Oscillating multidromion excitations in higher-dimensional nonlinear lattice with intersite and external on-site potentials using symbolic computation

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We show by an extensive method of quasi-discrete multiple-scale approximation that nonlinear multi-dimensional lattice waves subjected to intersite and external on-site potentials are found to be governed by $(N+1)$-dimensional nonlinear Schrödinger (NLS) equation. In particular, the resonant mode interaction of (2+1)-dimensional NLS equation has been identified and the theory allows the inclusion of transverse effect. We apply the exponential function method to the (2+1)-dimensional NLS equation and obtain the class of soliton solutions with a purely algebraic computational method. Notably, we discuss in detail the effects of the external on-site potentials on the explicit form of the soliton solution generated recursively. Under the action of the external on-site potentials, the model presents a rich variety of oscillating multidromion patterns propagating in the system.

Keywords: solitons, partial differential equations, lattice dynamics, exact solutions

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1. Introduction

A great deal of attention has been paid to the dynamics of nonlinear lattices with nonlinear intersite and on-site potentials since the pioneering works of Fermi, Pasta, and Ulam (FPU),[1] and of Zabusky and Kruskal.[2] Several pieces of work in this area are focused on models of one-dimensional (1D) lattices and their behavior is now well explored starting from the FPU[3–5] to many other analytical solvable models like the Toda lattice,[6] and so on. It is known that some nonlinear 1D lattice systems can support localized excitations such as solitons and discrete breathers.[7–11] The 1D lattice solitons which are described as the localized nonlinear excitations arise due to the balance between nonlinearity and dispersion,[7] but nowadays their generalization to multiple dimensions has not been completely solved. In the context of on-site potentials, Huang *et al*. have discussed a novel class of lattice solitons in 1D monoatomic lattices with nearest-neighbor (NN) interaction within the framework of quasidiscrete approximation.[12] The standing localized modes in 1D nonlinear lattices subjected to an external (on-site) potential with cubic and quartic nonlinearities are described in Ref. [13]. Moreover, the idea of intrinsic localized mode proposed by Sievers and Takeno[14,15] has initiated a large number of studies on 1D mono- and diatomic lattices and later on by many others.[16–19] However, the dynamical properties of higher dimensions and understanding the action of various factors involved still remain the major challenges in nonlinear lattices. Over the years, solitons have become a topic of intense investigation in several branches of higher-dimensional nonlinear science.[20–28]

For instance, soliton excitations in 2D lattices with in-plane displacement, competing and anisotropic exchange interactions have been reported in Ref. [29]. Interestingly, Flach *et al*. have investigated the intrinsic localized modes in 2D lattices.[30] Also, the formation of localized states in 2D lattices through the mechanism of modulational instability analysis has been studied in Ref. [31]. Conclaves *et al*. have reported an important phenomenon of resonant mode interactions observed in 2D diatomic lattices with a complex network.[32] The existence of intrinsic gap mode in 3D anharmonic diatomic lattices with realistic potentials has been explored in Refs. [33] and [34]. The solitary wave excitations in an isotropic 2D hexagonal lattice which can propagate in different directions have been illustrated.[35] More interestingly, vortices and dark solitons in 2D sine lattices and nonlinear Schrödinger (NLS) type lattice governing Frenkel excitations in atomic lattice chains have been found, respectively.[36,37] Furthermore, several authors have shown the existence of soliton-like excitations in 2D square, triangular and hexagonal lattices.[38–51] It has been reported in Ref. [52] that the nonlinear modulation of N-dimensional lattice system with cubic and quartic interatomic potentials are governed by the generalized $(N+1)$-dimensional Davey–Stewartson equations using

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the weakly nonlinear theory. Therefore, in recent years multi-dimensional lattices have attracted a considerable amount of attention, ever since its inception and experimental realization.

From the methodological viewpoint, it is worth noting that the higher-dimensional integrable systems, in particular, (2+1)-dimensional equations such as the Davey–Stewartson (DS) equations,[21,52] Kadomtsev–Petviashvili (KP) equations,[53,54] Nizhnik–Novikov–Veselov (NNV) equations,[55] (2+1)-dimensional long-wave or short-wave resonance interaction equation,[56] (2+1)-dimensional Burgers equations,[57] for which the soliton solutions are found by using different approaches. A direct search of exact solitary wave solutions opens the way to application in multi dimensions, which, hopefully, is expected to be even more interesting than in one-space dimension and in general highly complex to solve. However, the advent of computational methods aided with symbolic computation in both mathematical and theoretical physics has triggered the researchers to seek for the exact analytical solutions of the higher-dimensional nonlinear partial differential equations (NLPDEs).[52–57] Several methods have been proposed in the past such as inverse scattering transform (IST),[58] Hirota bilinearization,[59] binary Darboux transformation,[60] and Bäcklund transformation[61] to solve the NLPDEs. Recently, many effective methods like the sine–cosine method,[62,63] the homogeneous balance method,[64] the Jacobi elliptic function method,[65,66] the variational iteration method,[67] the homotopy perturbation method,[68] the tanh method,[69] the F-expansion method,[70] the modified extended tanh method,[71–74] and so on, are found to obtain the explicit form of the solitary wave solution. One important realization for finding the exact solutions to the NLPDEs is that it provides much physical information regarding the coherent structures and the transverse effects in higher dimensions.

In the present paper, we consider a system of multi-dimensional lattice waves presented by Huang *et al.*[52] In addition, N-dimensional on-site potentials with cubic and quartic nonlinearities are included. In order to investigate the underlying dynamics of the N-dimensional lattices, the paper is organized as follows. In Section 2, we formulate the model Hamiltonian and using the technique of quasidiscrete multiple-scale approach, and derive the N-dimensional nonlinear envelope equations for the slowly varying amplitudes of multiple scales. The relevant 2D lattice equations will also be deduced in this section. We then continue in Section 3 to derive soliton solutions of the (2+1)-dimensional NLS equation using the exponential function method. Finally, Section 4 is devoted to conclusion.

2. Model and the mathematical background

2.1. The envelope equations and an asymptotic expansion

We begin with a model of N-dimensional monoatomic scalar lattice with the mass M, harmonically coupled to their NNs, and subjected to a cubic and quartic external (on-site) potentials. Here, we consider that the system appropriately has small atomic vibrational amplitudes which can produce sufficient nonlinearity but no reconstruction or phase transition in the system. This in turn allows us to perform the Taylor expansion for the atomic interaction potential around the equilibrium and to truncate the series to the fourth order. The Hamiltonian of the system is then written as

$$\begin{aligned} H = \sum_{j=1}^{d} \Bigg[& \frac{1}{2} M \left(\frac{\mathrm{d}u(\boldsymbol{n})}{\mathrm{d}t} \right)^2 + \frac{1}{2} k_{2j} [u(\boldsymbol{n}+\boldsymbol{a}_j) - u(\boldsymbol{n})]^2 \\ & + \frac{1}{3} k_{3j} [u(\boldsymbol{n}+\boldsymbol{a}_j) - u(\boldsymbol{n})]^3 \\ & + \frac{1}{4} k_{4j} [u(\boldsymbol{n}+\boldsymbol{a}_j) - u(\boldsymbol{n})]^4 + \frac{1}{2} k_{5j} [u(\boldsymbol{n})]^2 \\ & + \frac{1}{3} k_{6j} [u(\boldsymbol{n})]^3 + \frac{1}{4} k_{7j} [u(\boldsymbol{n})]^4 \Bigg]. \end{aligned} \tag{1}$$

In the above expression, $u(\boldsymbol{n})$ denotes the displacement of atom from its equilibrium position located at the site $\boldsymbol{n} = \sum_{j=1}^{d} n_j \boldsymbol{a}_j$. Here, n_j and $\boldsymbol{a}_j$ represent the integer and the lattice vectors, respectively, and d is the dimension of the lattice. The coupling constants $k_{\rho,j}$ $(\rho = 2,3,4)$ characterize the strength of the harmonic, cubic, and quartic interatomic forces, respectively, $k_{5,j}$ is the frequency of small amplitude vibrations in a well of the substrate potentials, $k_{6,j}$ and $k_{7,j}$ are the anharmonicity parameters of the on-site potentials. We consider the anisotropy of the lattice in Ref. [52]. Following the similar procedure, the classical equation of motion is given by

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} u(\boldsymbol{n}) = & \sum_{j=1}^{d} J_{2j} [u(\boldsymbol{n}+\boldsymbol{a}_j) + u(\boldsymbol{n}-\boldsymbol{a}_j) - 2u(\boldsymbol{n})] \\ & + \sum_{j=1}^{d} J_{3j} \{ [u(\boldsymbol{n}+\boldsymbol{a}_j) - u(\boldsymbol{n})]^2 \\ & - [u(\boldsymbol{n}-\boldsymbol{a}_j) - u(\boldsymbol{n})]^2 \} \\ & + \sum_{j=1}^{d} J_{4j} \{ [u(\boldsymbol{n}+\boldsymbol{a}_j) - u(\boldsymbol{n})]^3 \\ & + [u(\boldsymbol{n}-\boldsymbol{a}_j) - u(\boldsymbol{n})]^3 \} - \sum_{j=1}^{d} J_{5j} [u(\boldsymbol{n})] \\ & - \sum_{j=1}^{d} J_{6j} [u(\boldsymbol{n})]^2 - \sum_{j=1}^{d} J_{7j} [u(\boldsymbol{n})]^3, \end{aligned} \tag{2}$$

where $J_{\rho' j} = k_{\rho' j}/M$ $(\rho' = \rho, 5, 6, 7)$. In recent years, multiple-scale methods are widely employed to study the dynamical properties of nonlinear lattices.[75–78] In order to include the effects of nonlinearity and discreteness simultaneously, we use the method of quasidiscrete multiple-scale

approximation[79–82] to reduce the equation and to evaluate the modulation of the amplitude in the lowest order of asymptotic expansion. In this treatment, we set

$$u(\boldsymbol{n}) = \sum_{\gamma=1} \varepsilon^{\gamma} u_{\gamma}(\boldsymbol{r}, \tau; \phi(\boldsymbol{n}, t)), \tag{3}$$

where ε is a small but finite parameter denoting the relative amplitude of the excitation, $\boldsymbol{r}$ and τ are slow variables defined by $\boldsymbol{r} = \varepsilon\,(\boldsymbol{n} - \boldsymbol{v}t)$ and $\tau = \varepsilon^2 t$ which are also called multiple-scaled variables. The parameter $\boldsymbol{v}$ introduced here is the velocity to be determined later by the solvability condition. The fast variable $\phi(\boldsymbol{n}, t) = \boldsymbol{q}\cdot\boldsymbol{n} - \omega t$ representing the phase of the carrier wave, is taken to be completely discrete. The quantity $\boldsymbol{q} = \sum_{j=1}^{d} q_j \boldsymbol{b}_j$ contained in the fast variable is the wave vector, with $\boldsymbol{b}_j$ being the vector of the reciprocal lattice: $\boldsymbol{b}_i \cdot \boldsymbol{a}_j = \delta_{ij}$ and ω the frequency of the respective harmonic. With this expansion, inserting Eq. (3) into Eq. (2) and equating the same powers of ε, the above equation of motion is reduced to

$$\begin{aligned} \hat{L}u_{\gamma} \equiv\ & \omega^2 \frac{\partial^2 u_{\gamma}}{\partial \phi^2} - \sum_j J_{2j}(u_{\gamma}^{(j)} + u_{\gamma}^{(-j)}) - \sum_j J_{5j} u_{\gamma} = M_{\gamma}, \\ & \gamma = 1, 2, \ldots, \end{aligned} \tag{4}$$

with

$$u_{\gamma}^{(\pm j)} \equiv u_{\gamma}(\boldsymbol{r}, \tau; \phi(\boldsymbol{n}, t) \pm q_j) - u_{\gamma}(\boldsymbol{r}, \tau; \phi(\boldsymbol{n}, t)), \tag{5}$$

$$M_1 = 0, \tag{6}$$

$$\begin{aligned} M_2 =\ & -2\omega(\boldsymbol{v}\cdot\nabla)\frac{\partial u_1}{\partial \phi} + \sum_j J_{2j} a_j \frac{\partial}{\partial x_j}(u_1^{(j)} - u_1^{(-j)}) \\ & + \sum_j J_{3j}[(u_1^{(j)})^2 - (u_j^{(-j)})^2] + \sum_j J_{6j} u_1^2, \end{aligned} \tag{7}$$

$$\begin{aligned} M_3 =\ & -(\boldsymbol{v}\cdot\nabla)^2 u_1 + 2\omega \frac{\partial^2 u_1}{\partial \phi \partial \tau} - 2\omega(\boldsymbol{v}\cdot\nabla)\frac{\partial u_2}{\partial \phi} \\ & + \sum_j J_{2j} a_j \frac{\partial}{\partial x_j}(u_2^{(j)} - u_2^{(-j)}) \\ & + \sum_j \frac{J_{2j}}{2}\left(a_j \frac{\partial}{\partial x_j}\right)^2 (u_1^{(j)} + u_1^{(-j)} + 2u_1) \\ & + 2\sum_j J_{3j}\Big[(u_1^{(j)} u_2^{(j)} - u_1^{(-j)} u_2^{(-j)}) \\ & + u_1^{(j)} a_j \frac{\partial}{\partial x_j}(u_1^{(j)} + u_1) + u_1^{(-j)} a_j \frac{\partial}{\partial x_j}(u_1^{(-j)} + u_1)\Big] \\ & + \sum_j J_{4j}\Big[(u_1^{(j)})^3 + (u_1^{(-j)})^3\Big] \\ & + 2\sum_j J_{6j} u_1 u_2 + \sum_j J_{7j} u_1^3, \end{aligned} \tag{8}$$

where $\nabla \equiv \partial/\partial \boldsymbol{r}$, $a_j = |\boldsymbol{a}_j|$, x_j is the j-th coordinate of the vector $\boldsymbol{r}$, and $\boldsymbol{r} = \sum_j x_j \boldsymbol{a}_j / a_j$. Moreover, the expression of M_{γ} for $\gamma = 4, 5, \ldots$ are not presented here. Now, we solve Eq. (4) order by order with the help of Eqs. (5)–(8). Also, in order to study the interaction between long-wavelength acoustic mode and high-frequency mode, we choose the lowest-order solution of Eq. (4) of the form

$$\begin{aligned} u_1 =\ & A_0(\boldsymbol{r}, \tau)\exp[\mathrm{i}\phi(\boldsymbol{n}, t)] \\ & + A_1(\boldsymbol{r}, \tau)\exp[\mathrm{i}\phi(\boldsymbol{n}, t)] + \text{c.c.}, \end{aligned} \tag{9}$$

where A_0 and A_1 are, respectively, the real and complex functions to be determined later. Here, c.c. denotes the complex conjugation. Hence, we have

$$\begin{aligned} u_1^{(\pm j)} =\ & A_0 \exp(\mathrm{i}\phi)[\exp(\pm \mathrm{i} q_j) - 1] \\ & + A_1 \exp(\mathrm{i}\phi)[\exp(\pm \mathrm{i} q_j) - 1] + \text{c.c.} \end{aligned} \tag{10}$$

Substituting Eq. (10) in the first order Eq. (6), we find that the system has the linear vibration with the frequency given by

$$\omega^2 \equiv [\omega(\boldsymbol{q})]^2 \equiv 2\sum_j J_{2j}(1 - \cos q_j) - \sum_j J_{5j}, \tag{11}$$

from which one can obtain the group velocity of the lattice wave as follows:

$$\boldsymbol{v}_g = \frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{q}} = \frac{1}{\omega}\sum_j J_{2j} \sin q_j \frac{\boldsymbol{a}_j}{a_j}. \tag{12}$$

Using Eq. (7) and the lowest-order solution (9), we calculate the element M_2 which yields the following second-order equation:

$$\begin{aligned} \hat{L}u_2 =\ & 2\mathrm{i}\omega\{[(\boldsymbol{v}_g - \boldsymbol{v})\cdot\nabla] \\ & \times [(A_0 + A_1)\exp(i\phi) - (A_0 + \bar{A}_1)\exp(-\mathrm{i}\phi)] \\ & + \chi_1^{(2)}[(A_0 + A_1)^2 \exp(2\mathrm{i}\phi) - (A_0 + \bar{A}_1)^2 \exp(-2\mathrm{i}\phi)] \\ & - \mathrm{i}\chi_2^{(2)}[(A_0 + A_1)\exp(\mathrm{i}\phi) \\ & + (A_0 + \bar{A}_1)\exp(-\mathrm{i}\phi)]^2\}, \end{aligned} \tag{13}$$

where

$$\chi_1^{(2)} = 2\sum_j \frac{J_{3j}}{\omega}(\cos q_j - 1)\sin q_j, \quad \chi_2^{(2)} = \sum_j \frac{J_{6j}}{2\omega}. \tag{14}$$

It is noticed that the term proportional to $\exp(\pm \mathrm{i}\phi)$ in the right-hand side of Eq. (13) is a secular term, which should be eliminated to ensure valid theory, thereby determining the solvability condition.[83] The secular term can be removed when the velocity introduced in the multiple slow variable is the group velocity of the carrier wave, i.e., $\boldsymbol{v} = \boldsymbol{v}_g$. In this way, we arrive at the solution u_2 given by

$$\begin{aligned} u_2 =\ & \mathrm{i}\alpha_1 (A_0 + A_1)^2 \exp(2\mathrm{i}\phi) + \text{c.c} \\ & + \alpha_2 [(A_0 + A_1)\exp(\imath\phi) + \text{c.c.}]^2, \end{aligned} \tag{15}$$

where

$$\alpha_1 = \frac{2\omega\chi_1^{(2)}}{4[\omega(\boldsymbol{q})]^2 - [\omega(2\boldsymbol{q})]^2}, \quad \alpha_2 = \frac{2\omega\chi_2^{(2)}}{4[\omega(\boldsymbol{q})]^2 - [\omega(2\boldsymbol{q})]^2}.$$

This is the criteria for the occurrence of second-order harmonic generation in the N-dimensional case, when the low-frequency mode can resonantly interact with the other modes. During this process the energy transfer can occur between

modes with the same direction of velocities or with different directions of velocities. In particular, the velocities are orthogonal.[19,32,52,84] In the case of second-order harmonic generation, when equation (15) satisfies the condition $\omega(2\boldsymbol{q}) = 2\omega(\boldsymbol{q})$, one arrives at the system of equations for the solvability condition of third order in the hierarchy of exponentials.

(i) e^0 :

$$\sum_m \delta m \frac{\partial}{\partial x_m}[(A_0+A_1)(A_0+\bar{A}_1)] = 0, \tag{16}$$

(ii) $e^{i\phi}$:

$$\begin{aligned} &i\frac{\partial}{\partial\tau}(A_0+A_1)+\frac{1}{2}\sum_{l,m}\omega_{lm}\frac{\partial^2}{\partial x_l\partial x_m}(A_0+A_1)\\ &+\chi(A_0+\bar{A}_1)(A_0+A_1)^2 = 0, \end{aligned} \tag{17}$$

where

$$\delta_m = \frac{2a_m}{\omega}J_{3m}(1-\cos(q_m)), \tag{18}$$

$$\omega_{lm} \equiv \frac{1}{\omega}[J_{2m}\cos(q_m)a_l a_m\delta_{lm} - v_l v_m], \tag{19}$$

and

$$\begin{aligned} \chi \;=\; &\frac{1}{2\omega}\sum_m[8iJ_{3m}(i\alpha_1+\alpha_2)(1-\cos(q_m))\sin(q_m)\\ &-24J_{4m}(1-\cos(q_m))^2\\ &+2J_{6m}(i\alpha_1+3\alpha_2)+3J_{7m}]. \end{aligned} \tag{20}$$

Now, for the sake of simplicity, we define a new variable $\psi = A_0 + A_1$ which is a complex function. By incorporating the new variable, the above system of equations can be written as

$$\sum_m \delta_m \frac{\partial}{\partial x_m}|\psi|^2 = 0, \tag{21}$$

and

$$i\frac{\partial\psi}{\partial\tau}+\frac{1}{2}\sum_{l,m}\omega_{lm}\frac{\partial^2}{\partial x_l\partial x_m}\psi+\chi|\psi|^2\psi = 0, \tag{22}$$

which denotes the envelope equation. From the above equations, it is implied that the delicate presence of nonlinearity and dispersion in Eq. (22) arising from the $\exp[i\phi]$ term balances each other, resulting in the formation of coherent structure contributes much to the dynamics of N-dimensional lattice waves. In general, the equation represents the well-known NLS type. In order to understand the dynamical behavior of the system, let us hereafter concentrate on Eq. (22). However, the explicit form of Eq. (21) will also be expressed.

2.2. Generalization of (2+1)-dimensional lattice equation

It is certainly of interest to extend the study to more dimensions, in particular the 2D case, by considering the lattice to be symmetric $J_{\rho',j} = J_{\rho'}$ ($\rho' = \rho, 5, 6, 7$ and $j = 1, 2$) and orthogonal $\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$=0 with the lattice constant $|a_j|$=1. In this representation, the coordinates $\boldsymbol{r} = (x_1, x_2)$ having the original cartesian system with the coordinate basis (1,0) and (0,1) are transformed to a newer system with the coordinate basis $\boldsymbol{e}_1 = (\lambda_1, \lambda_2)$ and $\boldsymbol{e}_2 = (-\lambda_2, \lambda_1)$, where

$$\boldsymbol{\lambda}_j = \frac{v_j}{v_g} = \frac{\sin q_j}{\sqrt{\sin^2 q_1 + \sin^2 q_2}}, \tag{23}$$

with v_j being the group velocity of the j-th component. While transforming, one of the direction of the newer basis $\boldsymbol{e}_1$ coincides with the group velocity of the carrier wave and the other orthogonal to it. Accordingly, we have the coordinates $x_1 = \varepsilon(n_1 - \lambda_1 v_g t)$ and $x_2 = \varepsilon(n_2 - \lambda_2 v_g t)$ and the associated 2D envelope equations are

$$\beta_1(|\psi|^2)_x + \beta_2(|\psi|^2)_y = 0, \tag{24}$$

$$i\psi_t + \gamma_{11}\psi_{xx} + \gamma_{22}\psi_{yy} + \gamma_{12}\psi_{xy} + \chi|\psi|^2\psi = 0, \tag{25}$$

where

$$\begin{aligned} x &= \boldsymbol{r}\cdot\boldsymbol{e}_1 = (\lambda_1 x_1 + \lambda_2 x_2) = \varepsilon[\lambda_1 n_1 + \lambda_2 n_2 - v_g t],\\ y &= \boldsymbol{r}\cdot\boldsymbol{e}_2 = (-\lambda_2 x_1 + \lambda_1 x_2) = \varepsilon[-\lambda_2 n_1 + \lambda_1 n_2],\\ \beta_1 &= \frac{2J_3}{\omega}[\lambda_1(1-\cos q_1) + \lambda_2(1-\cos q_2)],\\ \beta_2 &= \frac{2J_3}{\omega}[\lambda_1(1-\cos q_2) - \lambda_2(1-\cos q_1)],\\ \gamma_{11} &= \frac{1}{2\omega}[J_2(\lambda_1^2\cos q_1 + \lambda_2^2\cos q_2) - v_g^2],\\ \gamma_{22} &= \frac{1}{2\omega}[J_2(\lambda_2^2\cos q_1 + \lambda_1^2\cos q_2)],\\ \gamma_{12} &= \frac{J_2}{2\omega}\lambda_1\lambda_2[\cos q_2 - \cos q_1],\\ \chi &= \frac{1}{2m}\sum_m[8iJ_3(i\alpha_1+\alpha_2)\{(1-\cos q_1)\\ &\quad\times\sin q_1 + (1-\cos q_2)\sin q_2\}\\ &\quad-24J_4\{(1-\cos q_1)^2(1-\cos q_2)^2\}\\ &\quad+J_6(i\alpha_1+3\alpha_2)+3J_7],\\ \alpha_1 &= \frac{4J_2[\sin q_1(\cos q_1 - 1) + \sin q_2(\cos q_2 - 1)]}{4[\omega(\boldsymbol{q})]^2 - [\omega(2\boldsymbol{q})]^2},\\ \alpha_2 &= \frac{J_6}{4[\omega(\boldsymbol{q})]^2 - [\omega(2\boldsymbol{q})]^2}. \end{aligned}$$

Equations (24) and (25) are the 2D envelope equations which possess different coefficients for different values of wave vector $\boldsymbol{q}$ due to the anisotropy property of the lattice. As we already mentioned that the dynamics of the N-dimensional lattice wave is governed by the N-dimensional NLS equation (22) and the generalization of this equation derived in Section 2 represents the (2+1)-dimensional NLS equation which along with its various extensions possess soliton

solutions.[85–91] In addition, some special types of solutions such as dromions, breathers, instantons, and dromion solutions with oscillatory tails have been obtained in Refs. [85], [88], and [89]. With the advent of computational methods, several authors have shown the existence of exact soliton solution for the (2+1)-dimensional NLS equation with the variable coefficients.[90–92] In the vanishing case of the cross derivative term ψ_{xy} of Eq. (25), the equation resembles the one derived by Kates and Kaup[93] in the hydrodynamic disturbances on a static cosmological background. The exact soliton solution for the equation was discussed in Ref. [92] using the function transformation method. Also, in the vanishing case of mixed derivative term, equation (25) takes a similar form of the equation studied by Zhong *et al*.[90] and Hongyan *et al*.[91] with an additional term of external potential function and absorption coefficient (damping), respectively, in the context of nonlinear optics. In view of this, Hongyan *et al*. have shown that the propagation of an electromagnetic wave in a Kerr medium is associated with damped (2+1)-dimensional NLS equation and demonstrated the novel classes of exact solitary wave solutions with the help of a method which combines the two methods, namely, the tanh function and the symmetry group. Very recently, Zhong *et al*. have investigated the propagation of a light beam in the Kerr medium with an external trapping potential being described by the (2+1)-dimensional NLS equation and it admits a variety of exact solutions such as single and multiple dromions, ring- and parabola-shaped wave forms, peakons (square-shaped modes) and breathers using the technique of separation of variables and Hirota bilinearization.[90] The complex valued function ψ in Eq. (25) occurs in a variety of physical applications with different physical interpretations for ψ in their respective fields. For instance, in the context of nonlinear optics, ψ describes the electric field amplitude, and the envelope equation illustrates the propagation of optical pulse through a medium with Kerr nonlinearity. It may be an electromagnetic potential and the equation describes the collapse of Langmuir waves with collisional damping.[94–96] Due to its immense applications in the diverse fields of physics and mathematics, in the successive section we try to find out the exact solitary wave solution of Eq. (25) using exponential function method.

3. Application of exponential function method to (2+1)-dimensional NLS equation

Exactly solvable systems have played a significant role in understanding nonperturbative phenomena in mathematics and physics. It is therefore customary to seek for exact soliton solutions by employing powerful mathematical symbolic computational packages. Among a wealth of methods available for solving NLPDEs, one of the most straightforward, concise, and effective algorithms aided by symbolic computation is the exponential function method. It is a novel approach first proposed by He and Wu[97] to solve the nonlinear evolution equations with high nonlinearity. Further thorough numerous investigations in the past decades, He[98] has studied the concepts of the recently developed asymptotic methods including the exponential function method. The basic idea of the exponential function method was presented in He's monograph[99] and is used by many researchers to study various NLPDEs including higher-dimensional equations. This work has initiated enormous research interest[100,101] in the scientific community. The important remarks of the exponential function method has been described in Refs. [97] and [100], which illustrates that the method yields the generalized solutions of both solitary and periodic nature. Moreover, the expression of the exponential function is more general than the sine and tangent hyperbolic functions, so that one can find the explicit form of the more general solutions in the exponential function method. Also, the symbolic computational packages like Maple, Mathematica, and Matlab make the solution procedure of exponential function so simple, concise, and time conceptual. The implementation of the method to the nonlinear wave equation can be done both straightforwardly and by sub-equation way. However, the direct use of the method is preferable not only because of its convenience, but also because of the possibility of losing some information and solution in using sub-equation way. Therefore, the exponential function method is more convenient and effective than the extended Fan sub-equation method, and thus becomes a promising and powerful method in dealing with the nonlinear evolution equations. Very recently, Marinakis[102] and Zhang *et al*.[103] have constructed the N-soliton solutions for the nonlinear evolution equation by using exponential function method. It is worth pointing out that the method is also valid for difference–differential equations.[104,105] It is revealed from these studies that the free parameters in the solutions obtained using exponential function can recover some known solutions found by most of the existing methods. The subject of our main interest in this section is to construct solitary wave solutions for the N-dimensional lattice equation and to bring out the importance of on-site potential on the dynamical behavior of the system. To proceed, we start with the general model whose dynamics is governed by Eq. (25). Now, we shall find the class of soliton solutions for this equation by applying the exponential function method. In order to obtain some exact soliton solutions of Eq. (25), we first make the following transformations to transform the partial differential equation (PDE) into ordinary differential equation (ODE):

$$\begin{aligned}
&\psi(x,y,t) = u(\zeta)\exp[\mathrm{i}\theta],\\
&\zeta = k\left[x+ly-\left\{2\bar{p}\gamma_{11}+2\bar{q}l\gamma_{22}+(\bar{p}l+\bar{q})\gamma_{12}\right\}t\right],\\
&\theta = (\bar{p}x+\bar{q}y+st).
\end{aligned} \tag{26}$$

Now, we substitute the above relation into Eq. (25) and obtain the following equations:

$$k^2[\gamma_{11}+l^2\gamma_{22}+l\gamma_{12}]u'' - [\gamma\bar{p}^2+\bar{q}^2\gamma_{22}+\gamma_{12}\bar{p}\bar{q}+s]u+\chi u^3=0. \quad (27)$$

Furthermore, we assume a general solution of the above mentioned system containing exponential function in the form

$$u=\frac{\sum_{n'=-c'}^{d'} a_{n'}\exp(n'\zeta)}{\sum_{m'=-p'}^{q'} b_{m'}\exp(m'\zeta)}, \quad (28)$$

where n', m', c', d', p', and q' are unknown positive integers which can be freely chosen, $a_{n'}$ and $b_{m'}$ are unknown constants to be determined. It is possible to expand Eq. (27). In order to find out the values of c' and p', we balance the linear term $u_{\zeta\zeta}$ of the highest order with the highest-order nonlinear term u^3 in Eq. (27). In this way, we obtain

$$u_{\zeta\zeta}=\frac{A_1\exp[-(3p'+c')\zeta]+\cdots}{A_2\exp[-4p'\zeta]+\cdots},$$

and

$$u^3=\frac{A_3\exp[-(3c'+p')\zeta]+\cdots}{A_4\exp[-4p'\zeta]+\cdots},$$

where $A_{i'}$ ($i'=1$–4) are the determined coefficients taken for simplicity. Now, by solving the highest-order exponential terms in the above expansion, we have $-(c'+3p')=-(3c'+p')$ which yields $p'=c'$. Similarly, to determine the values of d' and q', we balance the linear term $u_{\zeta\zeta}$ of the lowest order in Eq. (27) with the lowest-order nonlinear term u^3, which yields

$$u_{\zeta\zeta}=\frac{\cdots+B_1\exp[(3q'+d')\zeta]}{\cdots+B_2\exp[4q'\zeta]},$$

and

$$u^3=\frac{\cdots+B_3\exp[(3d'+q')\zeta]}{\cdots+B_4\exp[4q'\zeta]},$$

where $B_{i'}$ are the coefficients determined. Upon equating the powers of exponential terms, we have $3q'+d'=3d'+q'$, which results in $q'=d'$. Let us choose $p'=c'=d'=q'=1$ for simplicity, then equation (28) reads

$$u(\zeta)=\frac{a_{-1}\exp(-\xi)+a_0+a_1\exp(\xi)}{b_{-1}\exp(-\xi)+b_0+b_1\exp(\xi)}. \quad (29)$$

Substituting Eq. (29) into Eq. (27), we obtain the following equation in the hierarchy of exponential powers:

$$R_1\exp(-3\zeta)+R_2\exp(-2\zeta)+R_3\exp(-\zeta)+R_4+R_5\exp(\zeta)+R_6\exp(2\zeta)+R_7\exp(3\zeta)=0, \quad (30)$$

where $R_{\bar{n}}$ ($\bar{n}=1,2,\ldots,7$) are the coefficients determined which are not written down explicitly here. Now equating the coefficients of all the powers of exponential terms to zero, we obtain the set of equations for a_0, b_0, a_{-1}, a_1, b_{-1}, b_1, $\bar{p}$, $\bar{q}$, s, l, and k as

$$R_1=0,\ R_2=0,\ \ldots,\ R_7=0. \quad (31)$$

With the aid of Matlab, we solve the above set of algebraic equations and obtain the following parameters:

$$\begin{aligned}
a_{-1} &= \frac{a_0b_0(2\gamma_{12}\bar{p}\bar{q}+2s+k^2\gamma_{11}+k^2l^2\gamma_{22}+k^2l\gamma_{12}+2\bar{q}^2\gamma_{22}+2\gamma_{11}\bar{p}^2)}{b_1(-s+4k^2\gamma_{11}-\bar{q}^2\gamma_{22}-\gamma_{12}\bar{p}\bar{q}-\gamma_{11}\bar{p}^2+4k^2l\gamma_{12}+4k^2l^2\gamma_{22})}, \quad a_1=0;\\
b_{-1} &= \Big[\frac{1}{2}(4\chi a_0^2k^2l\gamma_{12}-3\gamma_{11}^2\bar{p}^4b_0^2-3\bar{q}^4\gamma_{22}^2b_0^2+4\chi a_0^2k^2l^2\gamma_{22}+6k^4l^2\gamma_{22}b_0^2\gamma_{11}\\
&\quad +6k^4l^3\gamma_{22}b_0^2\gamma_{12}-\bar{q}^2-\gamma_{22}\chi a_0^2-6\bar{q}^2\gamma_{22}b_0^2\gamma_{11}\bar{p}^2-\gamma_{11}\bar{p}^2\chi a_0^2-6\gamma_{12}\bar{p}\bar{q}b_0^2s\\
&\quad -6\gamma_{12}\bar{p}\bar{q}^3b_0^2\gamma_{22}-6\gamma_{12}\bar{p}^3\bar{q}b_0^2\gamma_{11}+6k^4l\gamma_{12}b_0\gamma_{11}-3s^2b_0^2-\gamma_{12}\bar{p}\bar{q}\chi a_0^2\\
&\quad -3\gamma_{12}^2\bar{p}^2\bar{q}^2b_0^2-s\chi a_0^2-6sb_0^2\bar{q}^2\gamma_{22}-6sb_0^2\gamma_{11}\bar{p}^2+3k^2\gamma_{11}b_0^2+4\chi a_0^2k^2\gamma_{11}\\
&\quad +3k^4l^4\gamma_{22}^2b_0^2+3k^2l^2\gamma_{12}^2b_0^2)\Big]\Big/[b_1(-s-4k^2\gamma_{11}-\bar{q}^2\gamma_{22}-\gamma_{12}\bar{p}\bar{q}-\gamma_{11}\bar{p}^2+4k^2l\gamma_{12}+4k^2l^2\gamma_{22})\\
&\quad \times(\gamma_{11}\bar{p}^2+3k^2\gamma_{11}+3k^2l^2\gamma_{22}+3k^2l^2\gamma_{12}+\gamma_{12}\bar{p}\bar{q}+s+\bar{q}^2\gamma_{22})]; \quad b_1=b_1,
\end{aligned} \quad (32)$$

where a_0, b_0, and b_1 are the arbitrary free parameters. Upon substitution of Eq. (32) into Eqs. (26) and (29), we have the exact solitary wave solution for the (2+1)-dimensional NLS equation as

$$\psi=\left[\frac{a_{-1}\exp(-\zeta)+a_0}{b_{-1}\exp(-\zeta)+b_0+b_1\exp(\zeta)}\right]\exp(\mathrm{i}\theta). \quad (33)$$

Figures 1–7 present the impact of different interaction potentials on the nature of soliton solution. It is noticed from Fig. 1 that in the presence of harmonic potential $J_2=0.5$ alone, the system admits antikink evolution with the exponentially decaying localized structure at the center displaying dromions. From the figure, the obvious feature of the dromion is that it

oscillates along the y direction and is localized along the x direction which looks similar, as discussed by Zhong *et al.*[90] in the context of nonlinear optics. At different instants, these antikink dromions eventually have energy redistribution and it travels along the x direction with finite amplitude and velocity. One can visualize the antisteepening effect of antikink-like evolution from Fig. 1(d) and the corresponding contour plots are displayed in Fig. 1 for various instants $t = 0, 50, 100$, and 200 with the choices of parameters $k = 0.6$, $l = 0.1$, $a_0 = 0.01$, $b_0 = 0.6$, $\bar{p} = 0.4$, $\bar{q} = 0.56$, $q_1 = \pi/3$, $q_2 = \pi/3$, $b_1 = 0.5$, and $s = 0.1$. When the cubic potential $J_3 = 0.35$ is included, the system admits a similar form of mixed phases of dromion-like structure with antikink evolution. In this case, the amplitude of the coherent structure gets enhanced as seen in Fig. 2 compared to the previous case. For the quartic potential J_4, the same effect has been observed in Fig. 2. As the paper focuses on the effect of on-site potential, let us tune on the on-site potentials and see how the dynamics changes drastically. Upon the inclusion of on-site potential J_5, we could observe the oscillating antikink-like dromions having greater domain size as shown in Fig. 3 along with its contour plots. Now, in the presence of cubic or quartic on-site potential, the oscillating multidromions (Fig. 4) exhibits similar pattern to that given in Fig. 1. A close inspection of Fig. 5 and 6 shows the enhancement of multidromion width, when the strengths of all the interaction potentials are incorporated. Figures 7 depict the intensity profile of the oscillating multidromions. During the course of the travel for various instants of time, it acquires higher amplitude at $t = 25$ and 50. From the above discussion, we conclude that the inclusion of on-site potentials drastically changes the nature of the oscillating dromion width and these inferences may be of significance in describing some of the special physical problems.

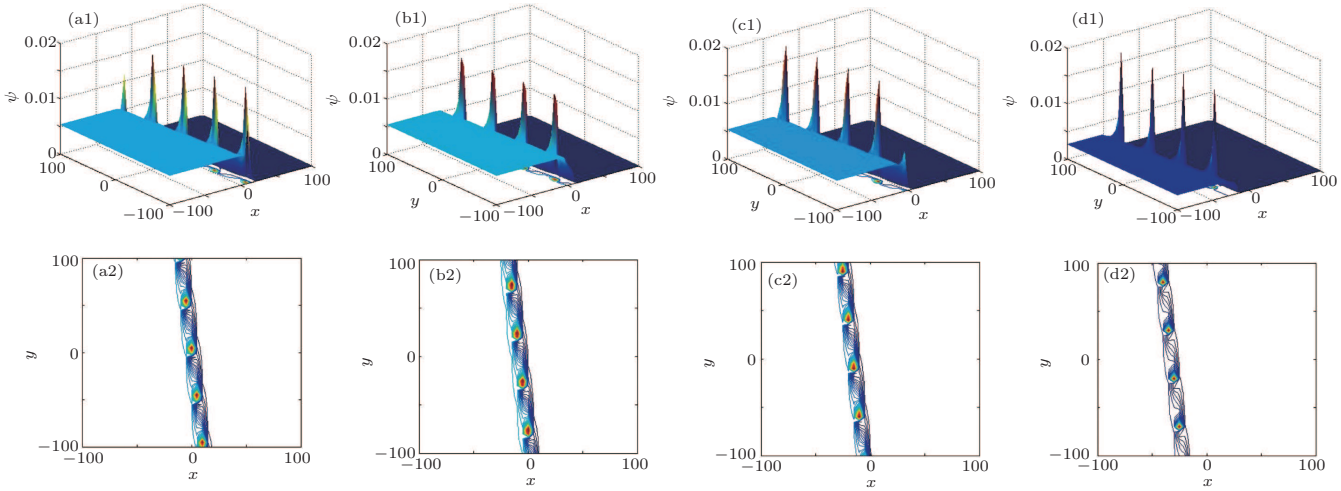


Fig. 1. (color online) Snapshots of oscillating multidromions for (a) $t = 0$, (b) $t = 50$, (c) $t = 100$, and (d) $t = 200$ with $J_2 = 0.5$.

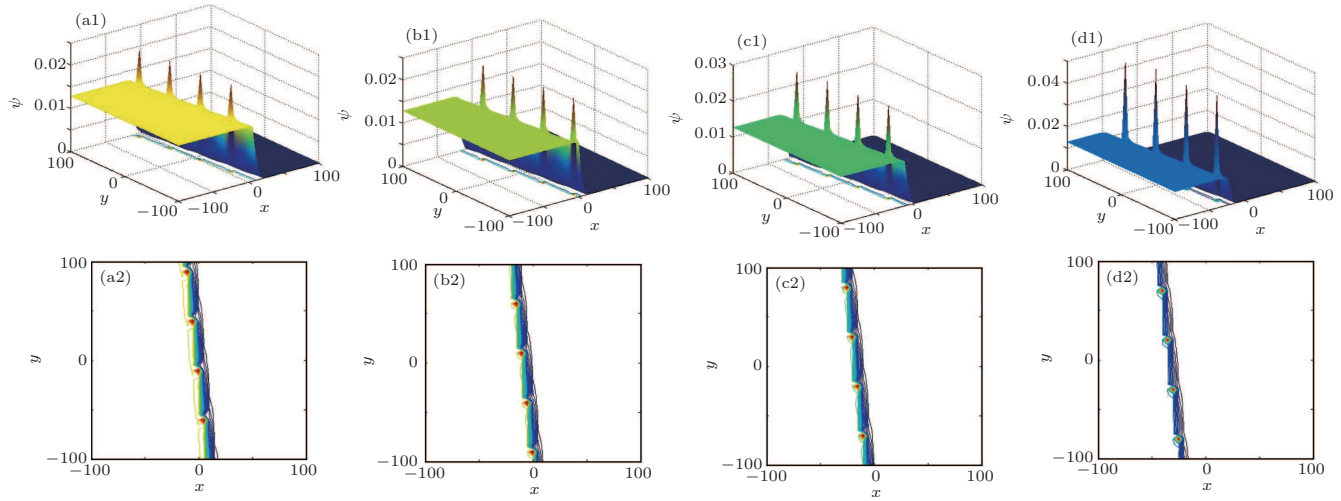


Fig. 2. (color online) Snapshots of oscillating multidromions for (a) $t = 0$, (b) $t = 50$, (c) $t = 100$, and (d) $t = 200$ with $J_2 = 0.5$ and $J_3 = 0.35$.

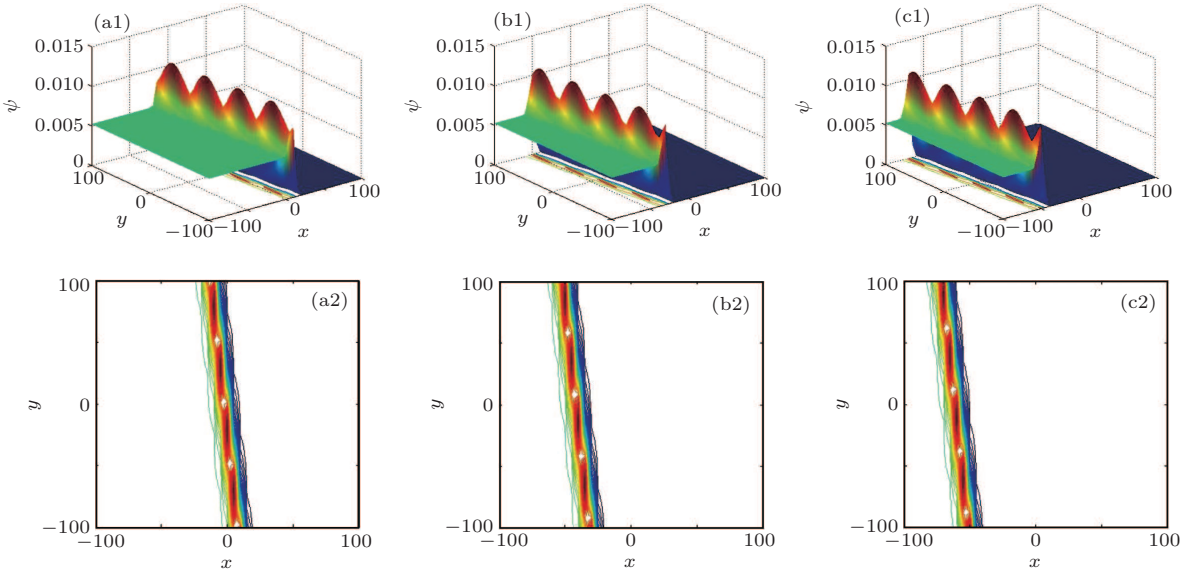


Fig. 3. (color online) Snapshots of oscillating multidromions for (a) $t = 0$, (b) $t = 50$, and (c) $t = 75$ with $J_2 = 0.5$ and $J_5 = 0.55$.

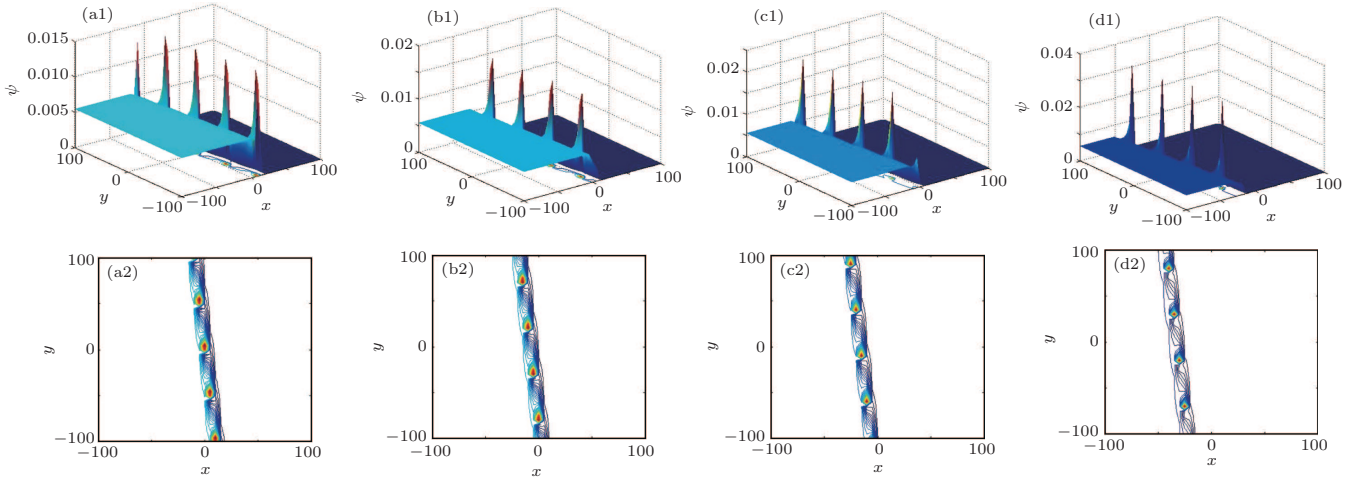


Fig. 4. (color online) Snapshots of oscillating multidromions for (a) $t = 0$, (b) $t = 50$, (c) $t = 100$, and (d) $t = 200$ with $J_2 = 0.5$ and $J_6 = 0.3$.

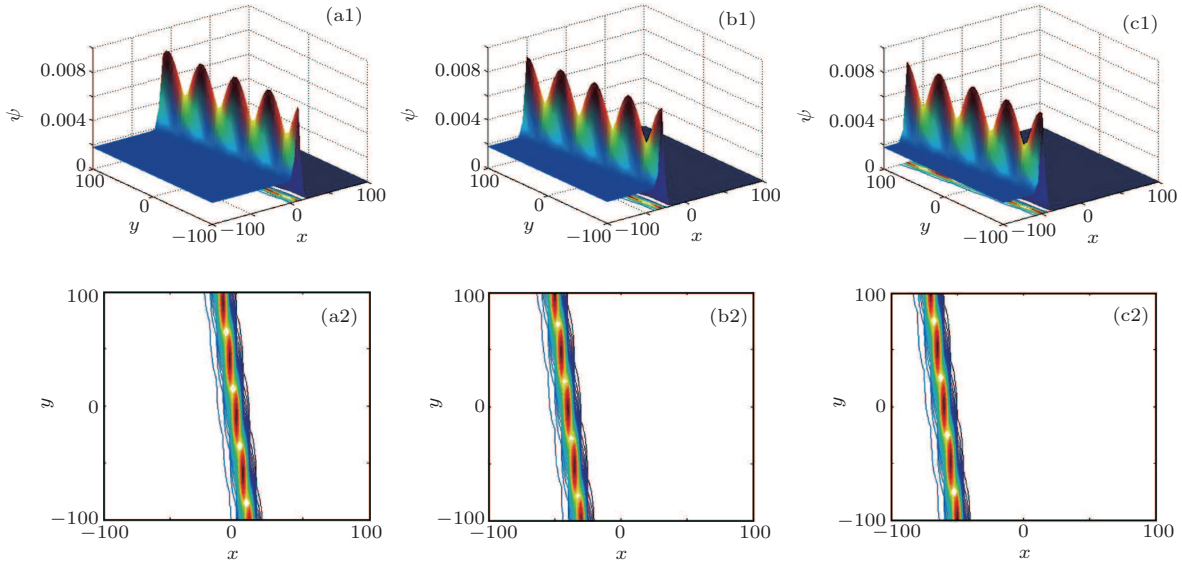


Fig. 5. (color online) Snapshots of oscillating multidromions for (a) $t = 0$, (b) $t = 50$, and (c) $t = 75$ with $J_2 = 0.5$, $J_3 = 0.35$, $J_4 = 0.4$, $J_5 = 0.55$, $J_6 = 0.3$, and $J_7 = 0.5$.

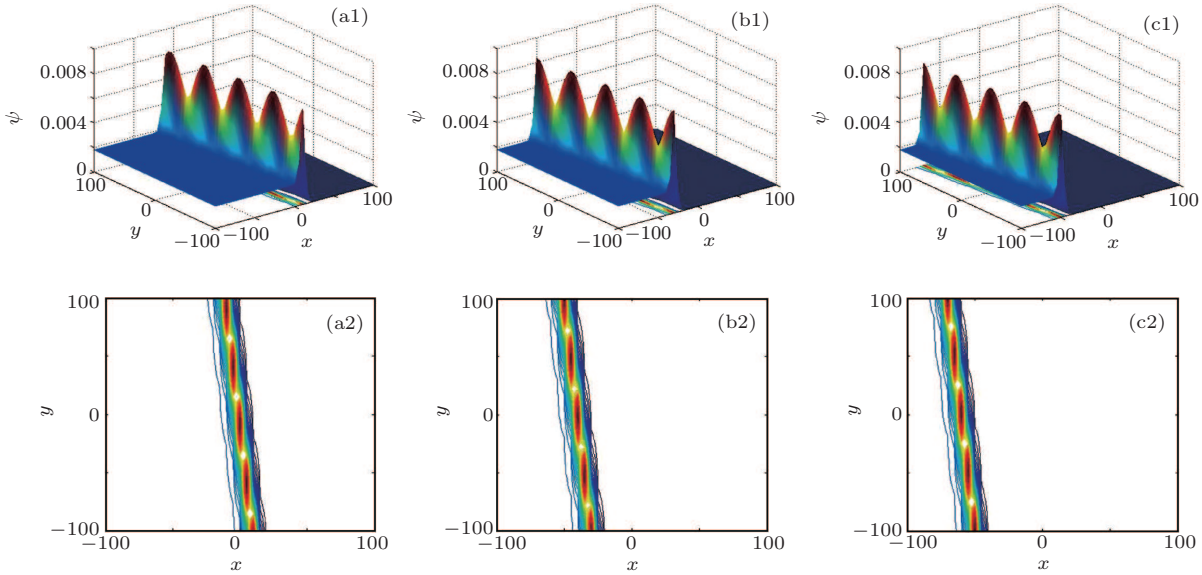


Fig. 6. (color online) Snapshots of oscillating multidromions for (a) $t = 0$, (b) $t = 50$, and (c) $t = 65$ with $J_2 = J_3 = J_4 = J_5 = J_7 = 0.5$ and $J_6 = 0.6$.

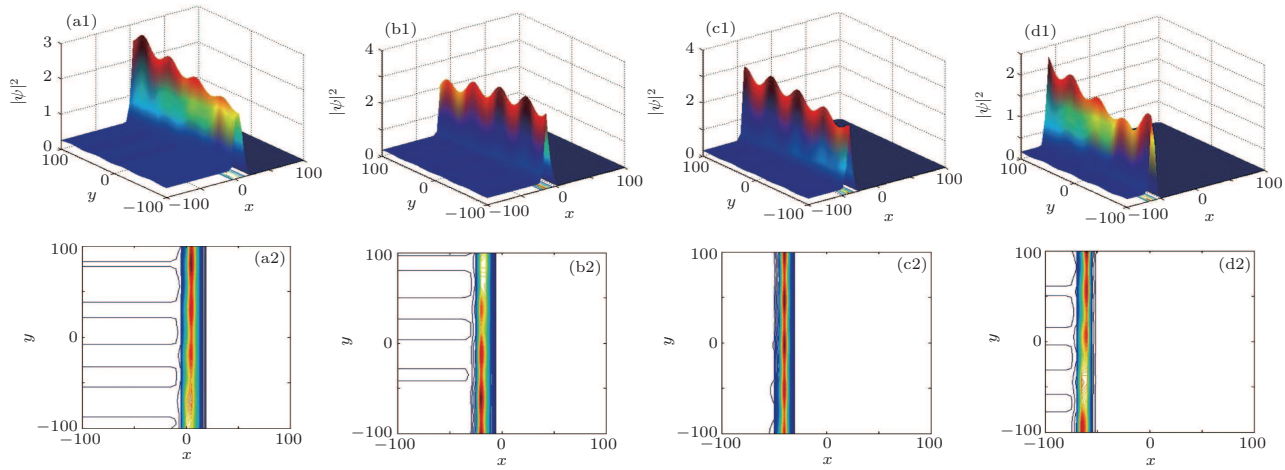


Fig. 7. (color online) Intensity profiles of multidromions for (a) $t = 0$, (b) $t = 25$, (c) $t = 50$, and (d) $t = 75$ with $J_2 = J_3 = J_4 = J_5 = J_6 = J_7 = 0.5$.

4. Conclusions

Investigating nonlinear excitations in higher dimensions attracts deep interest. It is shown in this work, while using the asymptotic expansion, a N-dimensional lattice system subjected to intersite and external on-site potentials reduces, under some considerations, to a N-dimensional NLS equation by incorporating the transverse effect. We also have extended our analysis to (2+1)-dimensional NLS equation and provided some exact soliton solutions for this equation based on the merits of the exponential function method aided by a symbolic computation. The solution includes oscillating multidromions and the width of the multidromion domain enlarges when the strength of all the higher-order external on-site potentials are taken into account. The investigation of (2+1)-dimensional NLS equation may have wider ramification in hydrodynamics and nonlinear optics and helps in understanding the complex physical systems involving multiple dimensions which may have possible potential applications. In general, the method employed is very promising and can be applicable to a wide variety of nonlinear problems in higher-dimensional systems to predict the existence of solitary wave solutions. We further foresee that our results may be of potential use in mathematical and applied physics of nonlinear science.

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