Block basis property of a class of 2×2 operator matrices and its application to elasticity*

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A necessary and sufficient condition is obtained for the generalized eigenfunction systems of 2×2 operator matrices to be a block Schauder basis of some Hilbert space, which offers a mathematical foundation of solving symplectic elasticity problems by using the method of separation of variables. Moreover, the theoretical result is applied to two plane elasticity problems via the separable Hamiltonian systems.

Keywords: symplectic elasticity, block Schauder basis, separable Hamiltonian system, operator matrices

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1. Introduction

The semi-inverse method is the classical solution method of elasticity problems. It is rather difficult to seek trial functions satisfying special boundary conditions.[1] In order to solve the analytical elasticity problems, Zhong[2,3] has been proposing the symplectic elasticity approach for deriving exact analytical solutions to some basic problems in solid mechanics and elasticity since the early 1990s. In recent years, the symplectic elasticity approach has been widely applied by many investigators to solid mechanical problems in Refs. [4]–[7] and the references cited therein. The essence of the symplectic elasticity method is the separation method of variables based on a Hamiltonian system, which needs the investigation of the basis property of the eigenfunction systems of Hamiltonian operator matrices.[8–13]

Applying the analogy theory of computational structural mechanics and optimal control [14], many elasticity equations can be written as the following separable linear Hamiltonian system:[15–17]

$$\dot{\boldsymbol{v}}=\mathscr{H}\boldsymbol{v}+\boldsymbol{f},\ \mathscr{H}=\begin{pmatrix}0 & B\\ C & 0\end{pmatrix},\ \dot{()}=\partial/\partial t, \tag{1}$$

where $\boldsymbol{f}$ denotes the external force vector and $\mathscr{H}$ is the off-diagonal Hamiltonian operators.[18,19] Set $\boldsymbol{v}=(v_1,v_2)^{\mathrm{T}}$ and $\boldsymbol{f}=(f_1,f_2)^{\mathrm{T}}$. Suppose that the linear operator B has a bounded inverse, then equation (1) is equivalent to

$$v_2=B^{-1}\dot{v}_1-B^{-1}f_1 \tag{2}$$

and

$$\ddot{v}_1=BCv_1+Bf_2+\dot{f}_1. \tag{3}$$

Hence, if solutions of Eq. (3) can be solved, the solutions of Eq. (1) can be determined by Eqs. (2) and (3). In this process, the basis property of the generalized eigenfunction systems of the product BC of two symmetric operators plays an important role.

Throughout this paper, the symbol X denotes a Hilbert space and $\mathscr{D}(T)$ represents the domain of linear operator T on X. And $\sigma_p(T)$ denotes the point spectrum of T. The inner product will be denoted by the uniform symbol $\langle\cdot,\cdot\rangle$. A closed linear operator T is invertible if T has a bounded inverse.

Definition 1[4] Let T be a linear operator on X. If there is a nonzero $\varphi^0\in\mathscr{D}(T)$ such that $T\varphi^0=\mu\varphi^0$, then μ is called the eigenvalue of T and φ^0 is an eigenfunction of T corresponding to μ. If there is $\varphi^1\in\mathscr{D}(T)$ such that

$$T\varphi^1=\mu\varphi^1+\varphi^0,\varphi^1\neq 0,$$

then φ^1 is the first-order Jordan eigenfunction of T corresponding to μ and φ^0. By induction, if the Jordan eigenfunction φ^{k-1} of order $k-1$ of T has been well defined, then Jordan eigenfunction φ^k of order k is defined by the formula

$$T\varphi^k=\mu\varphi^k+\varphi^{k-1},\varphi^k\neq 0.$$

The union of all eigenfunctions and Jordan eigenfuctions of T corresponding to all eigenvalues of T is called the generalized eigensystem of T.

Definition 2[20] A sequence $\{e_n\}_{n=1}^{+\infty}$ in X is called a Schauder basis of X if for every $x\in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^{+\infty}$ satisfying $x=\sum\limits_{n=1}^{+\infty}a_ne_n$.

Definition 3[21] A sequence $\{v_n\}_{n=1}^{+\infty}$ in X is a block Schauder basis (or Schauder basis with parentheses) if for every $x\in X$ there exist constant sequences $\{d_n\}_{n=1}^{+\infty}$ and monotone increasing integers $0=p_0<p_1<\ldots$ such that $x=$

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$\sum_{n=0}^{+\infty}\big(\sum_{k=p_n+1}^{p_{n+1}} d_k v_k\big)$. And a sequence $\{u_{\pm n}\}_{n=0}^{+\infty}$ in X is called a block Schauder basis if for every $y\in X$ there exist constant sequences $\{c_{\pm n}\}_{n=0}^{+\infty}$ such that

$$y=\sum_{n=0}^{+\infty}\Big[\sum_{k=r_n}^{r_{n+1}-1}(c_k u_k+c_{-k}u_{-k})\Big],$$

with monotone increasing integers $0=r_0<r_1<\ldots$.

The purpose of this paper is to study the block Schauder basis property of a generalized eigensystem of the following operator matrix:

$$\mathscr{M}=\begin{pmatrix}E & F\\ G & H\end{pmatrix}:\mathscr{D}(\mathscr{M})\subset X\times X\to X\times X, \tag{4}$$

where E, F, G, and H are self-adjoint operators, and establish the mathematical foundations of solving Eq. (3) by using separation of variables based on the product operator BC.

2. Main results

We begin with the following lemma.

Lemma 1 Let λ be an eigenvalue of $\mathscr{M}$ and $(x,y)^{\mathrm{T}}$ be the eigenfunction of $\mathscr{M}$ corresponding to λ. If F is invertible and $\langle F^{-1}x,x\rangle\neq 0$, then $y=\lambda F^{-1}x-F^{-1}Ex$ and

$$\lambda=\frac{\langle F^{-1}Ex,x\rangle+\langle HF^{-1}x,x\rangle\pm\sqrt{\Delta_x}}{2\langle F^{-1}x,x\rangle},$$

where

$$\begin{aligned}\Delta_x=&(\langle F^{-1}Ex,x\rangle+\langle HF^{-1}x,x\rangle)^2\\&-4\langle F^{-1}x,x\rangle(\langle HF^{-1}Ex,x\rangle-\langle Gx,x\rangle).\end{aligned}\tag{5}$$

Proof By using the assumptions, we have

$$Ex+Fy=\lambda x,\ Gx+Hy=\lambda y. \tag{6}$$

Noting that F is invertible, we obtain $y=\lambda F^{-1}x-F^{-1}Ex$ and

$$\lambda^2F^{-1}x-\lambda F^{-1}Ex-\lambda HF^{-1}x+HF^{-1}Ex-Gx=0, \tag{7}$$

consequently

$$\begin{aligned}&\lambda^2\langle F^{-1}x,x\rangle-\lambda\langle F^{-1}Ex,x\rangle-\lambda\langle HF^{-1}x,x\rangle\\&+\langle HF^{-1}Ex,x\rangle-\langle Gx,x\rangle=0.\end{aligned}\tag{8}$$

The assertion follows from the above equation and the assumption $\langle F^{-1}x,x\rangle\neq 0$.

In the following, we always suppose that $\mathscr{M}$ satisfies the assumptions as follows:

(i) $\mathscr{M}$ has at most countable eigenvalues;

(ii) F is invertible, E,F,G,H are self-adjoint operators and E,F,H are commuted;

(iii) If $(x_n^{\mathrm{T}},y_n^{\mathrm{T}})^{\mathrm{T}}$ is an eigenfunction of $\mathscr{M}$ corresponding to eigenvalue $\lambda=\lambda_n$, then

$$\langle F^{-1}x_n,x_m\rangle=\begin{cases}\neq 0, n=m\\ =0, n\neq m\end{cases}, \tag{9}$$

and

$$\langle F^{-1}Ex_n,x_m\rangle=\langle HF^{-1}x_n,x_m\rangle=0\text{ for }n\neq m. \tag{10}$$

Remark 1 Let $(x^{\mathrm{T}},y^{\mathrm{T}})^{\mathrm{T}}$ be the eigenfunction of $\mathscr{M}$ corresponding to λ. The assumption (ii) shows that equation (8) is a quadratic equation of real coefficients. Therefore, we can describe $\sigma_p(\mathscr{M})$ by using the signs of discriminant Δ_x in Eq. (5). Considering (iii) and combining the proof of Lemma 1, we obtain that x_n in assumption (iii) is a nonzero solution of the following equation

$$\begin{aligned}&\lambda^2F^{-1}x_n-\lambda F^{-1}Ex_n-\lambda HF^{-1}x_n\\&+HF^{-1}Ex_n-Gx_n=0\end{aligned}\tag{11}$$

and

$$\begin{aligned}\langle Gx_n,x_n\rangle=&\lambda^2\langle F^{-1}x_n,x_n\rangle-\lambda\langle F^{-1}Ex_n,x_n\rangle\\&-\lambda\langle HF^{-1}x_n,x_n\rangle+\langle HF^{-1}Ex_n,x_n\rangle.\end{aligned}\tag{12}$$

For convenience, we use the notations λ_n and λ_{-n} to denote the two roots of Eq. (12) $(n=0,1,2,\ldots)$. Further, we have

$$(\lambda_n+\lambda_{-n})\langle F^{-1}x_n,x_n\rangle=\langle F^{-1}Ex_n,x_n\rangle+\langle HF^{-1}x_n,x_n\rangle \tag{13}$$

and

$$(\lambda_n-\lambda_{-n})\langle F^{-1}x_n,x_n\rangle=\sqrt{\Delta_{x_n}}. \tag{14}$$

Herein, we assume that $\lambda_0=0$.

Lemma 2 If $\lambda_n\in\sigma_p(\mathscr{M})$ and x_n is a nonzero solution of Eq.(11) associated with $\lambda=\lambda_n$ $(n=0,1,\ldots)$, then $(x_n,\lambda_nF^{-1}x_n-F^{-1}Ex_n)^{\mathrm{T}}$ is an eigenfunction of $\mathscr{M}$ corresponding to λ_n. In addition, if $\Delta_{x_n}\neq 0$ and $\{x_n\}_{n=0}^{+\infty}$ is a Schauder basis of X, then λ_{-n} also is an eigenvalue of $\mathscr{M}$ and $(x_n,\lambda_{-n}F^{-1}x_n-F^{-1}Ex_n)^{\mathrm{T}}$ is the eigenfunction of $\mathscr{M}$ corresponding to λ_{-n}.

Proof If x_n is a nonzero solution of Eq. (11) associated with $\lambda=\lambda_n$, taking $y_n=\lambda_nF^{-1}x_n-F^{-1}Ex_n$, we have $Ex_n+Fy_n=\lambda_nx_n$ and $Gx_n+Hy_n=\lambda_ny_n$, i.e., $(x_n,\lambda_nF^{-1}x_n-F^{-1}Ex_n)^{\mathrm{T}}$ is the eigenfunction of $\mathscr{M}$ corresponding to λ_n.

Noting that $\Delta_{x_n}\neq 0$ and from Eqs. (9) and (13), we obtain

$$(\lambda_n+\lambda_{-n})\langle F^{-1}x_n,x_m\rangle-\langle F^{-1}Ex_n,x_m\rangle-\langle HF^{-1}x_nx_m\rangle=0$$

for any nonnegative integers n,m. Since $\{x_n\}_{n=0}^{+\infty}$ is a Schauder basis of X, therefore,

$$(\lambda_n+\lambda_{-n})F^{-1}x_n-F^{-1}Ex_n-HF^{-1}x_n=0. \tag{15}$$

By straightforward calculation and using Eq. (15), we can derive that $(x_n,\lambda_{-n}F^{-1}x_n-F^{-1}Ex_n)^{\mathrm{T}}$ is the eigenfunction of $\mathscr{M}$ corresponding to λ_{-n}.

For convenience, we abbreviate $(x_n, \lambda_n F^{-1}x_n - F^{-1}Ex_n)^{\mathrm{T}}$ and $(x_n, \lambda_{-n}F^{-1}x_n - F^{-1}Ex_n)^{\mathrm{T}}$ by $\boldsymbol{U}_n^0$ and $\boldsymbol{U}_{-n}^0$, respectively.

Lemma 3 Let $\boldsymbol{U}_n^0$ be the eigenfunction of $\mathscr{M}$ corresponding to λ_n $(n = 0, 1, \ldots)$. If $\Delta_{x_n} \neq 0$ and $\{x_n\}_{n=0}^{+\infty}$ is a Schauder basis of X, then the Jordan eigenfunction of $\mathscr{M}$ corresponding to λ_n and $\boldsymbol{U}_n^0$ does not exist, and the Jordan eigenfunction of $\mathscr{M}$ corresponding to λ_{-n} and $\boldsymbol{U}_{-n}^0$ does not occur.

Proof It is sufficient to consider that the assertion holds for the case of λ_n and $\boldsymbol{U}_n^0$. To this end, assume to the contrary that there exists a first-order Jordan eigenfunction $\boldsymbol{U}_n^1$ such that $\mathscr{M}\boldsymbol{U}_n^1 = \lambda_n \boldsymbol{U}_n^1 + \boldsymbol{U}_n^0$. Setting $\boldsymbol{U}_n^1 = (x_n^1, y_n^1)^{\mathrm{T}}$ yields

$$\begin{cases} Ex_n^1 + Fy_n^1 = \lambda_n x_n^1 + x_n, \\ Gx_n^1 + Hy_n^1 = \lambda_n y_n^1 + \lambda_n F^{-1}x_n - F^{-1}Ex_n. \end{cases} \tag{16}$$

Since F is invertible, $y_n^1 = \lambda_n F^{-1}x_n^1 - F^{-1}Ex_n^1 + F^{-1}x_n$. Again, from Eq. (16), we have

$$\begin{aligned} &\lambda_n^2 F^{-1}x_n^1 - \lambda_n F_n^{-1}Ex_n^1 - \lambda_n HF^{-1}x_n^1 + HF^{-1}Ex_n^1 \\ &- Gx_n^1 + 2\lambda_n F^{-1}x_n - HF^{-1}x_n - F^{-1}Ex_n = 0. \end{aligned}$$

Moreover, we conclude that

$$\begin{aligned} &\langle x_n^1, (\lambda_n - \overline{\lambda}_n)[(\lambda_n + \overline{\lambda}_n)F^{-1}x_n - F^{-1}Ex_n - HF^{-1}x_n]\rangle \\ &+ (\lambda_n - \lambda_{-n})\langle F^{-1}x_n, x_n\rangle = 0. \end{aligned}$$

If $\Delta_{x_n} > 0$, then $\lambda_n \in \mathbb{R}$, and then $\lambda_n - \overline{\lambda}_n = 0$. If $\Delta_{x_n} < 0$, then $\lambda_{-n} = \bar{\lambda}_n$. Since $\{x_n\}_{n=1}^{+\infty}$ is a Schauder basis of X, then equation (15) holds. Thus, for $\Delta_{x_n} \neq 0$, we obtain

$$\langle x_n^1, (\lambda_n - \overline{\lambda_n})[(\lambda_n + \overline{\lambda_n})F^{-1}x_n - F^{-1}Ex_n - HF^{-1}x_n]\rangle = 0,$$

and therefore $\langle F^{-1}x_n, x_n\rangle = 0$, a contradiction.

In a way similar to the proof of Lemma 3, the next lemma can be proved.

Lemma 4 Let $\boldsymbol{U}_n^0$ be an eigenfunction of $\mathscr{M}$ corresponding to λ_n (n=0, 1, ...). If $\Delta_{x_n} = 0$ and $\{x_n\}_{n=0}^{+\infty}$ is a Schauder basis of X, then $\mathscr{M}$ only has the first-order Jordan eigenfunction $(x_n^1, y_n^1)^{\mathrm{T}}$ corresponding to λ_n and $\boldsymbol{U}_n^0$, where $x_n^1 = x_n$ and $y_n^1 = \lambda_n F^{-1}x_n - F^{-1}Ex_n + F^{-1}x_n$.

Remark 2 The zero eigenvalue of the operator matrix arising from symplectic elasticity is usually very important. The generalized eigensystem of a zero eigenvalue formulates the physical solutions in Saint-Venant problems of structural statics and elastic statics.[4] If $\lambda_0 = 0 \in \sigma_p(\mathscr{M})$ and $(x_0, y_0)^{\mathrm{T}}$ is the eigenfunction of $\mathscr{M}$ corresponding to λ_0, then, by

$$Gx_0 = HF^{-1}Ex_0, \tag{17}$$

we have

$$\lambda_{-0} = \frac{\langle F^{-1}Ex_0, x_0\rangle + \langle HF^{-1}x_0, x_0\rangle}{2\langle F^{-1}x_0, x_0\rangle},$$

especially,

$$\begin{aligned} &\boldsymbol{U}_0^0 = (x_0, -F^{-1}Ex_0)^{\mathrm{T}}, \\ &\boldsymbol{U}_{-0}^0 = (x_0, \lambda_{-0}F^{-1}x_0 - F^{-1}Ex_0)^{\mathrm{T}}, \text{ if } \lambda_{-0} \neq 0; \\ &\boldsymbol{U}_0^0 = (x_0, -F^{-1}Ex_0)^{\mathrm{T}}, \\ &\boldsymbol{U}_0^1 = (x_0, -F^{-1}Ex_0 + F^{-1}x_0)^{\mathrm{T}}, \text{ if } \lambda_{-0} = 0, \end{aligned}$$

where x_0 is a nonzero solution of Eq. (17).

The following theorem is the main result of this paper. It concerns the block Schauder basis property of the generalized eigensystem of block operator matrix $\mathscr{M}$.

Theorem 1 The generalized eigensystem of $\mathscr{M}$ is a block Schauder basis of Hilbert space $X \times X$ if and only if the first components of generalized eigensystem of $\mathscr{M}$ is a Schauder basis of X.

Proof The necessity is obvious, it suffices to prove the sufficiency. By Lemma 2, the eigenfunction $\boldsymbol{U}_n^0$ of $\mathscr{M}$ corresponding to λ_n has the following representation:

$$\boldsymbol{U}_n^0 = (x_n, \lambda_n F^{-1}x_n - F^{-1}Ex_n)^{\mathrm{T}}, \tag{18}$$

where x_n is a nonzero solution of Eq. (11) associated with $\lambda = \lambda_n$.

Suppose that $\{x_n\}_{n=1}^{+\infty}$ is a Schauder basis of X. The assumption of the invertibility of F implies that $\{F^{-1}x_n\}$ also is a Schauder basis of X.

Case I: $\Delta_{x_n} \neq 0$ for every nonnegative integers n.

From Lemma 1, Lemma 2, and Lemma 3, the eigenfunctions $\boldsymbol{U}_{\pm n}^0$ of $\mathscr{M}$ corresponding to eigenvalues $\lambda_{\pm n}$ $(n = 0, 1, 2, \ldots)$ have the following forms:

$$\begin{aligned} &\boldsymbol{U}_n^0 = (x_n, \lambda_n F^{-1}x_n - F^{-1}Ex_n)^{\mathrm{T}}, \\ &\boldsymbol{U}_{-n}^0 = (x_n, \lambda_{-n}F^{-1}x_n - F^{-1}Ex_n)^{\mathrm{T}}. \end{aligned}$$

In order to prove that $\{U_{\pm n}\}_{n=1}^{+\infty}$ in $X \times X$ is a block Schauder basis, it is sufficient to prove that for every $(f, g)^{\mathrm{T}} \in X \times X$, there exists sequence of scalars $\{C_{\pm n}^0\}_{n=0}^{+\infty}$ such that

$$\begin{pmatrix} f \\ g \end{pmatrix} = \lim_{N \to +\infty} \sum_{n=0}^{N} (C_n^0 \boldsymbol{U}_n^0 + C_{-n}^0 \boldsymbol{U}_{-n}^0). \tag{19}$$

Indeed, setting

$$C_n^0 = \frac{\langle f, F^{-1}x_n\rangle[\langle F^{-1}x_n, x_n\rangle\lambda_{-n} - \langle F^{-1}Ex_n, x_n\rangle] - \langle F^{-1}x_n, x_n\rangle\langle g, x_n\rangle}{(\lambda_n - \lambda_{-n})\langle F^{-1}x_n, x_n\rangle^2}$$

and

$$C_{-n}^0 = -\frac{\langle f, F^{-1}x_n\rangle[\langle F^{-1}x_n, x_n\rangle\lambda_n - \langle F^{-1}Ex_n, x_n\rangle] - \langle F^{-1}x_n, x_n\rangle\langle g, x_n\rangle}{(\lambda_n - \lambda_{-n})\langle F^{-1}x_n, x_n\rangle^2},$$

and by straightforward calculation, we have

$$\sum_{n=0}^{N}(C_n^0\boldsymbol{U}_n^0+C_{-n}^0\boldsymbol{U}_{-n}^0)$$
$$=\sum_{n=0}^{N}\begin{pmatrix}\dfrac{\langle f,F^{-1}x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}x_n\\ \dfrac{\langle f,F^{-1}x_n\rangle}{\langle F^{-1}x_n,x_n\rangle^2}\alpha_n+\dfrac{\langle g,x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}F^{-1}x_n\end{pmatrix}, \quad (20)$$

where $\alpha_n=\langle F^{-1}Ex_n,x_n\rangle F^{-1}x_n-\langle F^{-1}x_n,x_n\rangle F^{-1}Ex_n$.

Noting that

$$\langle\alpha_n,x_n\rangle = \langle F^{-1}Ex_n,x_n\rangle\langle F^{-1}x_n,x_n\rangle - \langle F^{-1}x_n,x_n\rangle\langle F^{-1}Ex_n,x_n\rangle=0,$$

and by Eq. (9) in the assumption (iii), we have

$$\langle\alpha_n,x_m\rangle = \langle F^{-1}Ex_n,x_n\rangle\langle F^{-1}x_n,x_m\rangle - \langle F^{-1}x_n,x_n\rangle\langle F^{-1}Ex_n,x_m\rangle=0,\ n\neq m.$$

Since $\{x_n\}_{n=0}^{+\infty}$ is a Schauder basis of X, $\alpha_n=0$ for every nonnegative integers n. Therefore, we have

$$\sum_{n=0}^{N}(C_n^0\boldsymbol{U}_n^0+C_{-n}^0\boldsymbol{U}_{-n}^0)=\sum_{n=0}^{N}\begin{pmatrix}\dfrac{\langle f,F^{-1}x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}x_n\\ \dfrac{\langle g,x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}F^{-1}x_n\end{pmatrix}. \quad (21)$$

Moreover,

$$\lim_{N\to+\infty}\sum_{n=0}^{N}\frac{\langle f,F^{-1}x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}x_n=f,$$
$$\lim_{N\to+\infty}\sum_{n=0}^{N}\frac{\langle g,x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}F^{-1}x_n=g.$$

This proves that equation (19) is true.

Case II: $\Delta_{x_n}=0$ for every nonnegative integers n.

By Lemma 4, the generalized eigenfunctions of $\mathscr{M}$ corresponding to λ_n are

$$\boldsymbol{U}_n^0=\begin{pmatrix}x_n\\ \lambda_nF^{-1}x_n-F^{-1}Ex_n\end{pmatrix},$$
$$\boldsymbol{U}_n^1=\begin{pmatrix}x_n\\ \lambda_nF^{-1}x_n-F^{-1}Ex_n+F^{-1}x_n\end{pmatrix},$$

and the Jordan eigenfunction of order more than one does not occur.

In the sequel we prove that for every $(f,g)^{\mathrm{T}}\in X\times X$ there exists sequence of scalars $\{C_n^0\}_{n=1}^{+\infty}$ and $\{C_n^1\}_{n=1}^{+\infty}$ such that

$$\begin{pmatrix}f\\ g\end{pmatrix}=\lim_{N\to+\infty}\sum_{n=0}^{N}(C_n^0\boldsymbol{U}_n^0+C_n^1\boldsymbol{U}_n^1). \quad (22)$$

In fact, we take

$$C_n^0=\frac{\langle f,F^{-1}x_n\rangle(\langle F^{-1}x_n,x_n\rangle\lambda_n-\langle F^{-1}Ex_n,x_n\rangle+\langle F^{-1}x_n,x_n\rangle)-\langle F^{-1}x_n,x_n\rangle\langle g,x_n\rangle}{\langle F^{-1}x_n,x_n\rangle^2}$$

and

$$C_n^1=-\frac{\langle f,F^{-1}x_n\rangle(\langle F^{-1}x_n,x_n\rangle\lambda_n-\langle F^{-1}Ex_n,x_n\rangle)-\langle F^{-1}x_n,x_n\rangle\langle g,x_n\rangle}{\langle F^{-1}x_n,x_n\rangle^2}.$$

It is similar to Case I that

$$\lim_{N\to+\infty}\sum_{n=0}^{N}(C_n^0\boldsymbol{U}_n^0+C_n^1\boldsymbol{U}_n^1)$$
$$=\begin{pmatrix}\lim\limits_{N\to+\infty}\sum\limits_{n=0}^{N}\dfrac{\langle f,F^{-1}x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}x_n\\ \lim\limits_{N\to+\infty}\sum\limits_{n=0}^{N}\dfrac{\langle g,x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}F^{-1}x_n\end{pmatrix}=\begin{pmatrix}f\\ g\end{pmatrix}.$$

Case III: $\Delta_{x_n}\neq0$ for every nonnegative integer $n\neq n'$, but $\Delta_{x_{n'}}=0$.

In this case, it is sufficient to note that

$$(C_{n'}^0\boldsymbol{U}_{n'}^0+C_{n'}^1\boldsymbol{U}_{n'}^1)+\sum_{n=0,n\neq n'}^{N}(C_n^0\boldsymbol{U}_n^0+C_{-n}^0\boldsymbol{U}_{-n}^0)$$
$$=\sum_{n=0}^{N}\begin{pmatrix}\dfrac{\langle f,F^{-1}x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}x_n\\ \dfrac{\langle g,x_n\rangle}{\langle F^{-1}x_n,x_n\rangle}F^{-1}x_n\end{pmatrix}.$$

The proof is completed.

3. Application to symplectic elasticity

In this section, we apply Theorem 1 to two symplectic elastic models.

Example 1[22] Considering the rectangular thin plate on elastic foundation of double parameters. The governing equation is given by

$$\nabla^4W-\frac{G_s}{D}\nabla^2W+\frac{K}{D}W=\frac{q}{D}. \quad (23)$$

The bending moments are described by

$$M_x=-D\left(\frac{\partial^2W}{\partial x^2}+\nu\frac{\partial^2W}{\partial y^2}\right),$$
$$M_y=-D\left(\frac{\partial^2W}{\partial y^2}+\nu\frac{\partial^2W}{\partial x^2}\right), \quad (24)$$

where W denotes the vertical deflection of the plate, $D=Eh^3/[12(1-\nu^2)]$ is the flexural rigidity; E, ν, and h are the elastic modulus, Possion's ratio and thickness of the material, respectively; G_s and K denote the shear modulus and reaction modulus, respectively.

Introduce

$$M=-\frac{M_x+M_y}{D(1+\nu)},\quad \theta=\frac{\partial W}{\partial y},$$
$$\varphi=-\frac{\partial M}{\partial y},\quad N=\frac{D}{K}M, \tag{25}$$

then equations (23) and (24) can be rewritten as the separable Hamiltonian system

$$\frac{\partial \boldsymbol{U}}{\partial y}=\mathscr{H}\boldsymbol{U}+\boldsymbol{f}, \tag{26}$$

where $\boldsymbol{U}=(W,N,\theta,\varphi)^{\mathrm{T}}$, $\boldsymbol{f}=(0,0,0,-q/D)^{\mathrm{T}}$, and

$$\mathscr{H}=\begin{pmatrix}0 & \boldsymbol{B}\\ \boldsymbol{C} & 0\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}1 & 0\\ 0 & -\dfrac{D}{K}\end{pmatrix},$$
$$\boldsymbol{C}=\begin{pmatrix}-\dfrac{\partial^2}{\partial x^2} & \dfrac{K}{D}\\ \dfrac{K}{D} & \dfrac{K}{D}\left(\dfrac{\partial^2}{\partial x^2}-\dfrac{G_{\mathrm{s}}}{D}\right)\end{pmatrix}.$$

Let $\boldsymbol{U}_1=(W,N)^{\mathrm{T}}$, $\boldsymbol{U}_2=(\theta,\varphi)^{\mathrm{T}}$, and $\boldsymbol{f}_2=(0,-q/D)^{\mathrm{T}}$. Then equation (26) is equivalent to

$$\boldsymbol{U}_2=\boldsymbol{B}^{-1}\frac{\partial \boldsymbol{U}_1}{\partial y} \tag{27}$$

and

$$\frac{\partial^2 \boldsymbol{U}_1}{\partial y^2}=\boldsymbol{B}\boldsymbol{C}\boldsymbol{U}_1+\boldsymbol{g}, \tag{28}$$

where $\boldsymbol{g}=\boldsymbol{B}^{-1}\boldsymbol{f}_2=(0,q/K)^{\mathrm{T}}$ and

$$\boldsymbol{B}\boldsymbol{C}=\begin{pmatrix}-\dfrac{\partial^2}{\partial x^2} & \dfrac{K}{D}\\ -1 & -\dfrac{\partial^2}{\partial x^2}+\dfrac{G_{\mathrm{s}}}{D}\end{pmatrix}.$$

Therefore, the solution of Eq. (26) comes down to solving Eq. (28), we should consider the basis property of generalized eigensystem of block operator matrix

$$\mathscr{M}=\begin{pmatrix}E & F\\ G & H\end{pmatrix}=\begin{pmatrix}-\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} & \dfrac{K}{D}\\ -1 & -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}+\dfrac{G_{\mathrm{s}}}{D}\end{pmatrix}. \tag{29}$$

Without loss of generality, consider the following boundary conditions for simply supported edges

$$W=0,\ \frac{\partial^2 W}{\partial x^2}=0,\ x=0,1. \tag{30}$$

Set $X=L_2(0,1)$. According to Eqs. (25) and (30), the domain of $\mathscr{M}$ is given by

$$\mathscr{D}(\mathscr{M})=\mathscr{D}(E)\times\mathscr{D}(H), \tag{31}$$

where

$$\mathscr{D}(E)=\mathscr{D}(H)=\left\{v(x)\in X\,\middle|\,\begin{array}{l}v(0)=v(1)=v''(0)=v''(1)=0,\ v' \text{ is}\\ \text{absolutely continuous and } v',v''\in X\end{array}\right\}.$$

The eigenvalues of $\mathscr{M}$ can be obtained by direct calculation

$$\lambda_{\pm n}=(n\pi)^2+\frac{G_{\mathrm{s}}}{2D}\pm\sqrt{\left(\frac{G_{\mathrm{s}}}{2D}\right)^2-\frac{K}{D}},\ n=1,2,\ldots.$$

Moreover, $\mathscr{M}$ only has basic eigenfunctions corresponding to $\lambda_n,\lambda_{-n}(n=1,2,\ldots)$, and

$$\boldsymbol{U}_n^0=\left\{\sin(n\pi x),\ \frac{D}{K}\left[\frac{G_{\mathrm{s}}}{2D}-\sqrt{(\frac{G_{\mathrm{s}}}{2D})^2-\frac{K}{D}}\,\right]\sin(n\pi x)\right\}^{\mathrm{T}},$$
$$\boldsymbol{U}_{-n}^0=\left\{\sin(n\pi x),\ \frac{D}{K}\left[\frac{G_{\mathrm{s}}}{2D}+\sqrt{(\frac{G_{\mathrm{s}}}{2D})^2-\frac{K}{D}}\,\right]\sin(n\pi x)\right\}^{\mathrm{T}}.$$

It is easy to check that F is invertible and E, F, H are commute with each other. Further

$$\langle F^{-1}\sin(m\pi x),\sin(n\pi x)\rangle=\begin{cases}\neq 0, & n=m\\ =0, & n\neq m\end{cases}, \tag{32}$$
$$\langle F^{-1}E\sin(m\pi x),\sin(n\pi x)\rangle$$
$$=\langle HF^{-1}\sin(m\pi x),\sin(n\pi x)\rangle=0, n\neq m. \tag{33}$$

Thus $\mathscr{M}$ satisfies all assumptions of Theorem 1. Since $\{\sin(n\pi x)\}_{n=1}^{+\infty}$ is a Schauder basis of $L_2(0,1)$, the eigenfunction system of $\mathscr{M}$ is a block Schauder basis in $X\times X$ by Theorem 1.

Example 2[14] Consider the free vibration equation of rectangular thin plates

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 W=0 \tag{34}$$

subjected to the boundary conditions for simply supported edges

$$W=0,\ \frac{\partial^2 W}{\partial y^2}=0,\ y=0,1. \tag{35}$$

Introducing the bending moments

$$M_x=-D\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right),$$
$$M_y=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right),$$

and setting

$$M=-\frac{M_x+M_y}{D(1+\nu)},\quad \theta=\frac{\partial W}{\partial x},\quad \varphi=\frac{\partial M}{\partial y}, \tag{36}$$

we have the following separable Hamiltonian system

$$\frac{\partial \boldsymbol{V}}{\partial x} = \mathscr{H}\boldsymbol{V}, \tag{37}$$

where $\boldsymbol{V} = (W, M, \theta, \varphi)^{\mathrm{T}}$, $\mathscr{H} = \begin{pmatrix} 0 & \boldsymbol{B} \\ \boldsymbol{C} & 0 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and

$$\boldsymbol{C} = \begin{pmatrix} 0 & -\dfrac{\partial^2}{\partial y^2} \\ -\dfrac{\partial^2}{\partial y^2} & 1 \end{pmatrix}.$$

Let $\boldsymbol{V}_1 = (W, M)^{\mathrm{T}}$ and $\boldsymbol{V}_2 = (\theta, \varphi)^{\mathrm{T}}$. Then system (37) can be rewritten as

$$\boldsymbol{V}_2 = \boldsymbol{B}^{-1}\frac{\partial \boldsymbol{V}_1}{\partial x} \tag{38}$$

and

$$\frac{\partial^2 \boldsymbol{V}_1}{\partial x^2} = \boldsymbol{B}\boldsymbol{C}\boldsymbol{V}_1. \tag{39}$$

Thus the solution of Eq. (37) is closely dependent on Eq. (39), in which, the basis property of generalized eigensystem of block operator matrix

$$\mathscr{M} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} -\dfrac{\mathrm{d}^2}{\mathrm{d}y^2} & 1 \\ 0 & -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \end{pmatrix} \tag{40}$$

is considered first of all.

Set $X = L_2(0,1)$. In view of Eqs. (30) and (36), the domain of $\mathscr{M}$ is as Eq. (31).

By straightforward calculation, the eigenvalues of $\mathscr{M}$ are

$$\lambda_n = (n\pi)^2, n = 1, 2, \ldots.$$

The Jourdan eigenfunctions of $\mathscr{M}$ corresponding to λ_n have the following representations:

$$\boldsymbol{U}_n^0 = (\sin(n\pi x), 0)^{\mathrm{T}},$$
$$\boldsymbol{U}_n^1 = (\sin(n\pi x), \sin(n\pi x))^{\mathrm{T}},\ n = 1, 2, \ldots.$$

It can be checked that $\mathscr{M}$ satisfies the all assumptions of Theorem 1. Note that $\{\sin n\pi x\}_{n=1}^{+\infty}$ is a Schauder basis of $L_2(0,1)$, the generalized eigensystem of $\mathscr{M}$ is a block Schauder basis in $X \times X$ by Theorem 1.

4. Conclusions

The symplectic elasticity approach realizes a transition from the Euclid space to the Hilbert space, and implements a conversion from a Language system to a Hamiltonian system. Since many elasticity problems can be derived into a separable Hamiltonian system, the solving of the system is considered in this paper. In order to reduce the dimension of the problem and provide the corresponding mathematical foundation, we proved the generalized eigensystem of a 2×2 block operator matrice to be a block Schauder basis. Moreover, we apply the proved theorem to the thin plate on elastic foundation and the free vibration problem of thin plates via the separable Hamiltonian systems and show that the feasibility of solving the problems by using the method of separation of variables. It can be seen that the operator matrix $\mathscr{M}$ is non-self-adjoint in general. The presented approach and technique can be used as a reference for more complex non-self-adjoint problems.

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References

[1] Timoshenko S P and Woinowsky-Krieger S W 1959 *Theory of Plates and Shells* (New York: McGraw-Hill)
[2] Zhong W X 1991 *Journal of Dalian University of Technology* **31** 373 (in Chinese)
[3] Zhong W X 1992 *Acta Mech. Sin.* **24** 432 (in Chinese)
[4] Yao W A, Zhong W X and Lim C W 2009 *Symplectic Elasticity* (New Jersey: World Scientific) p. 5
[5] Lim C W, Cui S and Yao W A 2007 *Int. J. Solids Struct.* **44** 5396
[6] Leung A Y T, Xu X S, Zhou Z H and Wu Y F 2009 *Eng. Fract. Mech.* **76** 1866
[7] Zhao L, Chen W Q and Lü C F 2012 *Mech. Mater.* **54** 32
[8] Alatancang and Wu D Y 2009 *Sci. China Ser. A* **52** 173
[9] Hou G L and Alatancang 2012 *Sci. China Math.* **42** 57
[10] Qi G W, Hou G L and Alatancang 2011 *Chin. Phys. B* **20** 124601
[11] Wang H, Alatancang and Huang J J 2011 *Chin. Phys. B* **20** 100202
[12] Huang J J, Alatancang and Wang H 2010 *Chin. Phys. B* **19** 120201
[13] Wang H, Alatancang and Huang J J 2011 *Acta Math. Sci.* **31** 1266
[14] Zhong W X and Zhong X X 1993 *Numer. Meth. Part Differ. Equ.* **9** 63
[15] Liu Y and Li R 2010 *Appl. Math. Modelling* **34** 856
[16] Li R, Zhong Y and Tian B 2011 *Mech. Res. Commum.* **38** 111
[17] Pan B F, Li R, Su Y, Wang B and Zhong Y 2013 *Appl. Math. Lett.* **26** 355
[18] Tretter C 2008 *Spectral Theory of Block Operator Matrices and Applications* (London: Imperial College Press)
[19] Hou G L and Alatancang 2009 *Commun. Theor. Phys.* **51** 200
[20] Fabian M, Habala P, Hájek P, Montesinos V and Zizler V 2011 *Banach Space Theory: the Basis for Linear and Nonlinear Analysis* (New York: Springer) p. 182
[21] Wyss C 2010 arXiv:1005.5336
[22] Zhong Y and Zhang Y S 2005 *Acta Mech. Solida Sin.* **26** 325 (in Chinese)