Cluster exponential synchronization of a class of complex networks with hybrid coupling and time-varying delay*

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This paper deals with the cluster exponential synchronization of a class of complex networks with hybrid coupling and time-varying delay. Through constructing an appropriate Lyapunov–Krasovskii functional and applying the theory of the Kronecker product of matrices and the linear matrix inequality (LMI) technique, several novel sufficient conditions for cluster exponential synchronization are obtained. These cluster exponential synchronization conditions adopt the bounds of both time delay and its derivative, which are less conservative. Finally, the numerical simulations are performed to show the effectiveness of the theoretical results.

Keywords: complex network, cluster exponential synchronization, linear matrix inequality, time-varying delay

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1. Introduction

Complex dynamical networks have received increasing attention since they widely exist in various real-world fields, such as the Internet, the World Wide Web (WWW), food webs, electricity distribution networks, relationship networks, and metabolic pathways.[1–3] One of the most important phenomena in complex networks is the synchronization of dynamical nodes. Therefore, many researchers have focused on this topic and have developed several efficient synchronization techniques for networks of coupled systems.[4–9] In Ref. [7], a general criterion for pinning synchronization of complex dynamical networks was proposed. In Ref. [8], the synchronization problem for complex dynamical networks with switching topology was studied from the viewpoint of a switched system.

On the other hand, time delays in complex networks are necessary to be taken into consideration for modeling realistic complex networks because the network traffic congestion and the finite speed of signal transmission over the links may lead to the oscillation phenomenon or instability of the networks.[10–19] In Refs. [11] and [18], the global synchronization and global exponential synchronization of linearly hybrid coupled networks with time-varying delay were investigated, respectively. In Ref. [15], the stochastic synchronization was analyzed for a new array of coupled discrete stochastic complex networks with the nonlinearity occurring randomly.

There is a particular synchronization phenomenon named cluster synchronization, which requires that synchronization occurs in each group, but it does not synchronize among different groups. The cluster synchronization was investigated in Refs. [20]–[25]. Cao *et al.*[20] studied the cluster synchronization in an array of hybrid coupled neural networks with constant delay. Yao *et al.*[23] investigated a new cluster projective synchronization scheme in time delay coupled complex dynamical networks with nonidentical nodes. Wu *et al.*[24] studied cluster mixed synchronization of networks using some linear pinning control schemes. The convergence speed of exponential synchronization is larger than that of asymptotic synchronization. To the best of our knowledge, few authors have considered the cluster exponential synchronization for a complex network with discrete and distributed time-varying delay.

Motivated by the above discussion, we introduce a linearly hybrid coupled network with discrete time-varying delay, distributed time-varying delay, and a nonlinear term with distributed time-varying delay. Based on this network model, several novel sufficient conditions of cluster exponential synchronization are deduced by using the Lyapunov–Krasovskii functional and the linear matrix inequality (LMI). The delay-dependent sufficient conditions are obtained, which are less conservative. It should be especially emphasized that we do not assume that the coupling configuration matrices are symmetric or irreducible and the inner coupling matrices are not assumed to be diagonal or positive definite either. Finally, the effectiveness of the cluster exponential synchronization criteria can be demonstrated by utilizing numerical examples.

2. System description and preliminaries

Notation Throughout the paper, the symbol $*$ denotes entry induced by symmetry. The symbol $\otimes$ is Kronecker product.

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R^n denotes the n-dimensional Euclidean space. $\boldsymbol{R}^{m\times n}$ denotes a set of $m\times n$ real matrices. $\boldsymbol{X}^{\mathrm{T}}$ denotes the transpose of matrix $\boldsymbol{X}$. $\boldsymbol{X}>0$ ($\boldsymbol{X}<0$), where $\boldsymbol{X}\in\boldsymbol{R}^{n\times n}$, means that $\boldsymbol{X}$ is real positive definite matrix (negative definite matrix). $\boldsymbol{I}_n$ represents the n-dimensional identity matrix. Matrix dimensions are assumed to be compatible for algebraic operations, unless otherwise stated.

In this paper, we consider the following delayed complex dynamical network consisting of N coupled nodes:

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) &= -\boldsymbol{C}\boldsymbol{x}_i(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}_i(t))+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{x}_i(t-\tau(t)))\\ &\quad+\boldsymbol{D}\int_{t-\tau(t)}^{t}\boldsymbol{f}(\boldsymbol{x}_i(s))\mathrm{d}s+\boldsymbol{u}(t)+\sum_{j=1}^{N}G_{ij}^{(1)}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\\ &\quad+\sum_{j=1}^{N}G_{ij}^{(2)}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau(t))\\ &\quad+\sum_{j=1}^{N}G_{ij}^{(3)}\boldsymbol{\Gamma}_3\int_{t-\tau(t)}^{t}\boldsymbol{x}_j(s)\mathrm{d}s,\quad i=1,2,\ldots,N. \quad (1)\end{aligned}$$

Each node is an n-dimensional subsystem. Here, $\boldsymbol{x}_i(t)=[\boldsymbol{x}_{i1}(t),\boldsymbol{x}_{i2}(t),\ldots,\boldsymbol{x}_{in}(t)]^{\mathrm{T}}\in\mathrm{R}^n$ is the state vector, $\boldsymbol{C}=\mathrm{diag}[c_1,c_2,\ldots,c_n]$ is a diagonal matrix with positive diagonal entries $c_i>0$, $\boldsymbol{A}=(a_{ij})_{n\times n}$ is the connection weight matrix, $\boldsymbol{B}=(b_{ij})_{n\times n}$ and $\boldsymbol{D}=(d_{ij})_{n\times n}$ are the delayed connection weight matrices, $\boldsymbol{u}(t)=[u_1(t),u_2(t),\ldots,u_n(t)]^{\mathrm{T}}\in\mathrm{R}^n$ is an external input, $\boldsymbol{f}(\boldsymbol{x}_i(t))=[f_1(\boldsymbol{x}_{i1}(t)),f_2(\boldsymbol{x}_{i2}(t)),\ldots,f_n(\boldsymbol{x}_{in}(t))]^{\mathrm{T}}\in\mathrm{R}^n$ denotes the output of node i, and $\tau(t)$ is the time-varying delay (a bounded differential function of time t satisfies $0\le\tau(t)\le\tau$ and $\dot{\tau}(t)\le h<1$, where τ and h are positive real constants). The parameters $\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_2$, and $\boldsymbol{\Gamma}_3\in\mathrm{R}^{n\times n}$ denote the inner coupling matrices. The parameters $\boldsymbol{G}^{(1)}=\left(G_{ij}^{(1)}\right)_{N\times N}$, $\boldsymbol{G}^{(2)}=\left(G_{ij}^{(2)}\right)_{N\times N}$, and $\boldsymbol{G}^{(3)}=\left(G_{ij}^{(3)}\right)_{N\times N}$ are the coupling configuration matrices representing coupling strength and the topological structures of the network, which may not be identical but satisfy the following conditions:

$$G_{ij}^{(r)}\ge 0,\quad i\ne j,\quad G_{ii}^{(r)}=-\sum_{j=1,j\ne i}^{N}G_{ij}^{(r)},\quad r=1,2,3.$$

Denote $\boldsymbol{x}_i(s)=\boldsymbol{\phi}_i(t)\in\mathscr{C}([-\tau,0],\mathrm{R}^n)$ $(i=1,2,\ldots,N)$ as the initial conditions with system (1), where $\tau=\sup_{t\in\mathrm{R}}\tau(t)$, and $\mathscr{C}([-\tau,0],\mathrm{R}^n)$ is the set of continuous functions from $[-\tau,0]$ to R^n. For simplicity, the following fundamental assumptions are made.

Assumption 1 $f_i(\boldsymbol{x}_i)$ $(i=1,2,\ldots,n)$ are monotonically nondecreasing in $\mathbf{R}$.

Assumption 2 $f_i(\boldsymbol{x}_i)$ $(i=1,2,\ldots,n)$ are Lipschitz continuous. There exist constants $L_i>0$ such that $|f_i(x)-f_i(y)|\le L_i|x-y|$, $\forall\ x,y\in\mathbf{R}$. For convenience, we denote $\boldsymbol{L}=\mathrm{diag}(L_1,L_2,\ldots,L_n)$.

Assumption 3 The coupling configuration matrices $\boldsymbol{G}^{(r)}$ $(r=1,2,3)$ satisfy

$$\boldsymbol{G}^{(r)}=\begin{pmatrix}\boldsymbol{Z}_{11}^{(r)} & \boldsymbol{Z}_{12}^{(r)} & \ldots & \boldsymbol{Z}_{1k}^{(r)}\\ \boldsymbol{Z}_{21}^{(r)} & \boldsymbol{Z}_{22}^{(r)} & \ldots & \boldsymbol{Z}_{23}^{(r)}\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{Z}_{k1}^{(r)} & \boldsymbol{Z}_{k2}^{(r)} & \ldots & \boldsymbol{Z}_{kk}^{(r)}\end{pmatrix}, \quad (2)$$

where $\boldsymbol{Z}_{ii}^{(r)}\in\boldsymbol{R}^{m_i\times m_i}$, and $\boldsymbol{Z}_{ij}^{(r)}\in\boldsymbol{R}^{m_i\times m_j}$ $(i,j=1,2,\ldots,k)$. Assume that all rows in $\boldsymbol{Z}_{ij}^{(r)}$ are the same. $\boldsymbol{Z}_{ij}^{(r)}=[\boldsymbol{v}^{(r)},\boldsymbol{v}^{(r)},\ldots,\boldsymbol{v}^{(r)}]^{\mathrm{T}}$, and $\boldsymbol{v}^{(r)}=[\boldsymbol{v}_1^{(r)},\boldsymbol{v}_2^{(r)},\ldots,\boldsymbol{v}_{m_j}^{(r)}]^{\mathrm{T}}$ is a vector.

Definition 1 Let $\tau=\sup_{t\in\mathrm{R}}\tau(t)$. The set $\boldsymbol{S}=\{\boldsymbol{x}_1(s),\boldsymbol{x}_2(s),\ldots,\boldsymbol{x}_N(s)\ :\ \boldsymbol{x}_i(s)\in\mathscr{C}([-\tau,0],\mathrm{R}^n),\ \boldsymbol{x}_i(s)=\boldsymbol{x}_j(s)\ (i,j=1,2,\ldots,N)\}$, which is called the synchronization manifold.

Definition 2[11] Synchronization manifold S is said to be globally exponentially synchronized, if there exist $\varepsilon>0$, $T>t_0$, and $M>0$ such that $\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|\le M\mathrm{e}^{-\varepsilon t}$, where $\phi_i(t)\in\mathscr{C}([-\tau,0],\mathbf{R}^n)$ $(t>T,\ i,j=1,2,\ldots,N)$.

Definition 3[21] The set $\boldsymbol{S}'=\{\boldsymbol{x}_1(s),\boldsymbol{x}_2(s),\ldots,\boldsymbol{x}_N(s)\ :\ \boldsymbol{x}_i(s)\in\mathscr{C}([-\tau,0],\ \mathrm{R}^n),\ \boldsymbol{x}_1(s)=\boldsymbol{x}_2(s)=\cdots=\boldsymbol{x}_{m_1}(s),\ \boldsymbol{x}_{m_1+1}(s)=\boldsymbol{x}_{m_1+2}(s)=\cdots=\boldsymbol{x}_{m_1+m_2}(s),\ \ldots,\boldsymbol{x}_{m_1+m_2+\cdots+m_{k-1}+1}(s)=\boldsymbol{x}_{m_1+m_2+\cdots+m_{k-1}+2}(s)=\cdots=\boldsymbol{x}_{m_1+m_2+\cdots m_{k-1}+m_k}(s),\ m_1+m_2+\cdots+m_{k-1}+m_k=N\}$ is called the cluster synchronization manifold.

Definition 4 Synchronization manifold $\boldsymbol{S}'$ is said to be cluster exponential synchronization, if the $\boldsymbol{N}$ nodes split into several clusters, such as $\{(1,2,\ldots,m_1),(m_1+1,m_1+2,\ldots,m_1+m_2),\ldots,(m_1+m_2+\cdots+m_{k-1}+1,\ m_1+m_2+\cdots+m_{k-1}+2,\ldots,m_1+m_2+\cdots+m_{k-1}+m_k),m_1+m_2+\ldots+m_{k-1}+m_k=N\}$ such that the nodes in the same cluster exponentially synchronize with one another, and for the states $\boldsymbol{x}_i(t)$ and $\boldsymbol{x}_j(t)$ of arbitrary nodes i and j in the same cluster, if there exist $\varepsilon>0$, $T>t_0$, and $M>0$ such that $\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|\le M\mathrm{e}^{-\varepsilon t}$, where $\phi_i(t)\in\mathscr{C}([-\tau,0],\mathbf{R}^n)$ $(t>T)$.

Definition 5[5] Let $\hat{R}$ denote a ring and define $T(\hat{R},K)$ as the set of matrices with entries $\hat{R}$ such that the sum of the entries in each row is equal to K for some $K\in\hat{R}$.

Lemma 1[5] Let $\boldsymbol{G}$ be an $N\times N$ matrix in the set $\boldsymbol{T}(\hat{\boldsymbol{R}},\boldsymbol{K})$, then the $(N-1)\times(N-1)$ matrix $\boldsymbol{H}$ will satisfy $\boldsymbol{MG}=\boldsymbol{HM}$, where $\boldsymbol{H}=\boldsymbol{MGJ}$,

$$\boldsymbol{M}=\begin{pmatrix}1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1\end{pmatrix}_{(N-1)\times N},$$

$$\boldsymbol{J}=\begin{pmatrix}1&1&1&\cdots&1\\0&1&1&\cdots&1\\&&\ddots&&1\\&&\cdots&1&1\\0&0&\cdots&0&1\\0&0&0&\cdots&0\end{pmatrix}_{N\times(N-1)},$$

where 1 is the multiplicative identity of $\hat{\boldsymbol{R}}$. Moreover, matrix $\boldsymbol{H}$ can be rewritten explicitly as follows: $H_{ij}=\sum_{k=1}^{j}(G_{(ik)}-G_{(i+1,k)})$ for $i,j\in\{1,2,\ldots,N-1\}$.

Lemma 2[20] The $(N-k)\times(N-k)$ matrix $\tilde{\boldsymbol{H}}_r$ satisfies $\tilde{\boldsymbol{M}}\boldsymbol{G}^{(r)}=\tilde{\boldsymbol{H}}_r\tilde{\boldsymbol{M}}$, where

$$\tilde{\boldsymbol{Z}}_r=\begin{pmatrix}\boldsymbol{Z}_{11}^{(r)}&&&\\&\boldsymbol{Z}_{22}^{(r)}&&\\&&\ddots&\\&&&\boldsymbol{Z}_{kk}^{(r)}\end{pmatrix}_{N\times N},$$

$$\tilde{\boldsymbol{M}}=\begin{pmatrix}\boldsymbol{M}_1&&&\\&\boldsymbol{M}_2&&\\&&\ddots&\\&&&\boldsymbol{M}_k\end{pmatrix}_{(N-k)\times N},$$

$$\tilde{\boldsymbol{J}}=\begin{pmatrix}\boldsymbol{J}_1&&&\\&\boldsymbol{J}_2&&\\&&\ddots&\\&&&\boldsymbol{J}_k\end{pmatrix}_{N\times(N-k)},$$

$\tilde{\boldsymbol{H}}_r=\tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_r\tilde{\boldsymbol{J}}$, $\boldsymbol{Z}_{ii}^{(r)}\in\boldsymbol{R}^{m_i\times m_i}$, $\boldsymbol{M}_i\in\boldsymbol{R}^{(m_i-1)\times m_i}$, and $\boldsymbol{J}_i\in\boldsymbol{R}^{m_i\times(m_i-1)}$ $(r=1,2,3)$.

Lemma 3[26] The Kronecker product has the following properties:

(1) $(\alpha\boldsymbol{A})\otimes\boldsymbol{B}=\boldsymbol{A}\otimes(\alpha\boldsymbol{B})$,

(2) $(\boldsymbol{A}+\boldsymbol{B})\otimes\boldsymbol{C}=\boldsymbol{A}\otimes\boldsymbol{C}+\boldsymbol{B}\otimes\boldsymbol{C}$,

(3) $(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D})=(\boldsymbol{AC})\otimes(\boldsymbol{BD})$.

Lemma 4[27] Assume that the vector function $\omega:[0,r]\to\mathbf{R}^m$ is well defined for the following integration. For any symmetric matrix $\boldsymbol{W}\in\boldsymbol{R}^{m\times m}$ and scalar $r>0$, one has

$$r\int_0^r\omega(s)^{\mathrm{T}}\boldsymbol{W}\omega(s)\,\mathrm{d}s\geq\left(\int_0^r\omega(s)\,\mathrm{d}s\right)^{\mathrm{T}}\boldsymbol{W}\left(\int_0^r\omega(s)\,\mathrm{d}s\right).$$

Let $\boldsymbol{x}=[\boldsymbol{x}_1^{\mathrm{T}},\boldsymbol{x}_2^{\mathrm{T}},\ldots,\boldsymbol{x}_N^{\mathrm{T}}]^{\mathrm{T}}$, where $\boldsymbol{x}_i\in\mathbf{R}^n$ $(i=1,2,\ldots,N)$. Denote $d(x)=\|\boldsymbol{Mx}\|^2=(\boldsymbol{Mx})^{\mathrm{T}}\boldsymbol{Mx}$, and $d(x)$ is a nonnegative distance function, where $\boldsymbol{M}=\tilde{\boldsymbol{M}}\otimes\boldsymbol{I}_n$, and $\tilde{\boldsymbol{M}}$ is defined in Lemma 2.

3. Main results

In this section, several novel criteria are proposed for the cluster exponential synchronization of the system (1) based on the Lyapunov–Krasovskii functional and the linear matrix inequality. To facilitate further development, let $\bar{\boldsymbol{C}}=\boldsymbol{I}_N\otimes\boldsymbol{C}$, $\bar{\boldsymbol{C}}'=\boldsymbol{I}_{N-k}\otimes\boldsymbol{C}$, $\bar{\boldsymbol{A}}=\boldsymbol{I}_N\otimes\boldsymbol{A}$, $\bar{\boldsymbol{A}}'=\boldsymbol{I}_{N-k}\otimes\boldsymbol{A}$, $\bar{\boldsymbol{B}}=\boldsymbol{I}_N\otimes\boldsymbol{B}$, $\bar{\boldsymbol{B}}'=\boldsymbol{I}_{N-k}\otimes\boldsymbol{B}$, $\bar{\boldsymbol{D}}=\boldsymbol{I}_N\otimes\boldsymbol{D}$, $\bar{\boldsymbol{D}}'=\boldsymbol{I}_{N-k}\otimes\boldsymbol{D}$, $\bar{\boldsymbol{\Gamma}}_1=\boldsymbol{G}^{(1)}\otimes\boldsymbol{\Gamma}_1$, $\bar{\boldsymbol{\Gamma}}_2=\boldsymbol{G}^{(2)}\otimes\boldsymbol{\Gamma}_2$, $\bar{\boldsymbol{\Gamma}}_3=\boldsymbol{G}^{(3)}\otimes\boldsymbol{\Gamma}_3$, $\boldsymbol{x}(t)=[\boldsymbol{x}_1^{\mathrm{T}}(t),\boldsymbol{x}_2^{\mathrm{T}}(t),\ldots,\boldsymbol{x}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\boldsymbol{f}(\boldsymbol{x}(t))=[\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_1(t)),\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_2(t)),\ldots,\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_N(t))]^{\mathrm{T}}$, and $\boldsymbol{U}(t)=[\boldsymbol{u}^{\mathrm{T}}(t),\boldsymbol{u}^{\mathrm{T}}(t),\ldots,\boldsymbol{u}^{\mathrm{T}}(t)]^{\mathrm{T}}$. Then, the complex network (1) can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{x}}(t)=&-\bar{\boldsymbol{C}}\boldsymbol{x}(t)+\bar{\boldsymbol{A}}\boldsymbol{f}(\boldsymbol{x}(t))+\bar{\boldsymbol{B}}\boldsymbol{f}(\boldsymbol{x}(t-\tau(t)))\\&+\bar{\boldsymbol{D}}\int_{t-\tau(t)}^{t}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s+\boldsymbol{U}(t)+\bar{\boldsymbol{\Gamma}}_1\boldsymbol{x}(t)\\&+\bar{\boldsymbol{\Gamma}}_2\boldsymbol{x}(t-\tau(t))+\bar{\boldsymbol{\Gamma}}_3\int_{t-\tau(t)}^{t}\boldsymbol{x}(s)\,\mathrm{d}s.\end{aligned}\tag{3}$$

Theorem 1 Under Assumptions 2 and 3, the cluster synchronization manifold $\boldsymbol{S}'$ of the dynamic system (3) is cluster exponential synchronization if there exist positive definite matrices $\bar{\boldsymbol{Q}}_i\in\boldsymbol{R}^{(N-k)n\times(N-k)n}$ $(i=1,2,3,4,5)$, and a positive definite diagonal matrix $\bar{\boldsymbol{S}}=\mathrm{diag}[s_1,s_2,\ldots,s_{(N-k)n}]$, $\bar{\boldsymbol{S}}\in\boldsymbol{R}^{(N-k)n\times(N-k)n}$, such that

$$\boldsymbol{\Phi}=\begin{pmatrix}\boldsymbol{\Phi}_{11}&\bar{\boldsymbol{Q}}_1\bar{\boldsymbol{H}}_2&\bar{\boldsymbol{Q}}_1\bar{\boldsymbol{A}}'&\bar{\boldsymbol{Q}}_1\bar{\boldsymbol{B}}'&\bar{\boldsymbol{Q}}_1\bar{\boldsymbol{H}}_3&\bar{\boldsymbol{Q}}_1\bar{\boldsymbol{D}}'\\\bar{\boldsymbol{H}}_2^{\mathrm{T}}\bar{\boldsymbol{Q}}_1&-(1-h)\bar{\boldsymbol{Q}}_2&0&0&0&0\\\bar{\boldsymbol{A}}'^{\mathrm{T}}\bar{\boldsymbol{Q}}_1&0&\bar{\boldsymbol{Q}}_3-\bar{\boldsymbol{S}}+\tau\bar{\boldsymbol{Q}}_5&0&0&0\\\bar{\boldsymbol{B}}'^{\mathrm{T}}\bar{\boldsymbol{Q}}_1&0&0&-(1-h)\bar{\boldsymbol{Q}}_3&0&0\\\bar{\boldsymbol{H}}_3^{\mathrm{T}}\bar{\boldsymbol{Q}}_1&0&0&0&-\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_4&0\\\bar{\boldsymbol{D}}'^{\mathrm{T}}\bar{\boldsymbol{Q}}_1&0&0&0&0&-\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_5\end{pmatrix}<0,$$

where $\Phi_{11} = -\bar{Q}_1\bar{C}' - \bar{C}'^{\mathrm{T}}\bar{Q}_1 + \bar{Q}_1\bar{H}_1 + \bar{H}_1^{\mathrm{T}}\bar{Q}_1 + \bar{Q}_2 + \tau\bar{Q}_4 + \bar{L}\bar{S}\bar{L}$, $\beta \in (0,1)$, $L = \mathrm{diag}[L_1, L_2, \ldots, L_n] \in R^{n\times n}$, $\bar{L} = I_{N-k} \otimes L$, $\tilde{H}_1 = \tilde{M}\tilde{Z}_1\tilde{J}$, $\tilde{H}_2 = \tilde{M}\tilde{Z}_2\tilde{J}$, $\tilde{H}_3 = \tilde{M}\tilde{Z}_3\tilde{J}$, $\bar{H}_1 = \tilde{H}_1 \otimes \Gamma_1$, $\bar{H}_2 = \tilde{H}_2 \otimes \Gamma_2$, $\bar{H}_3 = \tilde{H}_3 \otimes \Gamma_3$, $\tilde{M}$, $\tilde{J}$, $\tilde{Z}_1$, $\tilde{Z}_2$, and $\tilde{Z}_3$, have the same structures as Lemma 2.

Proof Construct a new Lyapunov–Krasovskii functional as

$$\begin{aligned} V(x(t)) = {} & \mathrm{e}^{\varepsilon t}[V_1(x(t)) + V_2(x(t)) + V_3(x(t)) \\ & + V_4(x(t)) + V_5(x(t))], \end{aligned} \tag{4}$$

where

$$V_1(x(t)) = x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1 Mx(t), \tag{5}$$

$$V_2(x(t)) = \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)M^{\mathrm{T}}\bar{Q}_2 Mx(s)\mathrm{d}s, \tag{6}$$

$$V_3(x(t)) = \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))M^{\mathrm{T}}\bar{Q}_3 Mf(x(s))\mathrm{d}s, \tag{7}$$

$$V_4(x(t)) = \int_{t-\tau}^{t}\int_{\theta}^{t} (Mx(s))^{\mathrm{T}}\bar{Q}_4(Mx(s))\mathrm{d}s\mathrm{d}\theta, \tag{8}$$

$$V_5(x(t)) = \int_{t-\tau}^{t}\int_{\theta}^{t} (Mf(x(s)))^{\mathrm{T}}\bar{Q}_5(Mf(x(s)))\mathrm{d}s\mathrm{d}\theta. \tag{9}$$

Then, the time derivative of $V_1(x(t))$ along the complex network (3) can be directly computed as follows:

$$\begin{aligned} \dot{V}_1(x(t)) = {} & 2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1 M\dot{x}(t) \\ = {} & 2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1 M[-\bar{C}x(t) + \bar{A}f(x(t)) \\ & + \bar{B}f(x(t-\tau(t))) + \bar{D}\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s \\ & + U(t) + \bar{\Gamma}_1 x(t) + \bar{\Gamma}_2 x(t-\tau(t)) \\ & + \bar{\Gamma}_3\int_{t-\tau(t)}^{t} x(s)\mathrm{d}s]. \end{aligned} \tag{10}$$

From the structure of M, we have

$$\begin{aligned} M\bar{C} = {} & \bar{C}'M, M\bar{A} = \bar{A}'M, \\ M\bar{B} = {} & \bar{B}'M, MU(t) = 0, \\ M\bar{\Gamma}_r = {} & (\tilde{M}\otimes I_n)(G^{(r)}\otimes\Gamma_r) \\ = {} & (\tilde{M}G^{(r)})\Gamma_r \\ = {} & (\tilde{H}_r\tilde{M})\otimes\Gamma_r \\ = {} & (\tilde{H}_r\otimes\Gamma_r)(\tilde{M}\otimes I_n) \\ = {} & \bar{H}_r M \quad (r = 1,2,3). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \dot{V}_1(x(t)) = {} & -2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1\bar{C}'Mx(t) \\ & + 2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1\bar{A}'Mf(x(t)) \\ & + 2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1\bar{B}'Mf(x(t-\tau(t))) \\ & + 2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1\bar{D}'M\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s \\ & + 2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1\bar{H}_1 Mx(t) \\ & + 2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1\bar{H}_2 Mx(t-\tau(t)) \\ & + 2x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_1\bar{H}_3 M\int_{t-\tau(t)}^{t} x(s)\mathrm{d}s, \end{aligned} \tag{11}$$

$$\begin{aligned} \dot{V}_2(x(t)) = {} & x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_2 Mx(t) \\ & - (1-\dot{\tau}(t))x^{\mathrm{T}}(t-\tau(t))M^{\mathrm{T}}\bar{Q}_2 Mx(t-\tau(t)) \\ \le {} & x^{\mathrm{T}}(t)M^{\mathrm{T}}\bar{Q}_2 Mx(t) \\ & - (1-h)x^{\mathrm{T}}(t-\tau(t))M^{\mathrm{T}}\bar{Q}_2 \\ & \times Mx(t-\tau(t)), \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{V}_3(x(t)) = {} & f^{\mathrm{T}}(x(t))M^{\mathrm{T}}\bar{Q}_3 Mf(x(t)) \\ & - (1-\dot{\tau}(t))f^{\mathrm{T}}(x(t-\tau(t)))M^{\mathrm{T}}\bar{Q}_3 \\ & \times Mf(x(t-\tau(t))) \\ \le {} & f^{\mathrm{T}}(x(t))M^{\mathrm{T}}\bar{Q}_3 Mf(x(t)) \\ & - (1-h)f^{\mathrm{T}}(x(t-\tau(t)))M^{\mathrm{T}}\bar{Q}_3 \\ & \times Mf(x(t-\tau(t))), \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{V}_4(x(t)) = {} & \tau(Mx(t))^{\mathrm{T}}\bar{Q}_4 Mx(t) \\ & - \int_{t-\tau}^{t} (Mx(s))^{\mathrm{T}}\bar{Q}_4 Mx(s)\mathrm{d}s \\ \le {} & \tau(Mx(t))^{\mathrm{T}}\bar{Q}_4 Mx(t) \\ & - \beta\int_{t-\tau}^{t} (Mx(s))^{\mathrm{T}}\bar{Q}_4 Mx(s)\mathrm{d}s \\ & - (1-\beta)\int_{t-\tau(t)}^{t} (Mx(s))^{\mathrm{T}}\bar{Q}_4 Mx(s)\mathrm{d}s. \end{aligned}$$

According to Lemma 4, we have

$$\begin{aligned} \dot{V}_4(x(t)) \le {} & \tau(Mx(t))^{\mathrm{T}}\bar{Q}_4 Mx(t) \\ & - \beta\int_{t-\tau}^{t} (Mx(s))^{\mathrm{T}}\bar{Q}_4 Mx(s)\mathrm{d}s \\ & - \frac{1-\beta}{\tau}\left(\int_{t-\tau(t)}^{t} Mx(s)\mathrm{d}s\right)^{\mathrm{T}} \\ & \times \bar{Q}_4\left(\int_{t-\tau(t)}^{t} Mx(s)\mathrm{d}s\right). \end{aligned} \tag{14}$$

Similarly, we obtain

$$\dot{V}_5(x(t)) \le \tau(Mf(x(t)))^{\mathrm{T}}\bar{Q}_5 Mf(x(t))$$

$$
\begin{aligned}
&- \beta \int_{t-\tau}^{t} (\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s)))^{\mathrm{T}} \bar{\boldsymbol{Q}}_5 \boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s \\
&- \frac{1-\beta}{\tau} \left(\int_{t-\tau(t)}^{t} \boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s \right)^{\mathrm{T}} \\
&\times \bar{\boldsymbol{Q}}_5 \left(\int_{t-\tau(t)}^{t} \boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s \right). \qquad (15)
\end{aligned}
$$

From Assumption 2, it follows that

$$
\begin{aligned}
&[\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t))]^{\mathrm{T}} \bar{\boldsymbol{S}} [\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t))] \\
&= \sum_{i=1}^{m_1-1} [\boldsymbol{f}(\boldsymbol{x}_i(t)) - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))]^{\mathrm{T}} \boldsymbol{S}_i [\boldsymbol{f}(\boldsymbol{x}_i(t)) - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))] \\
&\quad + \sum_{i=m_1+1}^{m_1+m_2-1} [\boldsymbol{f}(\boldsymbol{x}_i(t)) \\
&\quad - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))]^{\mathrm{T}} \boldsymbol{S}_{i-1} [\boldsymbol{f}(\boldsymbol{x}_i(t)) - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))] + \cdots \\
&\quad + \sum_{i=m_1+m_2+\ldots+m_{k-1}+1}^{N-1} [\boldsymbol{f}(\boldsymbol{x}_i(t)) - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))]^{\mathrm{T}} \\
&\quad \times \boldsymbol{S}_{i-k+1} [\boldsymbol{f}(\boldsymbol{x}_i(t)) - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))] \\
&\le \sum_{i=1}^{m_1-1} [\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)]^{\mathrm{T}} \boldsymbol{L}\boldsymbol{S}_i \boldsymbol{L} [\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)] \\
&\quad + \sum_{i=m_1+1}^{m_1+m_2-1} [\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)]^{\mathrm{T}} \\
&\quad \times \boldsymbol{L}\boldsymbol{S}_{i-1} \boldsymbol{L} [\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)] + \ldots \\
&\quad + \sum_{i=m_1+m_2+\ldots+m_{k-1}+1}^{N-1} [\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)]^{\mathrm{T}} \\
&\quad \times \boldsymbol{L}\boldsymbol{S}_{i-k+1} \boldsymbol{L} [\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)] \\
&= [\boldsymbol{M}\boldsymbol{x}(t)]^{\mathrm{T}} \bar{\boldsymbol{L}} \bar{\boldsymbol{S}} \bar{\boldsymbol{L}} [\boldsymbol{M}\boldsymbol{x}(t)], \qquad (16)
\end{aligned}
$$

where $S_i = \mathrm{diag}(s_{(i-1)n+1}, \ldots, s_{in})$. Let

$$
\begin{aligned}
\boldsymbol{\xi} = \Bigg[&(\boldsymbol{M}\boldsymbol{x}(t))^{\mathrm{T}}, (\boldsymbol{M}\boldsymbol{x}(t-\tau(t)))^{\mathrm{T}}, (\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t)))^{\mathrm{T}}, \\
&(\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t-\tau(t))))^{\mathrm{T}}, \left(\int_{t-\tau(t)}^{t} \boldsymbol{M}\boldsymbol{x}(s)\,\mathrm{d}s \right)^{\mathrm{T}}, \\
&\left(\int_{t-\tau(t)}^{t} \boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s \right)^{\mathrm{T}} \Bigg]^{\mathrm{T}}.
\end{aligned}
$$

Then, combining Eqs. (11)–(16), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{5} \dot{\boldsymbol{V}}_i(\boldsymbol{x}(t)) \\
&\le \xi^{\mathrm{T}} \boldsymbol{\Phi} \boldsymbol{\xi} - \beta \int_{t-\tau}^{t} (\boldsymbol{M}\boldsymbol{x}(s))^{\mathrm{T}} \bar{\boldsymbol{Q}}_4 \boldsymbol{M}\boldsymbol{x}(s)\,\mathrm{d}s \\
&\quad - \beta \int_{t-\tau}^{t} (\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s)))^{\mathrm{T}} \bar{\boldsymbol{Q}}_5 \boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s. \qquad (17)
\end{aligned}
$$

From the definitions of $\boldsymbol{V}_i(\boldsymbol{x}(t))$ $(i = 1, 2, \ldots, 5)$, it is not difficult to have the following inequalities:

$$
\begin{aligned}
\boldsymbol{V}_1(\boldsymbol{x}(t)) &\le \lambda_{\max}(\bar{\boldsymbol{Q}}_1) (\boldsymbol{M}\boldsymbol{x}(t))^{\mathrm{T}} \boldsymbol{M}\boldsymbol{x}(t) \\
&= \lambda_{\max}(\bar{\boldsymbol{Q}}_1) \|\boldsymbol{M}\boldsymbol{x}(t)\|^2, \qquad (18) \\
\boldsymbol{V}_2(\boldsymbol{x}(t)) &\le \int_{t-\tau}^{t} \boldsymbol{x}^{\mathrm{T}}(s) \boldsymbol{M}^{\mathrm{T}} \bar{\boldsymbol{Q}}_2 \boldsymbol{M}\boldsymbol{x}(s)\,\mathrm{d}s, \qquad (19) \\
\boldsymbol{V}_3(\boldsymbol{x}(t)) &\le \int_{t-\tau}^{t} \boldsymbol{f}^{\mathrm{T}}(x(s)) \boldsymbol{M}^{\mathrm{T}} \bar{\boldsymbol{Q}}_3 \boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s, \qquad (20) \\
\boldsymbol{V}_4(\boldsymbol{x}(t)) &\le \tau \int_{t-\tau}^{t} (\boldsymbol{M}\boldsymbol{x}(s))^{\mathrm{T}} \bar{\boldsymbol{Q}}_4 (\boldsymbol{M}\boldsymbol{x}(s))\,\mathrm{d}s, \qquad (21) \\
\boldsymbol{V}_5(\boldsymbol{x}(t)) &\le \tau \int_{t-\tau}^{t} (\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s)))^{\mathrm{T}} \bar{\boldsymbol{Q}}_5 (\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s)))\,\mathrm{d}s. \qquad (22)
\end{aligned}
$$

From inequalities (17)–(22), noting that $\|\boldsymbol{M}\boldsymbol{x}(t)\|^2 \le \|\boldsymbol{\xi}\|^2$, we obtain

$$
\begin{aligned}
&\dot{\boldsymbol{V}}(\boldsymbol{x}(t)) \\
&= \mathrm{e}^{\varepsilon t} \left(\varepsilon \sum_{i=1}^{5} \boldsymbol{V}_i(\boldsymbol{x}(t)) + \sum_{i=1}^{5} \dot{\boldsymbol{V}}_i(\boldsymbol{x}(t)) \right) \\
&\le \mathrm{e}^{\varepsilon t} [-(\lambda_{\min}(-\boldsymbol{\Phi}) - \varepsilon \lambda_{\max}(\bar{\boldsymbol{Q}}_1)) \|\boldsymbol{M}\boldsymbol{x}(t)\|^2 \\
&\quad - \int_{t-\tau}^{t} \boldsymbol{x}^{\mathrm{T}}(s) \boldsymbol{M}^{\mathrm{T}} (\beta \bar{\boldsymbol{Q}}_4 - \varepsilon \bar{\boldsymbol{Q}}_2 - \varepsilon\tau \bar{\boldsymbol{Q}}_4) \boldsymbol{M}\boldsymbol{x}(s)\,\mathrm{d}s \\
&\quad - \int_{t-\tau}^{t} f^{\mathrm{T}}(\boldsymbol{x}(s)) \boldsymbol{M}^{\mathrm{T}} (\beta \bar{\boldsymbol{Q}}_5 - \varepsilon \bar{\boldsymbol{Q}}_3 - \varepsilon\tau \bar{\boldsymbol{Q}}_5) \\
&\quad \times \boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s]. \qquad (23)
\end{aligned}
$$

Taking

$$
\varepsilon = \min \left\{ \frac{\lambda_{\min}(-\boldsymbol{\Phi})}{\lambda_{\max}(\bar{\boldsymbol{Q}}_1)}, \frac{\beta \lambda_{\min}(\bar{\boldsymbol{Q}}_4)}{\lambda_{\max}(\bar{\boldsymbol{Q}}_2 + \tau \bar{\boldsymbol{Q}}_4)}, \frac{\beta \lambda_{\min}(\bar{\boldsymbol{Q}}_5)}{\lambda_{\max}(\bar{\boldsymbol{Q}}_3 + \tau \bar{\boldsymbol{Q}}_5)} \right\},
$$

we obtain $\dot{\boldsymbol{V}}(\boldsymbol{x}(t)) < 0$, which implies that $\boldsymbol{V}(\boldsymbol{x}(t)) < V(0)$, namely, $\boldsymbol{V}(\boldsymbol{x}(t))$ is a bounded function. Hence, $\mathrm{e}^{\varepsilon t} \left(\boldsymbol{x}^{\mathrm{T}}(t) \boldsymbol{M}^{\mathrm{T}} \bar{\boldsymbol{Q}}_1 \boldsymbol{M}\boldsymbol{x}(t) \right)$ is also bounded, yielding that $\lambda_{\min}(\bar{\boldsymbol{Q}}_1) \|\boldsymbol{M}\boldsymbol{x}(t)\|^2 \le x^{\mathrm{T}}(t) \boldsymbol{M}^{\mathrm{T}} \bar{\boldsymbol{Q}}_1 \boldsymbol{M}\boldsymbol{x}(t) = O(\mathrm{e}^{-\varepsilon t})$. Thus, we have

$$
\lambda_{\min}(\bar{\boldsymbol{Q}}_1) \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|^2 \le \lambda_{\min}(\bar{\boldsymbol{Q}}_1) \|\boldsymbol{M}\boldsymbol{x}(t)\|^2 \le O(\mathrm{e}^{-\varepsilon t}),
$$

where i and j are in the same cluster. According to definition (4), the complex network (3) is cluster exponential synchronization. The proof is completed.

Corollary 1 Under Assumptions 1–3, the cluster synchronization manifold $\boldsymbol{S}'$ of the dynamic system (3) is cluster exponential synchronization if there exist positive definite matrices $\bar{\boldsymbol{Q}}_i \in \boldsymbol{R}^{(N-k)n \times (N-k)n}$ $(i = 1, 2, 3, 4, 5)$, and a positive definite diagonal matrix $\bar{\boldsymbol{S}} = \mathrm{diag}[s_1, s_2, \ldots, s_{(N-k)n}]$, $\bar{\boldsymbol{S}} \in \boldsymbol{R}^{(N-k)n \times (N-k)n}$, such that

$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Phi}_{11} & \bar{\boldsymbol{Q}}_1\bar{\boldsymbol{H}}_2 & \bar{\boldsymbol{Q}}_1\bar{\boldsymbol{A}}' + \bar{\boldsymbol{L}}\bar{\boldsymbol{S}} & \bar{\boldsymbol{Q}}_1\bar{\boldsymbol{B}}' & \bar{\boldsymbol{Q}}_1\bar{\boldsymbol{H}}_3 & \bar{\boldsymbol{Q}}_1\bar{\boldsymbol{D}}' \\ \bar{\boldsymbol{H}}_2^{\mathrm{T}}\bar{\boldsymbol{Q}}_1 & -(1-h)\bar{\boldsymbol{Q}}_2 & 0 & 0 & 0 & 0 \\ \bar{\boldsymbol{A}}'^{\mathrm{T}}\bar{\boldsymbol{Q}}_1 + \bar{\boldsymbol{L}}\bar{\boldsymbol{S}} & 0 & \bar{\boldsymbol{Q}}_3 - 2\bar{\boldsymbol{S}} + \tau\bar{\boldsymbol{Q}}_5 & 0 & 0 & 0 \\ \bar{\boldsymbol{B}}'^{\mathrm{T}}\bar{\boldsymbol{Q}}_1 & 0 & 0 & -(1-h)\bar{\boldsymbol{Q}}_3 & 0 & 0 \\ \bar{\boldsymbol{H}}_3^{\mathrm{T}}\bar{\boldsymbol{Q}}_1 & 0 & 0 & 0 & -\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_4 & 0 \\ \bar{\boldsymbol{D}}'^{\mathrm{T}}\bar{\boldsymbol{Q}}_1 & 0 & 0 & 0 & 0 & -\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_5 \end{pmatrix} < 0,$$

where $\boldsymbol{\Phi}_{11} = -\bar{\boldsymbol{Q}}_1\bar{\boldsymbol{C}}' - \bar{\boldsymbol{C}}'^{\mathrm{T}}\bar{\boldsymbol{Q}}_1 + \bar{\boldsymbol{Q}}_1\bar{\boldsymbol{H}}_1 + \bar{\boldsymbol{H}}_1^{\mathrm{T}}\bar{\boldsymbol{Q}}_1 + \bar{\boldsymbol{Q}}_2 + \tau\bar{\boldsymbol{Q}}_4$, $\beta \in (0,1)$, $L = \mathrm{diag}[\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_n] \in \boldsymbol{R}^{n\times n}$, $\bar{\boldsymbol{L}} = \boldsymbol{I}_{N-k} \otimes \boldsymbol{L}$, $\tilde{\boldsymbol{H}}_1 = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_1\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{H}}_2 = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_2\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{H}}_3 = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_3\tilde{\boldsymbol{J}}$, $\bar{\boldsymbol{H}}_1 = \tilde{\boldsymbol{H}}_1 \otimes \boldsymbol{\Gamma}_1$, $\bar{\boldsymbol{H}}_2 = \tilde{\boldsymbol{H}}_2 \otimes \boldsymbol{\Gamma}_2$, $\bar{\boldsymbol{H}}_3 = \tilde{\boldsymbol{H}}_3 \otimes \boldsymbol{\Gamma}_3$, $\tilde{\boldsymbol{M}}$, $\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{Z}}_1$, $\tilde{\boldsymbol{Z}}_2$, and $\tilde{\boldsymbol{Z}}_3$, have the same structures as Lemma 2.

Proof As in the proof of Theorem 1, we use the same Lyapunov–Krasovskii functional (4) and calculate the time derivative of $\boldsymbol{V}(\boldsymbol{x}(t))$ along the system trajectories. In the proof we only need to substitute Eq. (16) by

$$\begin{aligned} & [\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t))]^{\mathrm{T}}\bar{\boldsymbol{S}}[\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t))] \\ \le & \sum_{i=1}^{m_1-1}[\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)]^{\mathrm{T}}\boldsymbol{L}\boldsymbol{S}_i[\boldsymbol{f}(\boldsymbol{x}_i(t)) - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))] \\ & + \sum_{i=m_1+1}^{m_1+m_2-1}[\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)]^{\mathrm{T}} \\ & \times \boldsymbol{L}\boldsymbol{S}_{i-1}[\boldsymbol{f}(\boldsymbol{x}_i(t)) - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))] + \cdots \\ & + \sum_{i=m_1+m_2+\cdots+m_{k-1}+1}^{N-1}[\boldsymbol{x}_i(t) - \boldsymbol{x}_{i+1}(t)]^{\mathrm{T}} \\ & \times \boldsymbol{L}\boldsymbol{S}_{i-k+1}[\boldsymbol{f}(\boldsymbol{x}_i(t)) - \boldsymbol{f}(\boldsymbol{x}_{i+1}(t))] \\ = & [\boldsymbol{M}\boldsymbol{x}(t)]^{\mathrm{T}}\bar{\boldsymbol{L}}\bar{\boldsymbol{S}}[\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t))]. \end{aligned} \tag{24}$$

Remark 1 In this paper, the cluster exponential synchronization of the system (1) is studied. It should be emphasized that the coupling configuration matrices are not necessarily assumed to be symmetric, which can be less restrictive than those conditions in Refs. [4], [7], [11], [14], [16]–[19], and [23]–[25].

Remark 2 In Ref. [20], the cluster synchronization in an array of hybrid coupled neural networks with constant time delay and $D = 0$ was investigated. Therefore, the main result in Ref. [20] is a special case of this paper.

By constructing a more general Lyapunov–Krasovskii functional and using four free matrices to express the cluster exponential synchronization criterion, we can obtain the following new theorem.

Theorem 2 Under Assumptions 2 and 3, the cluster synchronization manifold $\boldsymbol{S}'$ of the dynamic system (3) is cluster exponential synchronization if there exist positive definite matrices $\bar{\boldsymbol{Q}}_i \in \boldsymbol{R}^{(N-k)n\times(N-k)n}$ $(i = 1,2,\ldots,6)$, four matrices $\bar{\boldsymbol{N}}_i \in \boldsymbol{R}^{(N-k)n\times(N-k)n}$ $(i = 1,2,3,4)$, and a positive definite diagonal matrix $\bar{\boldsymbol{S}} = \mathrm{diag}[s_1, s_2, \ldots, s_{(N-k)n}]$, $\bar{\boldsymbol{S}} \in \boldsymbol{R}^{(N-k)n\times(N-k)n}$, such that

$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \bar{\boldsymbol{N}}_1 + \bar{\boldsymbol{N}}_3\bar{\boldsymbol{H}}_2 & \bar{\boldsymbol{N}}_3\bar{\boldsymbol{A}}' & \bar{\boldsymbol{N}}_3\bar{\boldsymbol{B}}' & \bar{\boldsymbol{N}}_3\bar{\boldsymbol{H}}_3 & \bar{\boldsymbol{N}}_1 & \bar{\boldsymbol{N}}_3\bar{\boldsymbol{D}}' \\ * & \tau\bar{\boldsymbol{Q}}_6 - \bar{\boldsymbol{N}}_4 - \bar{\boldsymbol{N}}_4^{\mathrm{T}} & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{H}}_2 + \bar{\boldsymbol{N}}_2 & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{A}}' & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{B}}' & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{H}}_3 & \bar{\boldsymbol{N}}_2 & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{D}}' \\ * & * & -(1-h)\bar{\boldsymbol{Q}}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \tau\bar{\boldsymbol{Q}}_5 + \bar{\boldsymbol{Q}}_3 - \bar{\boldsymbol{S}} & 0 & 0 & 0 & 0 \\ * & * & * & * & -(1-h)\bar{\boldsymbol{Q}}_3 & 0 & 0 & 0 \\ * & * & * & * & * & -\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_4 & 0 & 0 \\ * & * & * & * & * & * & -\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_6 & 0 \\ * & * & * & * & * & * & * & -\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_5 \end{pmatrix} < 0,$$

where $\boldsymbol{\Phi}_{11} = \bar{\boldsymbol{Q}}_2 + \tau\bar{\boldsymbol{Q}}_4 - \bar{\boldsymbol{N}}_1 - \bar{\boldsymbol{N}}_1^{\mathrm{T}} - \bar{\boldsymbol{N}}_3\bar{\boldsymbol{C}}' - (\bar{\boldsymbol{N}}_3\bar{\boldsymbol{C}}')^{\mathrm{T}} + \bar{\boldsymbol{N}}_3\bar{\boldsymbol{H}}_1 + (\bar{\boldsymbol{N}}_3\bar{\boldsymbol{H}}_1)^{\mathrm{T}} + \bar{\boldsymbol{L}}\bar{\boldsymbol{S}}\bar{\boldsymbol{L}}$, $\boldsymbol{\Phi}_{12} = \bar{\boldsymbol{Q}}_1 - \bar{\boldsymbol{N}}_2^{\mathrm{T}} - \bar{\boldsymbol{N}}_3 - (\bar{\boldsymbol{N}}_4\bar{\boldsymbol{C}}')^{\mathrm{T}} + (\bar{\boldsymbol{N}}_4\bar{\boldsymbol{H}}_1)^{\mathrm{T}}$, $\beta \in (0,1)$, $\boldsymbol{L} = \mathrm{diag}[L_1, L_2, \ldots, L_n] \in \boldsymbol{R}^{n\times n}$, $\bar{\boldsymbol{L}} = \boldsymbol{I}_{N-k} \otimes \boldsymbol{L}$, $\tilde{\boldsymbol{H}}_1 = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_1\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{H}}_2 = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_2\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{H}}_3 = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_3\tilde{\boldsymbol{J}}$, $\bar{\boldsymbol{H}}_1 = \tilde{\boldsymbol{H}}_1 \otimes \boldsymbol{\Gamma}_1$, $\bar{\boldsymbol{H}}_2 = \tilde{\boldsymbol{H}}_2 \otimes \boldsymbol{\Gamma}_2$, $\bar{\boldsymbol{H}}_3 = \tilde{\boldsymbol{H}}_3 \otimes \boldsymbol{\Gamma}_3$, $\tilde{\boldsymbol{M}}$, $\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{Z}}_1$, $\tilde{\boldsymbol{Z}}_2$, and $\tilde{\boldsymbol{Z}}_3$, have the same structures as Lemma 2.

Proof Consider the following Lyapunov–Krasovkii functional candidate for the system (3).

$$\begin{aligned}\boldsymbol{V}(\boldsymbol{x}(t)) &= \mathrm{e}^{\varepsilon t}[\boldsymbol{V}_1(\boldsymbol{x}(t))+\boldsymbol{V}_2(\boldsymbol{x}(t))+\boldsymbol{V}_3(\boldsymbol{x}(t))\\ &\quad+\boldsymbol{V}_4(\boldsymbol{x}(t))+\boldsymbol{V}_5(\boldsymbol{x}(t))+\boldsymbol{V}_6(\boldsymbol{x}(t))],\end{aligned}\tag{25}$$

where $\boldsymbol{V}(\boldsymbol{x}_i(t))$ $(i=1,2,\ldots,5)$ are defined in Eqs. (5)–(9), respectively, and

$$\boldsymbol{V}_6(\boldsymbol{x}(t))=\int_{t-\tau}^{t}\int_{\theta}^{t}(\boldsymbol{M}\dot{\boldsymbol{x}}(s))^{\mathrm{T}}\bar{\boldsymbol{Q}}_6(\boldsymbol{M}\dot{\boldsymbol{x}}(s))\,\mathrm{d}s\,\mathrm{d}\theta.\tag{26}$$

Calculating the time derivatives of $\boldsymbol{V}_1(\boldsymbol{x}(t))$ and $\boldsymbol{V}_6(\boldsymbol{x}(t))$, respectively, we have

$$\dot{\boldsymbol{V}}_1(\boldsymbol{x}(t)) = 2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{M}^{\mathrm{T}}\bar{\boldsymbol{Q}}_1\boldsymbol{M}\dot{\boldsymbol{x}}(t),\tag{27}$$

and

$$\begin{aligned}\dot{\boldsymbol{V}}_6(\boldsymbol{x}(t)) &\le \tau(\boldsymbol{M}\dot{\boldsymbol{x}}(t))^{\mathrm{T}}\bar{\boldsymbol{Q}}_6\boldsymbol{M}\dot{\boldsymbol{x}}(t)\\ &\quad-\beta\int_{t-\tau}^{t}(\boldsymbol{M}\dot{\boldsymbol{x}}(s))^{\mathrm{T}}\bar{\boldsymbol{Q}}_6\boldsymbol{M}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\\ &\quad-\frac{1-\beta}{\tau}\left(\int_{t-\tau(t)}^{t}\boldsymbol{M}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\right)^{\mathrm{T}}\\ &\quad\times\bar{\boldsymbol{Q}}_6\left(\int_{t-\tau(t)}^{t}\boldsymbol{M}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\right).\end{aligned}\tag{28}$$

From the Leibniz–Newton formula and complex network (3), for any matrix $\bar{\boldsymbol{N}}_i$ $(i=1,2,3,4)$, we obatin

$$\begin{aligned}&2[(\boldsymbol{M}\boldsymbol{x}(t))^{\mathrm{T}}\bar{\boldsymbol{N}}_1+(\boldsymbol{M}\dot{\boldsymbol{x}}(t))^{\mathrm{T}}\bar{\boldsymbol{N}}_2]\\ &\quad\times\left[\int_{t-\tau(t)}^{t}\boldsymbol{M}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s-(\boldsymbol{M}\boldsymbol{x}(t)-\boldsymbol{M}x(t-\tau(t)))\right]\\ &=0,\end{aligned}\tag{29}$$

and

$$\begin{aligned}&2[(\boldsymbol{M}\boldsymbol{x}(t))^{\mathrm{T}}\bar{\boldsymbol{N}}_3+(\boldsymbol{M}\dot{\boldsymbol{x}}(t))^{\mathrm{T}}\bar{\boldsymbol{N}}_4]\boldsymbol{M}\\ &\quad\times\Bigg[-\bar{\boldsymbol{C}}x(t)+\bar{\boldsymbol{A}}\boldsymbol{f}(\boldsymbol{x}(t))+\bar{\boldsymbol{B}}\boldsymbol{f}(\boldsymbol{x}(t-\tau(t)))\\ &\quad+\bar{\boldsymbol{D}}\int_{t-\tau(t)}^{t}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s+\boldsymbol{U}(t)+\bar{\boldsymbol{\Gamma}}_1\boldsymbol{x}(t)\\ &\quad+\bar{\boldsymbol{\Gamma}}_2\boldsymbol{x}(t-\tau(t))+\bar{\boldsymbol{\Gamma}}_3\int_{t-\tau(t)}^{t}\boldsymbol{x}(s)\,\mathrm{d}s-\dot{\boldsymbol{x}}(t)\Bigg]\\ &=0.\end{aligned}\tag{30}$$

Let

$$\begin{aligned}\boldsymbol{\xi} = \Bigg[&(\boldsymbol{M}\boldsymbol{x}(t))^{\mathrm{T}},(\boldsymbol{M}\dot{\boldsymbol{x}}(t))^{\mathrm{T}},(\boldsymbol{M}\boldsymbol{x}(t-\tau(t)))^{\mathrm{T}},\\ &(\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t)))^{\mathrm{T}},(\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(t-\tau(t))))^{\mathrm{T}},\\ &\left(\int_{t-\tau(t)}^{t}\boldsymbol{M}\boldsymbol{x}(s)\,\mathrm{d}s\right)^{\mathrm{T}},\left(\int_{t-\tau(t)}^{t}\boldsymbol{M}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\right)^{\mathrm{T}},\\ &\left(\int_{t-\tau(t)}^{t}\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s\right)^{\mathrm{T}}\Bigg]^{\mathrm{T}},\end{aligned}$$

then combining inequalities (12)–(16) and inequalities (25)–(30), we obatin

$$\begin{aligned}\sum_{i=1}^{6}\dot{\boldsymbol{V}}_i(\boldsymbol{x}(t)) &\le \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{\xi}-\beta\int_{t-\tau}^{t}(\boldsymbol{M}\boldsymbol{x}(s))^{\mathrm{T}}\bar{\boldsymbol{Q}}_4\boldsymbol{M}\boldsymbol{x}(s)\,\mathrm{d}s\\ &\quad-\beta\int_{t-\tau}^{t}(\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s)))^{\mathrm{T}}\bar{\boldsymbol{Q}}_5\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s\\ &\quad-\beta\int_{t-\tau}^{t}(\boldsymbol{M}\dot{\boldsymbol{x}}(s))^{\mathrm{T}}\bar{\boldsymbol{Q}}_6\boldsymbol{M}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s.\end{aligned}\tag{31}$$

From the definition of $\boldsymbol{V}_6(\boldsymbol{x}(t))$, we have the following inequality:

$$\boldsymbol{V}_6(\boldsymbol{x}(t))\le\tau\int_{t-\tau}^{t}(\boldsymbol{M}\dot{\boldsymbol{x}}(s))^{\mathrm{T}}\bar{\boldsymbol{Q}}_6(\boldsymbol{M}\dot{\boldsymbol{x}}(s))\,\mathrm{d}s.\tag{32}$$

Combining inequalities (18)–(22), (31), and (32), we have

$$\begin{aligned}&\dot{\boldsymbol{V}}(\boldsymbol{x}(t))\\ &=\mathrm{e}^{\varepsilon t}(\varepsilon\sum_{i=1}^{6}\boldsymbol{V}_i(\boldsymbol{x}(t))+\sum_{i=1}^{6}\dot{\boldsymbol{V}}_i(\boldsymbol{x}(t)))\\ &\le\mathrm{e}^{\varepsilon t}\Bigg[-(\lambda_{\min}(-\boldsymbol{\Phi})-\varepsilon\lambda_{\max}(\bar{\boldsymbol{Q}}_1))\|\boldsymbol{M}\boldsymbol{x}(t)\|^2\\ &\quad-\int_{t-\tau}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{M}^{\mathrm{T}}(\beta\bar{\boldsymbol{Q}}_4-\varepsilon\bar{\boldsymbol{Q}}_2-\varepsilon\tau\bar{\boldsymbol{Q}}_4)\boldsymbol{M}\boldsymbol{x}(s)\,\mathrm{d}s\\ &\quad-\int_{t-\tau}^{t}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(s))\boldsymbol{M}^{\mathrm{T}}(\beta\bar{\boldsymbol{Q}}_5-\varepsilon\bar{\boldsymbol{Q}}_3-\varepsilon\tau\bar{\boldsymbol{Q}}_5)\\ &\quad\times\boldsymbol{M}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s\\ &\quad-\int_{t-\tau}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{M}^{\mathrm{T}}(\beta\bar{\boldsymbol{Q}}_6-\varepsilon\tau\bar{\boldsymbol{Q}}_6)\boldsymbol{M}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\Bigg].\end{aligned}\tag{33}$$

Taking

$$\begin{aligned}\varepsilon = \min\Bigg\{&\frac{\lambda_{\min}(-\boldsymbol{\Phi})}{\lambda_{\max}(\bar{\boldsymbol{Q}}_1)},\frac{\beta\lambda_{\min}(\bar{\boldsymbol{Q}}_4)}{\lambda_{\max}(\bar{\boldsymbol{Q}}_2+\tau\bar{\boldsymbol{Q}}_4)},\\ &\frac{\beta\lambda_{\min}(\bar{\boldsymbol{Q}}_5)}{\lambda_{\max}(\bar{\boldsymbol{Q}}_3+\tau\bar{\boldsymbol{Q}}_5)},\frac{\beta\lambda_{\min}(\bar{\boldsymbol{Q}}_6)}{\lambda_{\max}(\tau\bar{\boldsymbol{Q}}_6)}\Bigg\}\end{aligned}$$

we have $\dot{\boldsymbol{V}}(\boldsymbol{x}(t))<0$, which implies that $\boldsymbol{V}(\boldsymbol{x}(t))<\boldsymbol{V}(0)$. The remaining proof is similar to that for Theorem 1. This completes the proof.

Corollary 2 Under Assumptions 1 and 3, the cluster synchronization manifold S' of the complex network (3) is cluster exponential synchronization if there exist positive definite matrices $\bar{\boldsymbol{Q}}_i\in\boldsymbol{R}^{(N-k)n\times(N-k)n}$ $(i=1,2,\ldots,6)$, four matrices $\bar{\boldsymbol{N}}_i\in\boldsymbol{R}^{(N-k)n\times(N-k)n}$ $(i=1,2,3,4)$, and a positive definite diagonal matrix $\bar{\boldsymbol{S}}=\mathrm{diag}[s_1,s_2,\ldots,s_{(N-k)n}]$, $\bar{\boldsymbol{S}}\in\boldsymbol{R}^{(N-k)n\times(N-k)n}$, such that

$$\boldsymbol{\Phi}=\begin{pmatrix}
\boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{13} & \bar{\boldsymbol{N}}_3\bar{\boldsymbol{A}}'+\bar{\boldsymbol{L}}\bar{\boldsymbol{S}} & \bar{\boldsymbol{N}}_3\bar{\boldsymbol{B}}' & \bar{\boldsymbol{N}}_3\bar{\boldsymbol{H}}_3 & \bar{\boldsymbol{N}}_1 & \bar{\boldsymbol{N}}_3\bar{\boldsymbol{D}}' \\
* & \tau\bar{\boldsymbol{Q}}_6-\bar{\boldsymbol{N}}_4-\bar{\boldsymbol{N}}_4^{\mathrm{T}} & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{H}}_2+\bar{\boldsymbol{N}}_2 & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{A}}' & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{B}}' & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{H}}_3 & \bar{\boldsymbol{N}}_2 & \bar{\boldsymbol{N}}_4\bar{\boldsymbol{D}}' \\
* & * & -(1-h)\bar{\boldsymbol{Q}}_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \tau\bar{\boldsymbol{Q}}_5+\bar{\boldsymbol{Q}}_3-2\bar{\boldsymbol{S}} & 0 & 0 & 0 & 0 \\
* & * & * & * & -(1-h)\bar{\boldsymbol{Q}}_3 & 0 & 0 & 0 \\
* & * & * & * & * & -\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_4 & 0 & 0 \\
* & * & * & * & * & * & -\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_6 & 0 \\
* & * & * & * & * & * & * & -\dfrac{1-\beta}{\tau}\bar{\boldsymbol{Q}}_5
\end{pmatrix}<0,$$

where $\boldsymbol{\Phi}_{11}=\bar{\boldsymbol{Q}}_2+\tau\bar{\boldsymbol{Q}}_4-\bar{\boldsymbol{N}}_1-\bar{\boldsymbol{N}}_1^{\mathrm{T}}-\bar{\boldsymbol{N}}_3\bar{\boldsymbol{C}}'-(\bar{\boldsymbol{N}}_3\bar{\boldsymbol{C}}')^{\mathrm{T}}+\bar{\boldsymbol{N}}_3\bar{\boldsymbol{H}}_1+(\bar{\boldsymbol{N}}_3\bar{\boldsymbol{H}}_1)^{\mathrm{T}}$, $\boldsymbol{\Phi}_{12}=\bar{\boldsymbol{Q}}_1-\bar{\boldsymbol{N}}_2^{\mathrm{T}}-\bar{\boldsymbol{N}}_3-(\bar{\boldsymbol{N}}_4\bar{\boldsymbol{C}}')^{\mathrm{T}}+(\bar{\boldsymbol{N}}_4\bar{\boldsymbol{H}}_1)^{\mathrm{T}}$, $\boldsymbol{\Phi}_{13}=\bar{\boldsymbol{N}}_1+\bar{\boldsymbol{N}}_3\bar{\boldsymbol{H}}_2$, $\beta\in(0,1)$, $\boldsymbol{L}=\mathrm{diag}[\boldsymbol{L}_1,\boldsymbol{L}_2,\ldots,\boldsymbol{L}_n]\in\boldsymbol{R}^{n\times n}$, $\bar{\boldsymbol{L}}=\boldsymbol{I}_{N-k}\otimes\boldsymbol{L}$, $\tilde{\boldsymbol{H}}_1=\tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_1\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{H}}_2=\tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_2\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{H}}_3=\tilde{\boldsymbol{M}}\tilde{\boldsymbol{Z}}_3\tilde{\boldsymbol{J}}$, $\bar{\boldsymbol{H}}_1=\tilde{\boldsymbol{H}}_1\otimes\boldsymbol{\Gamma}_1$, $\bar{\boldsymbol{H}}_2=\tilde{\boldsymbol{H}}_2\otimes\boldsymbol{\Gamma}_2$, $\bar{\boldsymbol{H}}_3=\tilde{\boldsymbol{H}}_3\otimes\boldsymbol{\Gamma}_3$, $\tilde{\boldsymbol{M}}$, $\tilde{\boldsymbol{J}}$, $\tilde{\boldsymbol{Z}}_1$, $\tilde{\boldsymbol{Z}}_2$, and $\tilde{\boldsymbol{Z}}_3$, have the same structures as Lemma 2.

Proof As in the proof of Theorem 2, we use the same Lyapunov–Krasovskii functional (25). In the proof we only need to substitute Eq. (16) by Eq. (24).

Remark 3 In Theorem 2 and Corolly 2, we increase more relaxed conditional matrices in the criteria, by the matrix expansion method, which can increase the flexibility of the solutions.

4. Numerical examples

In this section, we show simulation examples to demonstrate the advantages of the proposed method. Consider a two-dimensional network with both discrete and distributed time-varying delay as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= -\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(t))+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{x}(t-\tau(t)))\\ &\quad+\boldsymbol{D}\int_{t-\tau(t)}^{t}\boldsymbol{f}(\boldsymbol{x}(s))\boldsymbol{d}s+u(t),\end{aligned}\tag{34}$$

where

$$\boldsymbol{x}(t)=[\boldsymbol{x}_1(t),\boldsymbol{x}_2(t)]^{\mathrm{T}}$$

$$\boldsymbol{f}(\boldsymbol{x}(t))=[\tanh(\boldsymbol{x}_1(t)),\tanh(\boldsymbol{x}_2(t))]^{\mathrm{T}},\ u(t)=[0,0]^{\mathrm{T}}$$

$$\tau(t)=0.8+0.4\sin(t),$$

$$\dot{\tau}(t)=0.4\cos(t)\le 0.4<1,$$

$$\boldsymbol{C}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \boldsymbol{A}=\begin{pmatrix}2.0&11\\0.2&2.0\end{pmatrix},$$

$$\boldsymbol{B}=\begin{pmatrix}-1.6&0.2\\0.5&-1.6\end{pmatrix},\ \boldsymbol{D}=\begin{pmatrix}-1.6&-0.5\\-2.7&-0.8\end{pmatrix},$$

then the dynamical behavior of model (34) with the initial condition $x(s)=\begin{pmatrix}0.5\\0.4\end{pmatrix}$, is shown in Fig. 1.

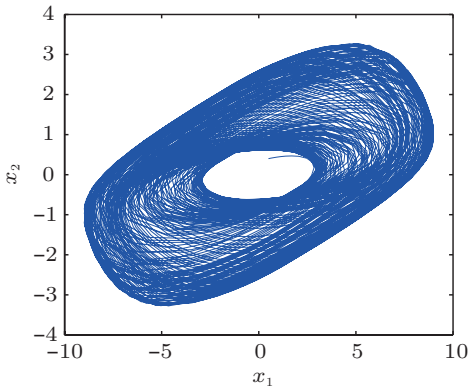


Fig. 1. (color online) Trajectory of model (34).

Now, we consider the following complex network model with time-varying delay and hybrid coupling:

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) &= -\boldsymbol{C}\boldsymbol{x}_i(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}_i(t))+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{x}_i(t-\tau(t)))\\ &\quad+\boldsymbol{D}\int_{t-\tau(t)}^{t}\boldsymbol{f}(\boldsymbol{x}_i(s))\mathrm{d}s+\boldsymbol{u}(t)+\sum_{j=1}^{N}\boldsymbol{G}_{ij}^{(1)}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\\ &\quad+\sum_{j=1}^{N}\boldsymbol{G}_{ij}^{(2)}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau(t))\\ &\quad+\sum_{j=1}^{N}\boldsymbol{G}_{ij}^{(3)}\boldsymbol{\Gamma}_3\int_{t-\tau(t)}^{t}\boldsymbol{x}_j(s)\mathrm{d}s,\\ &\quad i=1,2,\ldots,6,\end{aligned}\tag{35}$$

where $\boldsymbol{x}_i(t)=[\boldsymbol{x}_{i1}(t),\boldsymbol{x}_{i2}(t)]^{\mathrm{T}}$ is the state vector of the i-th node, and the coupling matrices are

$$\boldsymbol{\Gamma}_1=\begin{pmatrix}3.2&0\\0&3.2\end{pmatrix},\ \boldsymbol{\Gamma}_2=\begin{pmatrix}0.1&0\\0&0.1\end{pmatrix},\boldsymbol{\Gamma}_3=\begin{pmatrix}0.8&0\\0&0.8\end{pmatrix},$$

$$\boldsymbol{G}^{(r)} = \begin{pmatrix} -15 & 14.9 & 0.05 & 0.05 & 0 & 0 \\ 14.9 & -15 & 0.05 & 0.05 & 0 & 0 \\ 0 & 0 & -12 & 11.96 & 0.02 & 0.02 \\ 0 & 0 & 11.96 & -12 & 0.02 & 0.02 \\ 0.03 & 0.03 & 0 & 0 & -13 & 12.94 \\ 0.03 & 0.03 & 0 & 0 & 12.94 & -13 \end{pmatrix},$$

$r = 1, 2, 3$.

It is easy to verify that assumptions 1–3 hold for $L = \boldsymbol{I}_2$, $\tau = 1.2$, and $h = 0.4$.

The initial values are given as follows:

$$\boldsymbol{x}_1(s) = \begin{pmatrix} -0.5 \\ 0.2 \end{pmatrix}, \ \boldsymbol{x}_2(s) = \begin{pmatrix} -0.7 \\ 0.4 \end{pmatrix}, \ \boldsymbol{x}_3(s) = \begin{pmatrix} 0.5 \\ 0.6 \end{pmatrix},$$

$$\boldsymbol{x}_4(s) = \begin{pmatrix} -0.7 \\ 0.8 \end{pmatrix}, \ \boldsymbol{x}_5(s) = \begin{pmatrix} 0.4 \\ -0.5 \end{pmatrix}, \ \boldsymbol{x}_6(s) = \begin{pmatrix} -0.6 \\ -0.7 \end{pmatrix}.$$

Taking $\beta = 0.78$, from Theorem 1, one obtains the feasible solutions as follows:

$$\bar{\boldsymbol{Q}}_1 = \begin{pmatrix} 0.4586 & 0.0036 & 0 & 0 & 0 & 0 \\ 0.0036 & 0.4978 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5589 & 0.0045 & 0 & 0 \\ 0 & 0 & 0.0045 & 0.6281 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5219 & 0.0043 \\ 0 & 0 & 0 & 0 & 0.0043 & 0.5783 \end{pmatrix},$$

$$\bar{\boldsymbol{Q}}_2 = \begin{pmatrix} 6.5605 & -0.5800 & 0 & 0 & 0 & 0 \\ -0.5800 & 7.4267 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.9991 & -0.8509 & 0 & 0 \\ 0 & 0 & -0.8509 & 7.0953 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.2221 & -0.7431 \\ 0 & 0 & 0 & 0 & -0.7431 & 7.2335 \end{pmatrix},$$

$$\bar{\boldsymbol{Q}}_3 = \begin{pmatrix} 5.6506 & -0.3242 & 0 & 0 & 0 & 0 \\ -0.3242 & 6.1077 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.2711 & -0.4536 & 0 & 0 \\ 0 & 0 & -0.4536 & 5.8798 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.4237 & -0.4039 \\ 0 & 0 & 0 & 0 & -0.4039 & 5.9739 \end{pmatrix},$$

$$\bar{\boldsymbol{Q}}_4 = \begin{pmatrix} 26.4911 & -0.4652 & 0 & 0 & 0 & 0 \\ -0.4652 & 28.8207 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25.3621 & -0.8862 & 0 & 0 \\ 0 & 0 & -0.8862 & 28.6007 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.8602 & -0.6991 \\ 0 & 0 & 0 & 0 & -0.6991 & 28.7304 \end{pmatrix},$$

$$\bar{\boldsymbol{Q}}_5 = \begin{pmatrix} 6.5508 & -0.0543 & 0 & 0 & 0 & 0 \\ -0.0543 & 6.2261 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6.4515 & -0.0537 & 0 & 0 \\ 0 & 0 & -0.0537 & 5.9138 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.4982 & -0.0548 \\ 0 & 0 & 0 & 0 & -0.0548 & 6.0429 \end{pmatrix},$$

$$\bar{S} = \begin{pmatrix} 16.9384 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20.5942 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16.1718 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.9616 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16.4882 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20.8488 \end{pmatrix}.$$

According to Theorem 1, system (35) is cluster exponential synchronization. Figures 2 and 3 show the cluster synchronization states. The synchronization error between the first two nodes, the middle two nodes and the last two nodes in the complex network (35) are illustrated by Figs. 4–6, respectively, which are calculated by

$$e_1(t) = \sqrt{\sum_{i=1}^{2}(\boldsymbol{x}_{1i} - \boldsymbol{x}_{2i})^2}, \quad e_2(t) = \sqrt{\sum_{i=1}^{2}(\boldsymbol{x}_{3i} - \boldsymbol{x}_{4i})^2},$$

$$e_3(t) = \sqrt{\sum_{i=1}^{2}(\boldsymbol{x}_{5i} - \boldsymbol{x}_{6i})^2}.$$

Figure 7 shows the synchronization error of the dynamic system (35) by calculating

$$e(t) = \sum_{i=1}^{2}\sqrt{\sum_{j=2}^{6}(\boldsymbol{x}_{1i} - \boldsymbol{x}_{ji})^2}.$$

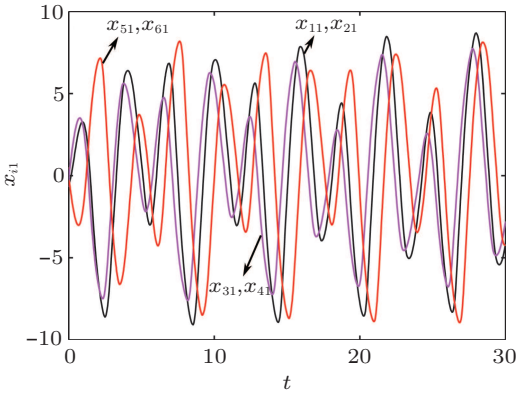


Fig. 2. (color online) States of system (35) $\boldsymbol{x}_{i1}$ $(1 \le i \le 6)$.

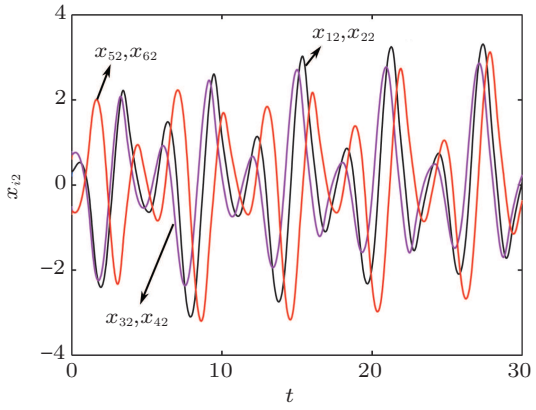


Fig. 3. (color online) States of system (35) $\boldsymbol{x}_{i2}$ $(1 \le i \le 6)$.

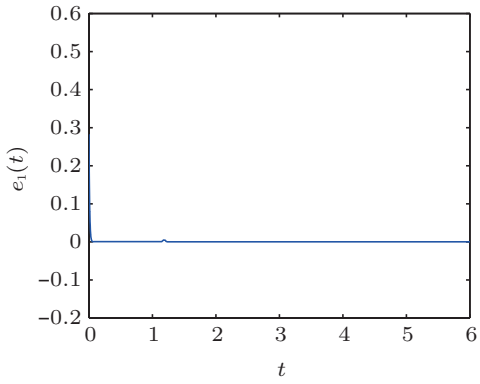


Fig. 4. Synchronization error $e_1(t)$ of the first two nodes.

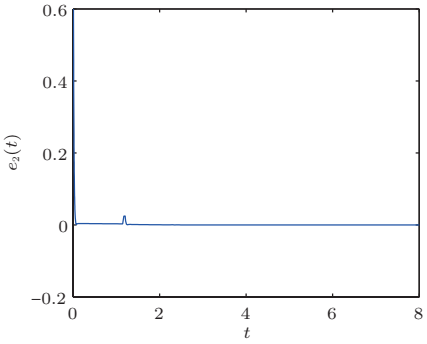


Fig. 5. Synchronization error $e_2(t)$ of the middle two nodes.

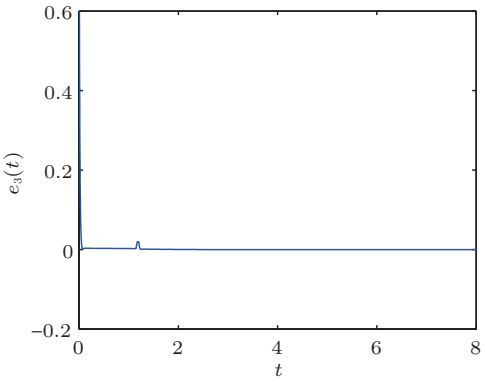


Fig. 6. Synchronization error $e_3(t)$ of the last two nodes.

From Theorem 1, we can obtain that the upper bound of delay is 1.39. Although our method can give a better effect than some other methods reported before, it is obvious that there is still some room for improvement. Thus, how to acquire less conservative results is still a challenging problem.

From Figs. 2–7, we can see that dynamic system (35) achieves cluster exponential synchronization.

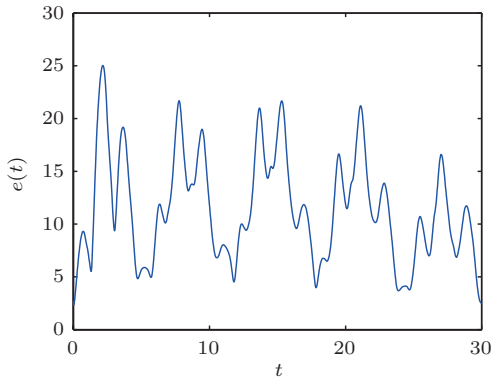


Fig. 7. Complete synchronization error $e(t)$ of coupled system (35).

5. Conclusion

In the last several years, the synchronization problem for complex networks has become a hot research topic. However, there are few results for coupled networks with time-varying delay. In this paper, the cluster exponential synchronization of coupled connected complex network with both discrete and distributed time-varying delay has been discussed. Several sufficient conditions have been derived for cluster exponential synchronization of the dynamic system (3) based on the Kronecker product technique of matrices and the Lyapunov–Krasovskii theory. We introduce the complex network of the nonlinear terms and coupling terms with the discrete and the distributed time-varying delay, and the upper of delay is obtained. Thus, the results extend and improve the earlier results.

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