Variable fluid properties and variable heat flux effects on the flow and heat transfer in a non-Newtonian Maxwell fluid over an unsteady stretching sheet with slip velocity

Ahmed M. Megahed†

Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

(Received 27 November 2012; revised manuscript received 29 January 2013)

The effects of variable fluid properties and variable heat flux on the flow and heat transfer of a non-Newtonian Maxwell fluid over an unsteady stretching sheet in the presence of slip velocity have been studied. The governing differential equations are transformed into a set of coupled non-linear ordinary differential equations and then solved with a numerical technique using appropriate boundary conditions for various physical parameters. The numerical solution for the governing non-linear boundary value problem is based on applying the fourth-order Runge–Kutta method coupled with the shooting technique over the entire range of physical parameters. The effects of various parameters like the viscosity parameter, thermal conductivity parameter, unsteadiness parameter, slip velocity parameter, the Deborah number, and the Prandtl number on the flow and temperature profiles as well as on the local skin-friction coefficient and the local Nusselt number are presented and discussed. Comparison of numerical results is made with the earlier published results under limiting cases.

Keywords: Maxwell fluid, unsteady stretching sheet, variable fluid properties, variable heat flux

PACS: 47.50.–d, 47.15.Cb, 47.45.Gx, 47.10.ad **DOI:** 10.1088/1674-1056/22/9/094701

1. Introduction

The flow of non-Newtonian fluids over a stretching surface has received considerable attention during the last several decades because of its numerous applications in engineering problems, such as chemical engineering and particularly, polymer fluids, manufacturing of plastic and rubber sheets, solidification of liquid crystals, glass blowing, hot rolling, crystal growing, continuous cooling and fibers spinning, exotic lubricants, and suspension solutions. Crane[1] was the first to study the motion set up in the ambient fluid due to a linearly stretching surface. Gupta and Gupta[2] have subsequently explored various aspects of the accompanying heat transfer occurring in the infinite fluid medium surrounding the stretching sheet. The flow field of a stretching wall with a power-law velocity variation was discussed by Banks.[3] Chen and Char[4] investigated the heat transfer characteristics over a continuous stretching sheet with variable surface temperature. The self-similar boundary flow with identically vanishing skin friction induced by a continuous plane surface with stretching velocity is considered by Magyari and Keller.[5] Mahapatra and Gupta[6] analyzed stagnation-point flow towards a stretching surface in the presence of free stream velocity.

In all these above studies,[1–6] the flow and temperature fields are considered to be at steady state. There have been relatively few papers published about the stretching force and surface temperature varying with time. The fluid velocity and skin-friction coefficient for an unsteady flow past a wall which starts to move impulsively from the rest, have been calculated by Pop and Na,[7] using both the series and numerical solution methods. They found that the unsteady flow approaches a steady flow situation after a long passage of time. The effect of the unsteadiness parameter on heat transfer and flow field over a stretching surface with and without heat generation was considered by Elbashbeshy and Bazid,[8,9] respectively. Ali and Magyari[10] studied the problem for an unsteady stretching surface condition by using a similarity method to transform governing time-dependent boundary layer equations to a set of ordinary differential equations. Ishak *et al.*[11] investigated boundary layer flow over a continuous stretching permeable surface. They found that the heat transfer rate at the surface increases with unsteadiness parameter. The effect of Hall currents on flow and heat transfer over an unsteady stretching surface in the presence of a strong magnetic field has been analyzed by El-Aziz.[12]

All the above studies[1–12] have dealt with the flow and heat transfer for a Newtonian fluid. A vast majority of reactions involved specially in polymer processing, food processing, biochemical industries, etc., are also typical examples of non-Newtonian behavior. The modeling studies deal with non-Newtonian fluids offer interesting challenges to the mathematicians, physicists, engineers and computer scientists. Rajagopal[13] has studied exact solutions for a class of unsteady unidirectional flows of a second-grade fluid under four different flow situations. Moreover, we cannot predict the effects of stress relaxation. The Maxwell model, which includes a subclass of rate-type fluids, can predict stress relaxation so it has become popular. Thus, some authors[14–17] studied the

†Corresponding author. E-mail: ah_mg_sh@yahoo.com

© 2013 Chinese Physical Society and IOP Publishing Ltd

problem of flow and heat transfer for the Maxwell fluid over a stretching surface by using a similarity method to transform the governing boundary layer equations into a set of ordinary differential equations. The fractional calculus approach in the constitutive relationship model of a generalized Maxwell fluid was introduced by Tan *et al.*[14] Three-dimensional flow of a non-Newtonian Maxwell fluid induced by a stretching surface with convective boundary conditions has been studied by Hayat *et al.*[15] The unsteady two-dimensional flow of an MHD non-Newtonian Maxwell fluid over a stretching surface with a prescribed surface temperature in the presence of a heat source/sink is investigated by Mukhopadhyay.[16] The fractional calculus were found to be quite flexible for describing the rheological and viscoelastic properties of fluids.[17] Mahmoud and Megahed[18] investigated the effects of variable viscosity and thermal conductivity on the flow and heat transfer of an electrically conducting power-law fluid within a thin liquid film over an unsteady stretching sheet in the presence of a transverse magnetic field.

It is important to determine the extent to which a surface heat flux will affect the boundary layer response. Dutta *et al.*[19] determined the temperature distribution in the flow over a stretching surface subject to uniform heat flux. Chang and Lee[20] investigated the problem of free convection flow and heat transfer over a vertical plate with uniform and constant heat flux in a thermally stratified micropolar fluid. Cheng[21] discussed the Soret and Dufour effects on the boundary layer flow due to natural convection heat and mass transfer over a vertical cone in a fluid-saturated porous medium with constant wall heat and mass fluxes.

In all the above-mentioned studies, no attention has been given to the effects of partial slip on the flow past a stretching surface. Undoubtedly, the subject matter of partial slip remains a hotly debated and challenging topic, which has provoked more than a thousand papers prior to 2010 (see Refs. [22]–[27]). Slip velocity may occur on the stretching boundary when the fluid is particulate such as emulsions, suspensions, foams and polymer solutions. Also, the fluids that exhibit boundary slip have important technological applications such as in the polishing of artificial heart valves and internal cavities. Rao and Rajagopal[23] studied the effect of the slip boundary condition on the flow of fluids in a channel. Of late, there has been a revival of interest in the flow problems with partial slip.[24] Polymer melts often exhibit macroscopic wall slip, which in general is governed by a nonlinear and monotone relationship between the slip velocity and traction.[25] Very recently, Mahmoud[26] studied the effects of surface slip and heat generation (absorption) on the flow and heat transfer of a non-Newtonian power-law fluid on a continuously moving surface. The purpose of the present work is to study the effects of variable fluid properties and variable heat flux on the flow and heat transfer of a non-Newtonian Maxwell fluid over an unsteady stretching sheet in the presence of slip velocity. To achieve this study, we use the well-known numerical technique called the shooting method.

This paper is organized as follows. In Section 2, the formulation of the problem is introduced. In Section 3, we summarize the procedure solution using the shooting method. In Section 4, the results and discussion are given. A conclusion is given in Section 5.

2. Formulation of the problem

Consider an unsteady, two-dimensional laminar boundary layer flow of a non-Newtonian Maxwell fluid over a stretching sheet immersed in an incompressible fluid. The x axis is chosen along the plane of the sheet and the y axis is taken normal to the plane. We assume that the surface starts stretching from rest with the velocity $U(x,t)$. The surface of the sheet is held at a surface heat flux $q_w(x,t)$, where t is the time.

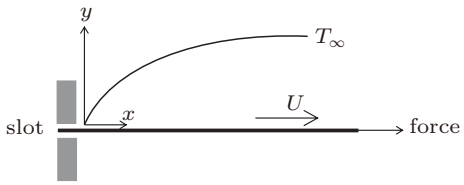


Fig. 1. Flow geometry and coordinate system.

The governing time-dependent boundary layer equations for mass, momentum, and energy conservation are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right], \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\rho c_{\rm p}}\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right), \tag{3}$$

where u and v are the velocity components along x and y directions, respectively; ρ is the fluid density; T is the temperature of the fluid; λ_1 is the relaxation time; $c_{\rm p}$ is the specific heat at constant pressure; μ is the fluid viscosity; κ is the fluid thermal conductivity. The appropriate boundary conditions for the present problem are

$$u=U+\gamma_1\frac{\partial u}{\partial y},\ v=0,\ -\kappa_\infty\frac{\partial T}{\partial y}=q(x,t),\ \text{ as }\ y=0; \tag{4}$$

$$u\to 0,\ T\to T_\infty,\ \text{ as }\ y\to\infty, \tag{5}$$

where U is the surface velocity of the stretching sheet, $\gamma_1=\sqrt{1-at}$ is the velocity slip factor which changes with time, κ_∞ is the thermal conductivity at the ambient, and T_∞ is the free stream temperature.

The flow is caused by stretching the elastic surface at $y=0$ such that the continuous sheet moves in the x direction with the velocity

$$U=\frac{bx}{1-at}, \tag{6}$$

where a and b are positive constants with dimension (t^{-1}). We must observe that our problem is valid only for $at < 1$.

The mathematical analysis of the problem is simplified by introducing the following dimensionless coordinates:

$$\eta = \left[\frac{b}{(\mu_\infty/\rho)}\right]^{1/2}(1-at)^{-1/2}y, \tag{7}$$

$$\psi = \left(\frac{\mu_\infty b}{\rho}\right)^{1/2}(1-at)^{-1/2}xf(\eta), \tag{8}$$

$$T = T_\infty + T_0\left(\frac{dx^2}{\kappa_\infty\sqrt{\rho b/\mu_\infty}}\right)(1-at)^{-3/2}\theta(\eta), \tag{9}$$

$$\theta = \frac{T-T_\infty}{\triangle T}, \quad \triangle T = T_0\left(\frac{dx^2}{\kappa_\infty\sqrt{\rho b/\mu_\infty}}\right)(1-at)^{-3/2}, \tag{10}$$

where f is the dimensionless stream function, ψ is the stream function which is defined in the classical form as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, η is the similarity variable, μ_∞ is the viscosity at the ambient, T_0 is a (positive or negative; heating or cooling) reference temperature, d is a constant, and θ is the dimensionless temperature of the fluid.

The viscosity μ and thermal conductivity κ of the fluid are assumed to vary with the temperature as follows:[18]

$$\mu = \mu_\infty e^{-\alpha\theta}, \tag{11}$$

$$\kappa = \kappa_\infty(1+\varepsilon\theta), \tag{12}$$

where α is the viscosity parameter, and ε is the thermal conductivity parameter.

The surface heat flux $q(x,t)$ at the stretching sheet varies with the power of distance x from the slit and with the power of time factor t[27]

$$q(x,t) = -\kappa_\infty\frac{\partial T}{\partial y} = T_0\frac{dx^2}{(1-at)^2}. \tag{13}$$

Using Eqs. (7)–(13), the mathematical problem defined in Eqs. (1)–(3) is then transformed into a set of the following ordinary differential equations and their associated boundary conditions:

$$e^{-\alpha\theta}(f''' - \alpha\theta' f'') + ff'' - f'^2 - S(f' + \frac{\eta}{2}f'') - De(f^2 f''' - 2ff'f'') = 0, \tag{14}$$

$$(1+\varepsilon\theta)\theta'' + \varepsilon\theta'^2 + Pr(f\theta - 2f'\theta - \frac{S}{2}(3\theta + \eta\theta')) = 0, \tag{15}$$

$$f(0) = 0, \quad f'(0) = 1 + \gamma f''(0), \quad \theta'(0) = -1, \tag{16}$$

$$f' \to 0, \ \theta \to 0, \quad \text{as} \quad \eta \to \infty, \tag{17}$$

where a prime denotes differentiation with respect to η, $S = a/b$ is the unsteadiness parameter, $\gamma = \sqrt{b\rho/\mu_\infty}$ is the slip parameter, $Pr = \mu_\infty c_p/\kappa_\infty$ is the Prandtl number and $De = \lambda_1(t)b/(1-at)$ is the Deborah number. For similarity, solution $\lambda_1(t)$ can be taken in the form $\lambda_1(t) = \lambda_0(1-at)$ for some constant λ_0, then $De = b\lambda_0$.

The physical quantities of interest are the local skin-friction coefficient Cf_x and the local Nusselt number Nu_x which are defined as

$$Cf_x = -2Re_x^{-1/2}f''(0), \tag{18}$$

$$Nu_x = -\frac{Re_x^{1/2}}{\theta(0)}. \tag{19}$$

3. Numerical solution using the shooting method

In order to gain a physical insight, Eqs. (14) and (15) constitute a system of nonlinear coupled ordinary differential equations. This system, along with the boundary conditions (16) and (17), is integrated numerically by means of Runge–Kutta method with systematic estimate of $f''(0)$ and $\theta(0)$ with the Newton–Raphson shooting technique until the boundary conditions at the infinity $f'(\eta_\infty)$ and $\theta(\eta_\infty)$ decay exponentially to zero. At every position, the iteration process continues until the convergence criterion for all the variables, 10^{-6}, is achieved. The step size $\nabla\eta = 0.001$ is used to obtain the numerical solution and the boundary condition $\eta \to \infty$ is approximated by $\eta_{\max} = 8$. In order to get a clear insight of physical problem, numerical results are displayed with the help of graphical illustrations.

4. Results and discussion

A boundary layer problem for the variable fluid properties and variable heat flux on the flow and heat transfer of a Maxwell fluid over an unsteady stretching sheet with slip velocity was solved numerically by applying the shooting method. The correctness of the present numerical method is checked with the results obtained by Abel *et al.*[22] for the values of skin-friction coefficient $-f''(0)$ in the limiting condition ($S = \alpha = \gamma = 0$). Thus, it is seen from Table 1, that the numerical results using the numerical proposed method are in excellent agreement with those published previously.

Table 1. Comparison of $-f''(0)$ values when $S = \alpha = \gamma = 0$ with Abel *et al.*[22] in the absence of magnetic number.

De	Abel *et al.*[22]	Present work
0.0	0.999962	0.999978
0.2	1.051948	1.051945
0.4	1.101850	1.101848
0.6	1.150163	1.150160
0.8	1.196692	1.196690
1.2	1.285257	1.285253
1.6	1.368641	1.368641
2.0	1.447617	1.447616

In Fig. 2, velocity profiles are shown for different values of S. From this figure, it is seen that the velocity along the sheet decreases with the increase of unsteadiness parameter S and this implies an accompanying reduction of the thickness of the momentum boundary layer. Figure 3 represents the temperature profiles for the same parameter values as in Fig. 2. For all values of S considered, the temperature is found to decrease monotonically with the distance from the sheet. This

shows the important fact that the rate of cooling is much faster for higher values of unsteadiness parameter whereas it may take longer time of cooling for smaller values of unsteadiness parameter. It is noteworthy that the impact of S on the temperature profiles is more pronounced than on the velocity profiles in Fig. 2.

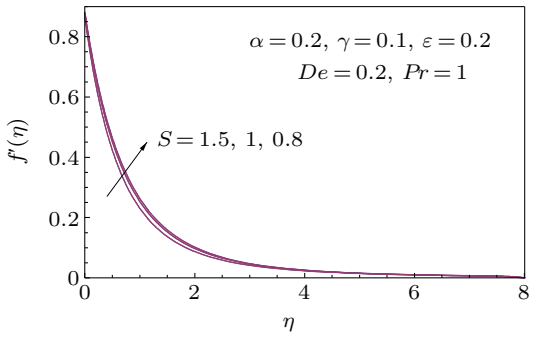


Fig. 2. (color online) Behavior of the velocity distribution for various values of S.

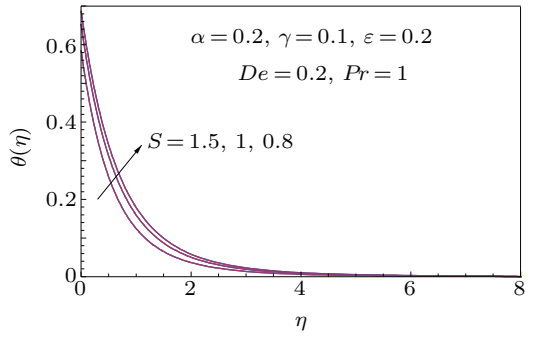


Fig. 3. (color online) Behavior of the temperature distribution for various values of S.

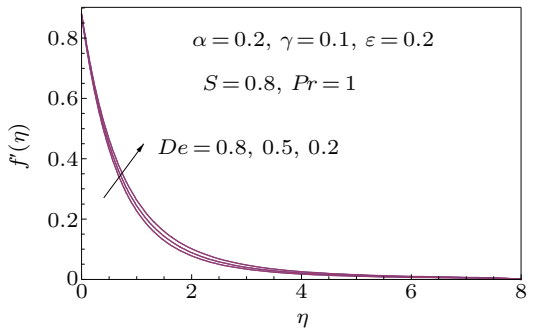


Fig. 4. (color online) Behavior of the velocity distribution for various values of De.

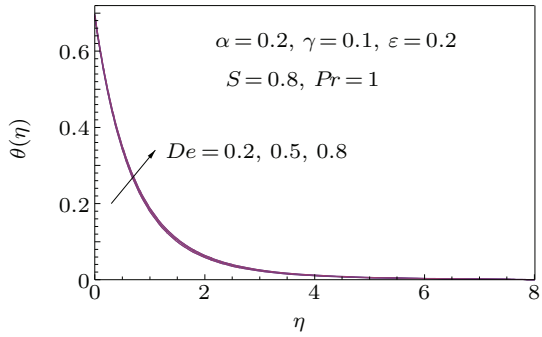


Fig. 5. (color online) Behavior of the temperature distribution for various values of De.

The influence of the Deborah number De on the velocity $f'(\eta)$ is shown in Fig. 4. It is found that an increase in the Deborah number De is noticed to decrease the velocity at any given point above the sheet. Figure 5 demonstrates that at any point the dimensionless temperature $\theta(\eta)$ increases with increasing De. Therefore, cooling of the heated sheet can be improved by choosing a coolant having a small Deborah number.

Figure 6 shows the effect of the Prandtl number on the temperature profiles above the sheet. It is noticed that an increase in the Prandtl number is seen to decrease the fluid temperature $\theta(\eta)$ above the sheet. This is because fluid with a higher value of Pr posses a large heat capacity, and hence intensifies the heat transfer. It is not surprising to realize that the thermal boundary becomes thinner for larger Prandtl numbers. Therefore, cooling of the heated sheet can be improved by choosing a coolant with a large Prandtl number.

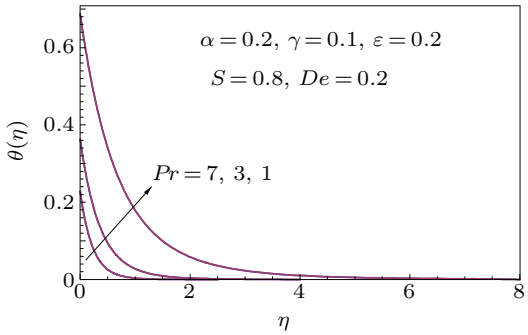


Fig. 6. (color online) Behavior of the temperature distribution for various values of Pr.

Effects of viscosity parameter α on the velocity and temperature profiles are clearly exhibited in Figs. 7 and 8, respectively. Fluid velocity decreases with increasing values of viscosity parameter but the temperature increases in this case. This is due to the fact that increasing the viscosity parameter leads to an increase in the skin-friction coefficient which causes a decrease in the velocity of the fluid. Physically, the unsteady stretching rate and the thermal viscosity cause a rise in the friction, when friction increases, the area of the stretching surface in contact with the flow increases therefore generated heat from the friction on the surface is transferred to the flow. This leads to a rise in the surface temperature and the flow is heated.

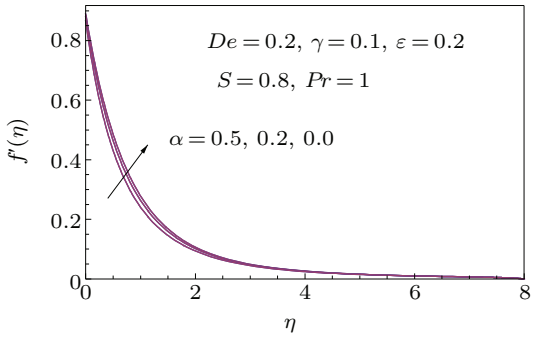


Fig. 7. (color online) Behavior of the velocity distribution for various values of α.

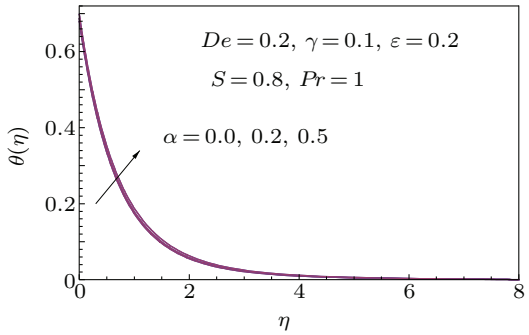


Fig. 8. (color online) Behavior of the temperature distribution for various values of α.

Figure 9 displays the effects of thermal conductivity parameter ε on the temperature profiles. Fluid temperature is found to increase with increasing values of ε which leads to a fall in the rate of heat transfer from the flow to the surface. Therefore, the rate of cooling is much faster for the coolant material having small thermal conductivity parameter.

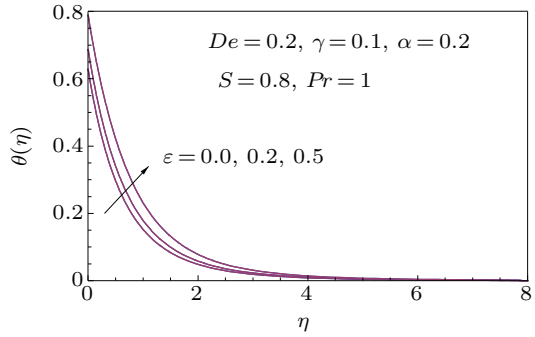


Fig. 9. (color online) Behavior of the temperature distribution for various values of ε.

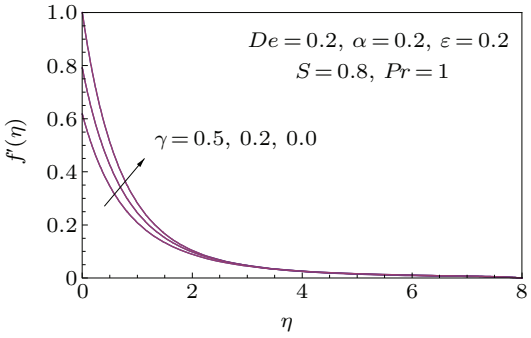


Fig. 10. (color online) Behavior of the velocity distribution for various values of γ.

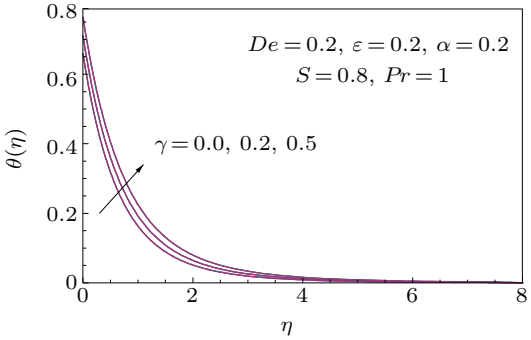


Fig. 11. (color online) Behavior of the temperature distribution for various values of γ.

Figures 10 and 11 depict the effect of the slip parameter γ on the dimensionless velocity and the dimensionless temperature profiles. It is found that higher velocity slips occur for a larger value of γ and the wall drag force becomes smaller for a large value of γ, while the dimensionless temperature in the boundary layer region increases as the slip parameter increases, respectively.

The numerical values of the local skin-friction and the local Nusselt number in terms of $f''(0)$ and $1/\theta(0)$ for various values of unsteadiness parameter S, viscosity parameter α, thermal conductivity parameter ε, the Deborah number De, slip velocity parameter γ, and the Prandtl number Pr are tabulated in Table 2. It can be seen that the local skin-friction coefficient increases by increasing the viscosity parameter, thermal conductivity parameter and the Deborah number, whereas the local Nusselt number decreases with the increasing values of the same parameters. However, it is found that both the skin-friction coefficient and the local Nusselt number increases as the unsteadiness parameter increases. Moreover, it is noticed that increases in the values of the slip velocity parameter leads to a decrease in the local Nusselt number. This is because the slip velocity mechanism will increase the friction near the surface and thus temperature at the surface increases, thereby decreasing the heat transfer at the sheet. However, an increase in the Prandtl number causes an increase in the local Nusselt number. This is because fluid with a higher value of Prandtl number posses a large heat capacity, and hence intensifies the heat transfer.

Table 2. Values of $-f''(0)$ and $1/\theta(0)$ for different values of S, α, ε, De, γ, and Pr.

S	α	ε	De	γ	Pr	$-f''(0)$	$1/\theta(0)$
0.8	0.2	0.2	0.2	0.1	1	1.19850	1.45677
1.0	0.2	0.2	0.2	0.1	1	1.24292	1.53423
1.5	0.2	0.2	0.2	0.1	1	1.34575	1.71291
0.8	0.0	0.2	0.2	0.1	1	1.10991	1.47205
0.8	0.2	0.2	0.2	0.1	1	1.19850	1.45677
0.8	0.5	0.2	0.2	0.1	1	1.34290	1.43238
0.8	0.2	0.0	0.2	0.1	1	1.19172	1.59285
0.8	0.2	0.2	0.2	0.1	1	1.19850	1.45677
0.8	0.2	0.5	0.2	0.1	1	1.21014	1.26801
0.8	0.2	0.2	0.2	0.1	1	1.19850	1.45677
0.8	0.2	0.2	0.5	0.1	1	1.23500	1.44606
0.8	0.2	0.2	0.8	0.1	1	1.27016	1.43588
0.8	0.2	0.2	0.2	0.0	1	1.41639	1.51884
0.8	0.2	0.2	0.2	0.2	1	1.04458	1.40881
0.8	0.2	0.2	0.2	0.5	1	0.76480	1.31036
0.8	0.2	0.2	0.2	0.2	1	1.19850	1.45677
0.8	0.2	0.2	0.2	0.2	3	1.16185	2.76234
0.8	0.2	0.2	0.2	0.2	7	1.15216	4.39793

5. Conclusions

In this paper, we investigated the problem of unsteady heat and fluid flow of non-Newtonian Maxwell fluid with variable fluid properties and slip velocity over a stretching sheet. The resulting non-linear system of ordinary differential equations is solved numerically using the shooting method. We observe that the effect of Deborah number and viscosity parameter on the Maxwell fluid over the stretching sheet is the suppression of the velocity field, which in turn causes the enhancement of the temperature field. Also, it is found that increasing the unsteadiness parameter causes a fall in the flow velocity and temperature. On the other hand, the unsteadiness parameter increases the skin-friction coefficient and the local Nusselt number. Likewise, it is observed that an increase in the Prandtl number results in decreasing thermal boundary layer thickness and less uniform temperature distribution across the boundary layer. The reason is that smaller values of Prandtl number are equivalent to increasing the thermal conductivities, and therefore heat is able to diffuse away from the heated surface more rapidly than for higher values of Prandtl number. Also, it is noticed that increases in the values of both the thermal conductivity parameter and the slip velocity parameter lead to a decrease in the local Nusselt number.

References

[1] Crane L J 1970 *Z. Angew. Math. Phys.* **21** 645
[2] Gupta P S and Gupta A S 1977 *Can. J. Chem. Eng.* **55** 744
[3] Banks W H H 1983 *J. Mec. Theor. Appl.* **2** 375
[4] Chen C K and Char M 1988 *J. Math. Anal. Appl.* **135** 568
[5] Magyari E and Keller B 1999 *J. Phys. D: Appl. Phys.* **32** 2876
[6] Mahapatra T R and Gupta A S 2003 *Can. J. Chem. Eng.* **81** 258
[7] Pop I and Na T 1996 *Mech. Res. Commun.* **23** 413
[8] Elbashbeshy E M A and Bazid M A A 2003 *Appl. Math. Comp.* **138** 239
[9] Elbashbeshy E M A and Bazid M A A 2004 *Heat Mass Tran.* **41** 1
[10] Ali M E and Magyari E 2007 *Int. J. Heat Mass Tran.* **50** 188
[11] Ishak A, Nazar R and Pop I 2009 *Nonlinear Anal. Real World Appl.* **10** 2909
[12] El-Aziz M A 2010 *Meccanica* **45** 97
[13] Rajagopal K R 1982 *Int. J. Non-Linear Mech.* **17** 369
[14] Tan W C, Xiao P W and Yu X M 2003 *Int. J. Non-Linear Mech.* **38** 645
[15] Hayat T, Shehzad S A and Alsaedi A 2012 *Int. J. Phys. Sci.* **7** 761
[16] Mukhopadhyay S 2012 *Chin. Phys. Lett.* **29** 054703
[17] Fetecau C, Athar M and Fetecau C 2009 *Comput. Math. Appl.* **57** 596
[18] Mahmoud M A A and Megahed A M 2009 *Can. J. Phys.* **87** 1065
[19] Dutta B K, Roy P and Gupta A S 1985 *Int. Commun. Heat Mass Trans.* **28** 1234
[20] Cheng L C and Zong Y 2008 *Mech. Res. Commun.* **35** 421
[21] Ching Y C 2011 *Int. Commun. Heat Mass Trans.* **38** 44
[22] Abel M S, Tawade J and Nandeppanavar M M 2012 *Meccanica* **47** 385
[23] Rao I J and Rajagopal K R 1999 *Acta Mech.* **135** 113
[24] Ariel P D 2008 *Z. Angew. Math. Mech.* **88** 320
[25] Abbas Z Z, Wang Y, Hayat T and Oberlack M 2009 *Int. J. Numer. Meth. Fluids* **59** 443
[26] Mostafa A A M 2011 *Math. Comput. Model.* **54** 1228
[27] Megahed A M 2011 *Eur. Phys. J. Plus* **126** 82