \mathrm{diag}\{\delta_1, \ldots, \delta_m\}$ such that $f(u, t) \in \mathrm{QUAD}(\boldsymbol{\Delta}, \boldsymbol{P}, \mathscr{B})$, the following claims are true.

(i) The manifold $\boldsymbol{M}$ is globally attractive for the star-global network (6) if

$$\frac{1}{2}\zeta_j < \theta < \frac{n}{n+1}.$$

(ii) If $n \geq 2\zeta_j - 1$, the threshold for the global attractiveness of the manifold $\boldsymbol{M}$ is

$$\theta \geq \max\left\{\frac{n}{n+1}, \frac{(\zeta_j - 1)(\delta_j - n\varepsilon\gamma_j)}{2\zeta_j - n - 1}\right\}. \tag{10}$$

(iii) As a special case, if $n \geq \max\{2\zeta_j + 1, 3\}$, the threshold for the global attractiveness of the manifold $\boldsymbol{M}$ is

$$\theta \geq \frac{1}{4}(n^2 - 2n).$$

Proof The proof for the claims (i) and (ii) are analogous to that in Corollary 1. According to the claim (ii), the claim (iii) is true if the inequality (10) holds.

In case of $\zeta_j < 1$, we have

$$\frac{(\zeta_j - 1)(\delta_j - n\varepsilon\gamma_j)}{2\zeta_j - n - 1} < 0.$$

Furthermore, it is easy to see that $2\zeta_j + 1 < 3$, and then $n \geq \max\{2\zeta_j + 1, 3\} = 3$. Therefore, the condition in the claim (iii) implies that

$$\begin{aligned}\theta &\geq \frac{1}{4}(n^2 - 2n)\\ &\geq \frac{n}{n+1}\end{aligned}$$

$$= \max\left\{\frac{n}{n+1}, \frac{(\zeta_j-1)(\delta_j-n\varepsilon\gamma_j)}{2\zeta_j-n-1}\right\}.$$

The condition in the claim (ii) is satisfied. In case of $\zeta_j \geq 1$, we have $n \geq 2\zeta_j+1 \geq 3$, and then $(n^2-2n)/4 \geq n/(n+1)$. Denoting $n' = 2\zeta_j$, we conclude from $n \geq n'$ that

$$\theta \geq \frac{1}{4}(n^2-2n) \geq \frac{1}{4}(n'^2-2n') \geq \frac{(n-\zeta_j)(\zeta_j-1)}{n+1-2\zeta_j}.$$

According to the claim (ii), the claim (iii) is proved.

Based on the results mentioned above, a meaningful question arises naturally: why the critical value $\theta = n/(n+1)$ plays a key role in synchronization of the network (6)? In order to make the question clear, two assertions should be stated at first.

i) The synchronizability of a star-coupled network is independent of its size, but the synchronizability of a globally coupled network is increasing with its size.

ii) The synchronizability of a globally coupled network is superior to the synchronizability of a star-coupled network.

For the star-global network (6), the parameter θ represents the coupling weight parameter between the two subnetworks. In case of the parameter $\theta < n/(n+1)$, the superior synchronizability of $\boldsymbol{G}^{\dagger}(\theta)$ cannot contribute enough to that of $\boldsymbol{G}^{*}(\theta)$. Thus, the synchronizability of the network (6) is determined by the inferior one of $\boldsymbol{G}^{*}(\theta)$, which is independent of network size. While in case of the parameter $\theta \geq n/(n+1)$, the superior synchronizability of $\boldsymbol{G}^{\dagger}(\theta)$ contributes greatly to the one of $\boldsymbol{G}^{*}(\theta)$. The oscillators in $\boldsymbol{G}^{*}$ are driven primarily by the oscillators in $\boldsymbol{G}^{\dagger}(\theta)$. Thus, the synchronizability of the network (6) is determined by the superior one of $\boldsymbol{G}^{\dagger}(\theta)$, which increases with the network size.

Based on the two corollaries, we can conclude that even if the coupling strength ε is very small, synchronization can be realized by increasing the coupling weight parameter θ or the network size n. That is to say, synchronization can be realized by increasing the network size and the parameter θ no matter how powerless the inner couplings in $\boldsymbol{G}^{\dagger}(\theta)$ and $\boldsymbol{G}^{*}(\theta)$ are.

4. Numerical examples

Consider the star-global network (6) composed of $2n$ v-coupled Lorenz systems,

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i,t) + \varepsilon\sum_{j=1}^{2n} a_{ij}(\theta)\boldsymbol{\Gamma}\boldsymbol{x}_j, \tag{11}$$

where $\boldsymbol{\Gamma} = \mathrm{diag}(0,1,0)$, $\boldsymbol{x}_i = (u_i, v_i, w_i)^{\mathrm{T}} \in \boldsymbol{R}^3$ satisfying that

$$\begin{aligned} \dot{u}_i &= 10(v_i-u_i),\\ \dot{v}_i &= 28u_i - v_i - u_i w_i, \quad i = 1,\cdots,2n,\\ \dot{w}_i &= -8w_i/3 + u_i v_i. \end{aligned} \tag{12}$$

According to the results in Ref. [20], we obtain the absorbing domain of the system (12) as

$$\mathscr{B} = \left\{(u_i)^2 + (v_i)^2 + (w_i-38)^2 < 1541\right\}.$$

By taking $\epsilon = 10^{-3}$, $\boldsymbol{P} = \mathrm{diag}(46.23,1,1)$, and $\boldsymbol{\Delta} = \mathrm{diag}(0,274.63,0)$, the right hand of Lorenz system has been proved to satisfy $\boldsymbol{f}(\boldsymbol{u},t) \in \mathrm{QUAD}(\boldsymbol{\Delta},\boldsymbol{P},\mathscr{B})$.[17]

Choose the initial conditions $u_i(0)$, $v_i(0)$, $w_i(0)$, randomly with uniform distribution on $[-10,10]$, and define the full synchronization errors as

$$e_{i,1} = \frac{1}{2n}\sum_{i=1}^{2n}||\boldsymbol{x}_i - \boldsymbol{x}_1||,$$

and the partial synchronization errors

$$\delta e_{i,i_0} = \frac{1}{\mathrm{Card}(G_{\mathrm{d}})}\sum_{i\in G_{\mathrm{d}}}||x_i - x_{i_0}||,$$

where $i_0 = \min\{i : i \in G_{\mathrm{d}}\}$, $\mathrm{Card}(G_{\mathrm{d}})$ is the cardinal number of the set G_{d}, the following simulations are carried out to show the validity of our results.

4.1. Validity of Theorem 1: $n = 5$

Set $\theta = 0.8$, Theorem 1(i) implies that the threshold for full synchronization is $\varepsilon \geq 171.64$. Instead of setting $\theta = 2$, Theorem 1(ii) implies that the threshold for full synchronization is $\varepsilon \geq 126.46$. Figure 2 shows that the two obtained thresholds of ε are both sufficient for full synchronization of the system (11).

4.2. Validity of Corollary 1(i): $\theta = 0.1$, $\varepsilon = 2$

Set $n = 5$, the partial synchronization error $\delta e_{i,1} = 0.2\sum_{i=1}^{5}||\boldsymbol{x}_i - \boldsymbol{x}_1||$ is shown to approach zero by the sub-figure of Fig. 3. Figure 3 shows that the full synchronization error does not approach zero. Therefore, partial synchronization, instead of full synchronization, occurs. Set $n = 100$, similar synchronization phenomena can be observed in Fig. 3.

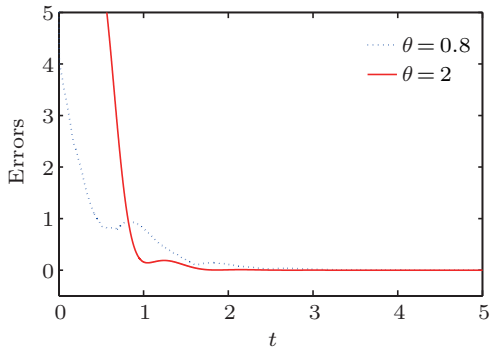


Fig. 2. (color online) Full synchronization of the system (11). The parameters n, θ, and ε are taken according to Theorem 1.

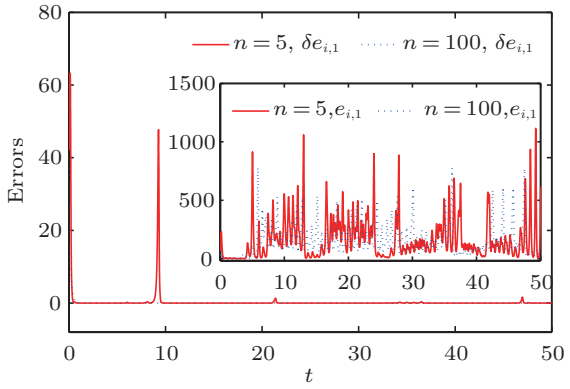


Fig. 3. (color online) Partial synchronization of the system (11). Full synchronization does not occur for either $n = 5$ or $n = 100$.

4.3. Validity of Corollary 1(ii): $\varepsilon = 0.1$

According to Corollary 1(ii), we obtain that $\theta = 2747$. Set $n = 5$, the partial synchronization errors $\delta e_{i+5,i} = 0.5||x_{i+5} - x_i||$ $(i = 1\text{–}5)$ are plotted in the sub-figure of Fig. 4. Combining with the time evolutions of the full synchronization error $e_{i,1}$, we can conclude that the system (11) realizes partial synchronization instead of full synchronization. However, figure 4 shows that full synchronization occurs when the network size $n = 100$. That is to say, synchronization can be induced by increasing the network size.

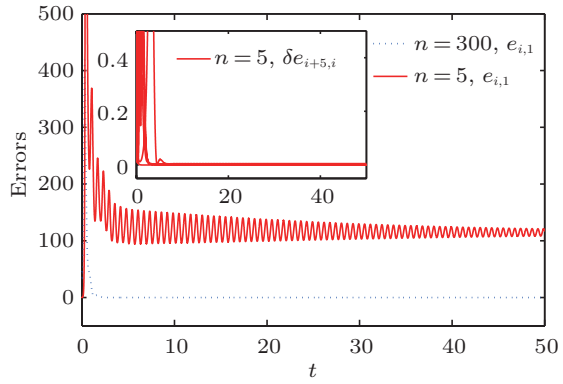


Fig. 4. (color online) Full synchronization induced by increasing the network size. Partial synchronization occurs when $n = 5$, and full synchronization occurs when $n = 100$.

4.4. Validity of Corollary 1(ii) and Corollary 2

Set $\varepsilon = 50$ and $\theta = 5.49$, it follows from Corollary 1(ii) that full synchronization occurs if $n \geq 31$. The time evolutions of the full synchronization error $e_{i,1}$ is plotted in Fig. 5.

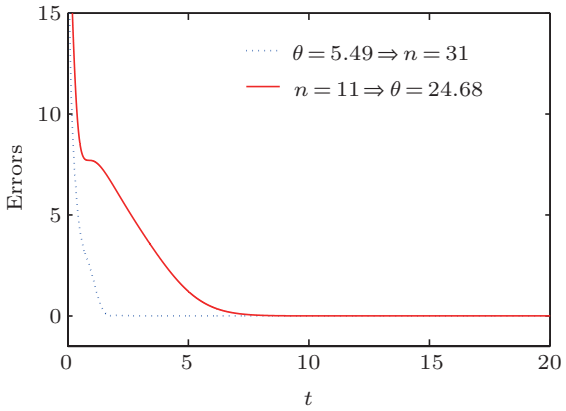


Fig. 5. (color online) Full synchronization of the system (11). The parameters n, θ, and ε are taken according to Corollary 1(ii) and Corollary 2(ii).

Corollary 2(i) can be validated by Simulation 1. Set $\varepsilon = 50$ and $n = 11$, Corollary 2(ii) implies that full synchronization occurs if $\theta \geq 24.68$, which is also conformed by Fig. 5.

4.5. Synchronization induced by θ

In this simulation, we suppose that the inner couplings in $G^{\dagger}$ and G^{*} are powerless. For instance, we set $\varepsilon = 0.01$ and $n = 50$. However, Figure 6 shows that full synchronization occurs when the parameter θ is large enough. That is to say, synchronization can also be induced by increasing the parameter θ.

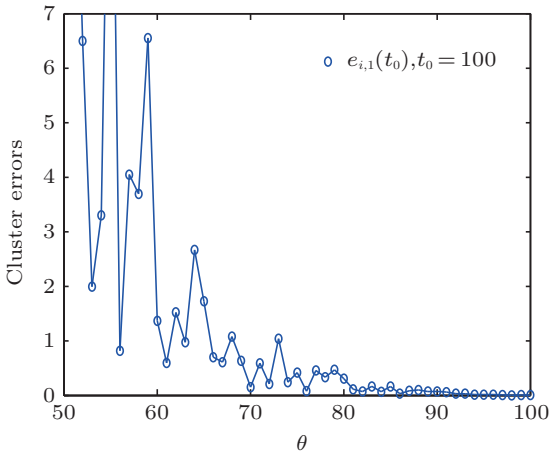


Fig. 6. (color online) Dependence of the partial synchronization errors $e_{i,1}(t_0), t_0 = 100$ on the coupling weight parameter θ for the system (11), where $\varepsilon = 0.01$, $n = 50$.

5. Conclusion

In this paper, we have discussed the synchronization of a star-global network and theoretically proposed three control schemes. The results indicate that even if the inner couplings in the subnetworks are weak, synchronization occurs providing that the couplings between the subnetworks are strong enough, or the network size is large enough. The schemes might be useful for synchronizing several subnetworks with weak couplings.

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