Accelerating an adiabatic process by nonlinear sweeping*

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We investigate the acceleration of an adiabatic process with the same survival probability of the ground state by sweeping a parameter nonlinearly, fast in the wide gap region and slowly in the narrow gap region, in contrast to the usual linear sweeping. We find the expected acceleration both in the Landau–Zener tunneling model and in the adiabatic quantum computing model for factorizing the number $N = 21$.

Keywords: quantum calculation, quantum adiabatic algorithm, factorization, nonlinear sweeping

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1. Introduction

Due to its robustness against noises, adiabatic quantum computation (AQC) is one of the promising experimentally feasible schemes.[1–7] However, the average energy gap between two adjacent eigenstates of a quantum computer decreases exponentially with the increasing number of qubits.[8,9] To satisfy the quantum adiabaticity criteria required by AQC,[10–12] the total evolution time, which is inversely proportional to the energy gap between the ground state and the lowest excited state, may increase exponentially. Such a time scaling may eventually counteract the speedup benefit of the quantum computer.

A possible way to circumvent the requirement of the long adiabatic evolution time is using rapid adiabatic passage method, which varies the energy levels either continuously or discontinuously according to certain designed rules.[13–18] It is proved that the rapid adiabatic passage is capable of transferring an initial ground state to a final ground state within a much shorter time. The price paid is that all the energy level information, which is usually beyond the reach of current techniques for a many-qubit quantum computer, must be provided in order to design these rules. As an alternative way, we may accelerate an adiabatic process in a moderate evolution time but require less information about the energy levels. For instance, we may require only the extremes of the gap during the total evolution.

In this paper, we investigated nonlinear sweeping methods to accelerate adiabatic processes, including the famous Landau–Zener (LZ) tunneling model and the AQC model,[1,4,6,19–21] with only the knowledge of the extremes of the gap. For the LZ model, we developed a quadratic sweeping scheme and numerically confirmed the acceleration effect of the nonlinear sweeping. By utilizing the same idea, we then designed a more complex nonlinear sweeping scheme for a single-qubit AQC model. Finally, we evaluated the effectiveness of this nonlinear sweeping method by factorizing the number $N = 21$.

2. Results and discussion

We consider the well-known LZ tunneling model of a two-level system described by

$$H(t) = \omega_x \sigma_x + \omega_z(t)\sigma_z, \tag{1}$$

where $\omega_{x,z}$ are the Zeeman splitting energy along the x and z direction, respectively, and $\sigma_{x,z}$ the Pauli matrix.[19,20] For a symmetric system as shown in Fig. 1(a) , the time dependence of a linear sweeping of $\omega_z(t)$ is

$$\omega_z(t) = -\omega_0 + vt, \tag{2}$$

where $v = 2\omega_0/T$ denotes the sweeping velocity with T being the total evolution time. For a large sweeping velocity v, an initial ground state is transferred to the final excited state with a large probability given by the Landau–Zener formula.[19–21] While for a small enough sweeping velocity v, the system would stay in the ground state according to the quantum adiabatic theorem.[10,11]

It is obvious that the linear sweeping of $\omega_z(t)$ is not optimized for the maximal fidelity F, which measures the survival probability of the final ground state and is defined as

$$F = |\langle\psi(T)|\psi_g\rangle|^2. \tag{3}$$

Here $|\psi_g\rangle$ denotes the instantaneous ground state of the Hamiltonian $H = H(T)$, and $|\psi(T)\rangle$ represents the actually evolved wave function at the end. An intuition for the optimization is sweeping fast in the large gap region but sweeping slowly in the small gap region. The gap Δ is defined as the energy

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difference between the ground state ω_g and the lowest excited state ω_e, i.e., $\Delta = \omega_e - \omega_g$. It is straightforward to construct a nonlinear sweeping form which satisfies the above intuition,

$$\omega_z(t) = \begin{cases} -\omega_0\left(\frac{t}{t_c} - 1\right)^2, & t \le t_c, \\ \omega_0\left(\frac{t}{t_c} - 1\right)^2, & t > t_c, \end{cases} \tag{4}$$

with $t_c = T/2$. The gap Δ is minimal at t_c. The corresponding instantaneous sweeping velocity is

$$v(t) = 2\omega_0\left|\frac{t}{t_c} - 1\right|, \tag{5}$$

which is large near $t = 0$ and T, where the gap is large, but small near $t = t_c$, where the gap is small. In fact, the velocity is roughly proportional to the gap if t is away from t_c. The velocity for the whole evolution can be adjusted by changing the total time T. The longer the T is, the smaller the velocity will be.

We present in Fig. 1(b) the fidelity F of the final ground state under either the linear sweeping form (Eq. (2)) or the nonlinear sweeping form (Eq. (4)). It clearly shows that the fidelity with nonlinear sweeping is much higher than that with linear sweeping, provided that the total evolution time T is the same. In other words, the adiabatic LZ process can be accelerated by adopting a nonlinear sweeping form.

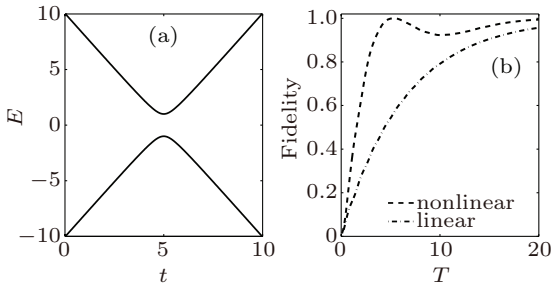


Fig. 1. (a) The instantaneous energy level diagram for a two-level LZ model and (b) the fidelity of the final ground state. The parameters are $\omega_0 = 10$, $\omega_x = 1$, and $T = 10$ for panel (a). In panel (b), the dash-dotted line denotes the fidelity under the linear sweeping and the dashed line the nonlinear sweeping. The fidelity for the nonlinear sweeping is higher than that for the linear sweeping.

It should be noted that other more complex nonlinear forms also accelerate the adiabatic process, especially the shortcut to adiabatic passage or the hyperbolic tangent form.[13–15] These methods usually require all the information of the energy levels, which is not always available. While in our nonlinear formula (4), only the position of the minimal gap t_c is required. In this sense, the nonlinear sweeping method is much easier to be implemented in experiments.

We next consider a single-qubit AQC model whose Hamiltonian is also time-dependent,

$$H(t) = [1 - s(t)]\omega_x\sigma_x + s(t)\omega_z\sigma_z, \tag{6}$$

where $s(t)$ increases monotonically from $s(0) = 0$ to $s(T) = 1$. By sweeping slowly $s(t)$ from 0 to 1, the two-level system transfers from an initial ground state to its final ground state. Compared with the LZ model (Eq. (1)), both factors before σ_x and σ_z are time-dependent. The linear sweeping and the quadratic sweeping form adopted in experiments[6,7] are, respectively,

$$s(t) = \frac{t}{T}, \tag{7}$$

$$s(t) = \left(\frac{t}{T}\right)^2. \tag{8}$$

It is obvious that neither the linear sweeping nor the quadratic sweeping satisfies the requirement of fast sweeping in a large gap region and slow sweeping in a small gap region. We find the optimal sweeping form as follows.

To fully utilize the optimized sweeping scheme developed in the above LZ model, we alternate the single-qubit AQC Hamiltonian (Eq. (6)) into a form similar to the LZ model (Eq. (1)), where only one parameter before the Pauli matrix is time-dependent,

$$H(t) = \omega_\perp\sigma_\perp + \omega_n(s(t))\sigma_n, \tag{9}$$

where

$$\begin{aligned} \sigma_\perp &= \sigma_z\sin\theta + \sigma_x\cos\theta, \\ \sigma_n &= \sigma_z\cos\theta - \sigma_x\sin\theta, \\ \omega_\perp &= \omega_x\cos\theta, \\ \omega_n(t) &= s(t)\Omega - \omega_x\sin\theta, \end{aligned}$$

with $\cos\theta = \omega_z/\Omega$, $\sin\theta = \omega_x/\Omega$, and $\Omega = \sqrt{\omega_z^2 + \omega_x^2}$. Obviously, $\sigma_\perp$ and σ_n are along two orthogonal directions.

The instantaneous energy gap is

$$\Delta(s) = 2\sqrt{s^2\omega_z^2 + (1-s)^2\omega_x^2}. \tag{10}$$

The minimal gap lies at

$$s_c = \frac{\omega_x^2}{\omega_x^2 + \omega_z^2}, \tag{11}$$

which satisfies

$$\left.\frac{d\Delta}{ds}\right|_{s_c} = 0. \tag{12}$$

Another way to obtain the above results of s_c is setting $\omega_n(s_c) = 0$. We notice that the gap is roughly proportional to s to the lowest order, i.e., $\Delta \propto |s - s_c|$.

Similar to what have been done in the LZ model, we require that the sweeping velocity $v(t) = d\omega_n/dt$ is proportional to the energy gap Δ. In this form, $\omega_n(s(t))$ sweeps fast in the large gap region and slowly in the small gap region. It is then straightforward to obtain the following relation:

$$\frac{d\omega_n(s(t))}{dt} = A|s(t) - s_c|, \tag{13}$$

by noticing $\Delta \propto |s - s_c|$. The value of A is in principle determined by the boundary conditions at $s(t) = 0, s_c$ and 1, but later we allow A to change freely to optimize $s(t)$.

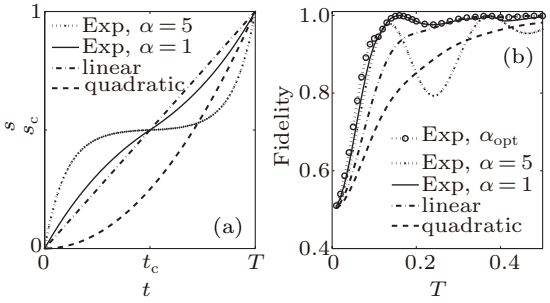


Fig. 2. (a) Various forms of $s(t)$; (b) dependence of the fidelity on the total evolution time. Dotted line with circles: exponential-like with optimized α; dotted lines: exponential-like with $\alpha = 5$; solid lines: exponential-like with $\alpha = 1$; dash-dotted lines: linear; dashed lines: quadratic. The parameters are $\omega_x = 18$ and $\omega_z = 30$. The exponential-like schemes are better than the linear or the quadratic one.

Equation (13) implies that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{A}{\Omega}|s - s_\mathrm{c}|. \tag{14}$$

It is easy to obtain the relation $s(t)$, by applying the boundary conditions that $s(0) = 0$, $s(T) = 1$, and $s(t_\mathrm{c}) = s_\mathrm{c}$,

$$s(t) = \begin{cases} \dfrac{s_\mathrm{c}}{1 - \mathrm{e}^{-\alpha}}(1 - \mathrm{e}^{-\alpha t/t_\mathrm{c}}), & t \le t_\mathrm{c}, \\ 1 - \dfrac{1 - s_\mathrm{c}}{\mathrm{e}^{\alpha/s_\mathrm{c}} - \mathrm{e}^{\alpha}}(\mathrm{e}^{\alpha/s_\mathrm{c}} - \mathrm{e}^{\alpha t/t_\mathrm{c}}), & t > t_\mathrm{c}. \end{cases} \tag{15}$$

We have introduced an adjustable parameter α, proportional to A/Ω, to vary the curvature of $s(t)$ (see Fig. 2(a)). When $\alpha = 0$, $s(t)$ is linear; when $\alpha = 1$, $s(t)$ is a two-interval exponential-like piecewise function; when $\alpha \to \infty$, $s(t)$ is step-like function. To determine t_c, we have assumed the simplest form $t_\mathrm{c}/T = s_\mathrm{c}$.

We present in Fig. 2(b) the fidelity of the final ground state under the time-dependent Hamiltonian (Eq. (6)) for various sweeping schemes. Since the initial state of the two-level system is an eigenstate of σ_x, which overlaps with the final instantaneous ground state, the fidelity is 0.5 even if T approaches zero. As obviously shown in Fig. 2(b), the quadratic scheme is the worst choice. The fidelity with the linear scheme is between that with a quadratic scheme and that with a exponential-like scheme. By optimizing α for each T in the exponential-like schemes, the fidelity reaches its highest value. We also notice that for some values of α in the exponential-like schemes, the fidelity oscillates if the total evolution time T becomes large, e.g., $\alpha = 5$. These oscillations might be due to the over-long-time stay of the system near the minimal gap region where the gap is almost constant.[22]

Finally, we consider an experimentally realized three-qubit AQC model, i.e., factorizing the number $N = 21$.[6] The time-dependent Hamiltonian is an interpolation of an easy-realized initial Hamiltonian H_0 and a problem-solved Hamiltonian H_P,

$$H(t) = [1 - s(t)]H_0 + s(t)H_\mathrm{P}, \tag{16}$$

where $s(t)$ changes from 0 to 1 as the system evolves from 0 to T,

$$\begin{aligned} H_0 &= g(\sigma_{1x} + \sigma_{2x} + \sigma_{3x}), \\ H_\mathrm{P} &= [N - (2I - \sigma_{1z})(4I - \sigma_{2z} - 2\sigma_{3z})]^2. \end{aligned}$$

In the above equations, g is the Zeeman splitting of a spin in a uniform magnetic field along the x direction, I is the 2-by-2 identity matrix, and $\sigma_{i\alpha}$ ($i = 1, 2, 3$ and $\alpha = x, z$) denotes the α-component of Pauli matrix for the i-th spin.[6] The ground state of H_P, $|\downarrow_1\rangle \otimes |\downarrow_2\downarrow_3\rangle$ interpreted as $3 \otimes 7$, is the solution to the factors of $N = 21$. The 3-spin system is prepared initially in the ground state of H_0. After an adiabatic evolution under the total Hamiltonian $H(t)$, the system adiabatically transfers to the ground state of $H(T) = H_\mathrm{P}$ and the number N is eventually factorized.

Usually, two sweeping schemes might be experimentally adopted.[6,7] One is the linear scheme as defined by Eq. (7) and the other is the quadratic scheme given by Eq. (8). We have shown in the two-level model that both the linear and the quadratic sweeping schemes are not optimal. We may use the exponential-like scheme to accelerate the AQC process if the energy gap between the ground state and the lowest excited state is approximately in an inverted parabolic shape with respect to s. Fortunately, the gap for $N = 21$ is indeed very similar to that for the single qubit AQC model system, as shown in Fig. 3(a). Thus, we directly apply the exponential-like sweeping scheme to the adiabatic process of $N = 21$ factorization.

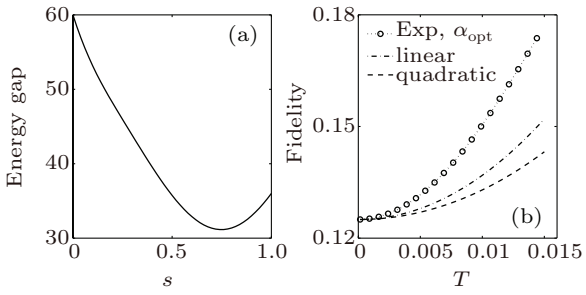


Fig. 3. (a) The instantaneous energy gap versus s for $N = 21$. The minimal gap position is near $s_\mathrm{c} \approx 0.74$. (b) The dependence of the fidelity on the total evolution time for factorizing N with optimized exponential-like sweeping scheme (dotted line with circles), linear sweeping scheme (dash-dotted line), and quadratic sweeping scheme (dashed line). The parameter is $g = 30$. Among the three schemes, the optimized exponential-like one shows the highest fidelity at the same evolution time T and is the best choice.

We present in Fig. 3(b) the fidelities of the evolved final state to the ground state of H_P at different total evolution time T with the linear sweeping scheme, the quadratic scheme, and the exponential-like scheme. We limit ourselves in the short T region where the difference is prominent among these schemes (For long enough total evolution time T, the fidelity approaches 1 and the three sweeping schemes coincides. However, the high fidelity is unnecessary in the factorization problem since checking the answer is incredibly easy). In Fig. 4,

we show how we obtain the optimized value of the parameter $\alpha_{\rm opt}$, which depends on T and is in the range between 0 and 5. It is clearly shown in Fig. 3 that the fidelity with the exponential-like sweeping scheme is higher than that with either the linear or the quadratic sweeping scheme for the same total evolution time T. Thus, to reach the same final fidelity, the exponential-like sweeping scheme uses the shortest time and accelerates the AQC process.

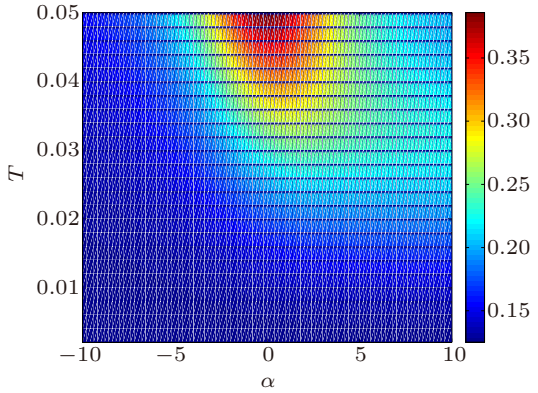


Fig. 4. (color online) Dependence of the fidelity on the total evolution time T and the parameter α. Other parameters are the same as that in Fig. 3. For a given T, the optimized value $\alpha_{\rm opt}$ corresponds to the highest fidelity.

For larger N, more qubits are required to factorize the number. The energy levels inevitably become so complex that the energy gap Δ may show multiple extremes. In this situation, a more complicated piecewise function $s(t)$ with an exponential-like form in each interval might be more appropriate. More explorations along this direction are worthwhile in the future.

3. Conclusion

In conclusion, we develop a nonlinear sweeping scheme for an adiabatic process, which changes quickly in the large gap region and slowly in the small gap region. This scheme improves the survival probability of the final ground state and may thus accelerate the adiabatic process. We confirm the usefulness of this nonlinear scheme in the Landau–Zener tunneling model, the single-qubit AQC model, and the AQC of factorizing the number $N = 21$. The proposed nonlinear scheme may be useful in AQC experiments to shorten the adiabatic evolution time (we do not expect that the nonlinear sweeping scheme proposed in this paper could change the scaling behavior of the exponentially increased evolution time for a general adiabatic quantum computation algorithm).

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