Topological properties of one-dimensional hardcore Bose–Fermi mixture*

Jia Yong-Fei(贾永飞)[a], Guo Huai-Ming(郭怀明)[a]†, Qin Ji-Hong(秦吉红)[b], Chen Zi-Yu(陈子瑜)[a], and Feng Shi-Ping(冯世平)[c]

[a] *Department of Physics, Beihang University, Beijing* 100191, *China*
[b] *Department of Physics, University of Science and Technology Beijing, Beijing* 100083, *China*
[c] *Department of Physics, Beijing Normal University, Beijing* 100875, *China*

(Received 7 April 2013; revised manuscript received 2 May 2013)

We study the topological properties of a one-dimensional (1D) hardcore Bose–Fermi mixture using the exact diagonalization method. We firstly add a hardcore boson to a fermionic system and by examining the edge states we find that the quasi-particle manifests the topological properties of the system. Then we study a mixture with 7 fermions and 1 boson. We find that the mixture also exhibits topological properties and its behaviors are similar to that of the corresponding fermionic system. We present a qualitative explanation to understand such behaviors using the mapping between a hardcore boson and a spinless fermion. These results show the existence of topological properties in a 1D hardcore Bose–Fermi mixture and may be realized using cold atoms trapped in optical lattices experimentally.

Keywords: Bose–Fermi mixture, topological insulators

PACS: 03.65.Vf, 71.27.+a **DOI:** 10.1088/1674-1056/22/9/090308

1. Introduction

Topological insulators (TIs) have been the focus of recent studies in condensed matter physics.[1–5] Their many unusual physical properties are related to the Dirac cone on the surface,[6,7] which makes them have potential applications in spintronics and quantum computing.[8] After understanding the topological properties of the non-interacting system, researchers began to explore the interplay between the topology and interactions.[9,10] Various numerical methods were applied to these studies and consistent results were obtained.[11–16] Meanwhile, formulas for calculating the topological invariant of interacting systems were developed.[17–20] Also, classifications of different interacting topological systems were made.[21,22] Besides these, the fractional Chern insulators were predicted in interacting topological flat bands.[23–33]

Though there have been many theoretical studies, only a few of them can be realized experimentally, for which a main reason is the lack of ideal TI materials. However, due to the achievements in the field of ultracold gases, loading atoms in optical lattices has enabled the creation of quantum systems that are unexpected in real materials.[34] This also shines a light on the field of TIs and many non-interacting and interacting models of TIs thus can be engineered and studied in optical lattices.[35–38] Another advantage of these experiments is the realization of Bose or even Bose–Fermi models besides Fermi ones.[39,40] Usually, bosons in the topological bands do not show topological properties anymore due to the breakdown of the band structure.[11] Then, it is interesting to ask: what is the situation for a Bose–Fermi mixture?

In this paper, we study the topological properties of a one-dimensional (1D) hardcore Bose–Fermi mixture using the exact diagonalization (ED) method. Firstly, we study the case of adding a boson to a half-filling fermionic system. We find that when it is added to a topological system, the quasi-particle shows the features of edge states, indicating that the system still manifests its topological properties even in the presence of a different kind of particle. While for the case of the trivial system, the quasi-particle is bulk like. Next, we study the topological properties of a mixture with 7 fermions and 1 boson. It is found that the mixture also has topological properties and its behaviors are similar to the corresponding fermionic system except having different critical values. Finally, we present a qualitative explanation to understand such behaviors using the mapping between the hardcore boson and the spinless fermion. These results show the existence of topological properties in the 1D hardcore Bose–Fermi mixture.

2. Model and method

The Hamiltonian of the Bose–Fermi mixture in 1D interacting topological band is described as[41,42]

$$\begin{aligned} H &= H_{\rm f} + H_{\rm b} + H_{\rm int}, \\ H_{\rm f(b)} &= \sum_i (M+2B)\Psi^{\dagger}_{i,{\rm f(b)}}\sigma_z\Psi_{i,{\rm f(b)}} \\ &\quad - \sum_{i,\hat{x}} B\Psi^{\dagger}_{i,{\rm f(b)}}\sigma_z\Psi_{i+\hat{x},{\rm f(b)}} \end{aligned} \tag{1}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11274032, 11104189, 11004006, 11274033, 11074023, and 11274044) and FOK YING TUNG Education Foundation, China.

†Corresponding author. E-mail: hmguo@buaa.edu.cn

© 2013 Chinese Physical Society and IOP Publishing Ltd

$$-\sum_{i,\hat{x}} \mathrm{sign}(\hat{x})\mathrm{i}A\Psi^{\dagger}_{i,\mathrm{f(b)}}\sigma_x\Psi_{i+\hat{x},\mathrm{f(b)}},$$

$$H_{\mathrm{int}} = U\sum_i (n^c_{i,\mathrm{b}}+n^c_{i,\mathrm{f}})(n^d_{i,\mathrm{b}}+n^d_{i,\mathrm{f}}),$$

with the constraint

$$n^{c(d)}_{i,\mathrm{b}} + n^{c(d)}_{i,\mathrm{f}} \leq 1, \tag{2}$$

where σ_x, σ_z are Pauli matrices, $\Psi_{i,\mathrm{f(b)}} = (c_{i,\mathrm{f(b)}}, d_{i,\mathrm{f(b)}})^{\mathrm{T}}$ with $c_{i,\mathrm{f(b)}}(d_{i,\mathrm{f(b)}})$ fermion (hardcore boson) annihilating operator at the site $\boldsymbol{r}_i$ and $n^c_{i,\mathrm{f(b)}}(n^d_{i,\mathrm{f(b)}})$ is the number operator of the orbit $c(d)$ for fermion (hardcore boson). In the fermionic version of the non-interacting model ($U = 0$), depending on the values of the parameters A, B, and M the system can be a trivial insulator or a non-trivial insulator at half-filling. The non-trivial insulator is protected by the chiral symmetry: $H_{\mathrm{f}} = -PH_{\mathrm{f}}P^{-1}$ with P transforming c, d to $d, -c$ on each site. The inclusions of hardcore bosons and interactions also maintain the symmetry. In the following, we set $A = B = 1$ and the system is in a topological phase for $-4 < M < 0$. The fermions obey the anticommutation relation $\{c_{i,\mathrm{f}}, c^{\dagger}_{j,\mathrm{f}}\}(\{d_{i,\mathrm{f}}, d^{\dagger}_{j,\mathrm{f}}\}) = \delta_{ij}$, while the hardcore bosons obey the commutation relation $[c_{i,\mathrm{b}}, c^{\dagger}_{j,\mathrm{b}}]([d_{i,\mathrm{b}}, d^{\dagger}_{j,\mathrm{b}}]) = 0$ at sites $i \neq j$ but the anticommutation relation $\{c_{i,\mathrm{b}}, c^{\dagger}_{i,\mathrm{b}}\}(\{d_{i,\mathrm{b}}, d^{\dagger}_{i,\mathrm{b}}\}) = 1$ at sole site i.

In the following, we use the ED method to study the physical properties of Hamiltonian Eq. (1). Generally, for a finite chain of N sites with the constraint Eq. (2), the full Hilbert space has a dimension of 9^N. Since in the Hamiltonian the particle numbers of fermions and bosons are conserved separately, the Hamiltonian can be diagonalized in a subspace with a fixed number of particles, which greatly reduces the dimension of the matrix of the Hamiltonian . All subsequent calculations are with fixed numbers of hardcore bosons and fermions, and are carried out on the systems with the length $N = 8$.

To characterize the topological properties of the system, we calculate the quasi-particle energy spectrum (QPES), the distribution of the quasi-particle, and the Berry phase, using the ground-state energy and the wave function obtained with the ED method. The energy of a quasi-particle when adding or removing a particle in a many-body system with N particles is defined as $\Delta E_N = \pm(E^0_{N\pm1} - E^0_N)$, where E^0_N is the ground energy of a system with N particles. Similarly, the distribution of added or removed quasi-particles is defined as $\Delta n_i = \pm(\langle\psi^0_{N\pm1}|\hat{n}_i|\psi^0_{N\pm1}\rangle - \langle\psi^0_N|\hat{n}_i|\psi^0_N\rangle)$ with $\hat{n}_i = n^c_{i,\mathrm{f}} + n^d_{i,\mathrm{f}} + n^c_{i,\mathrm{b}} + n^d_{i,\mathrm{b}}$ the operator of total particle number at site i and ψ^0_N the ground-state wave function of a system with N particles. The Berry phase of the ground-state of the interacting system can be calculated using the twisted boundary condition which is defined as[43–47]

$$\gamma = \oint \mathrm{i}\langle\psi_\theta|\frac{\mathrm{d}}{\mathrm{d}\theta}|\psi_\theta\rangle, \tag{3}$$

where θ is the twisted boundary phase which takes values from 0 to 2π and ψ_θ is the corresponding ground-state many-body wave function.

If the mixture has nontrivial topological properties, the states in the gap of the QPES should appear when the boundary condition changes from periodic boundary conditions (PBC) to open boundary conditions (OBC), and the in-gap modes mainly distribute near the edges of the chains. In addition, the Berry phase has a quantized value π.

3. Adding a boson to a half-filling fermionic topological system

One evidence of the topological properties is the existence of the edge states, i.e., states distributing mainly near the edges appear in the gap when the boundary becomes open. As has been stated, the edge states can be characterized by the QPES and the distribution of the quasi-particle when adding or removing a same kind of particle. It is also interesting to ask whether the edge states remain when adding a different kind of particle (such as hardcore boson) to a half-filling fermionic topological system.

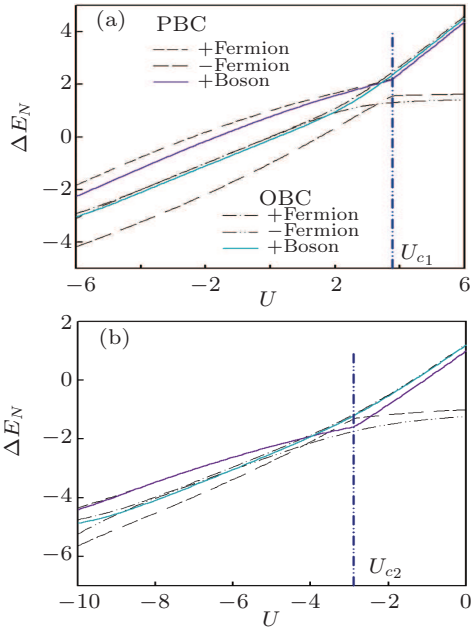


Fig. 1. (color online) The energies of the quasi-particle when adding a boson to a half-filling fermionic system with PBC and OBC. (a) The phase at $U = 0$ is topological ($M = -1$); (b) the phase at $U = 0$ is trivial ($M = 1$). For comparison, the corresponding ones of adding or removing a fermion are shown in dashed lines. Here, $U_{c_1} = 3.77$ and $U_{c_2} = -2.89$ are determined from the calculations of the Berry phase.

We firstly consider the non-interacting case. When a boson is added to a topological phase at the half-filling case ($M = -1$), as shown in Figs. 1(a) and 2(a) at $U = 0$, the excitation energy of the quasi-particle is in the gap, and its distribution is mainly near the edges. However, when a boson is added to a trivial phase [$M = 1$, see Figs. 1(b) and 2(b) at $U = 0$], there is no in-gap state and the distribution has a bulk-like feature. The results show that even in the mixture, the system still has topological properties. We further confirm the above results in the presence of on-site Hubbard interactions.

As has been known,[38] when starting from a topological phase at $U = 0$, the topological phase persists till a critical value $U_{c_1} > 0$ (repulsive). While when the starting phase is trivial at $U = 0$, the topological phase can be generated by a large enough attractive interaction. In Fig. 1, we show the QPES when adding a boson in (E, U) plane and the range where in-gap states appear is consistent with the topological properties of the interacting fermionic system. In Fig. 2, we show the distribution of the quasi-particle at several typical values of the interactions and the results give a further confirmation.

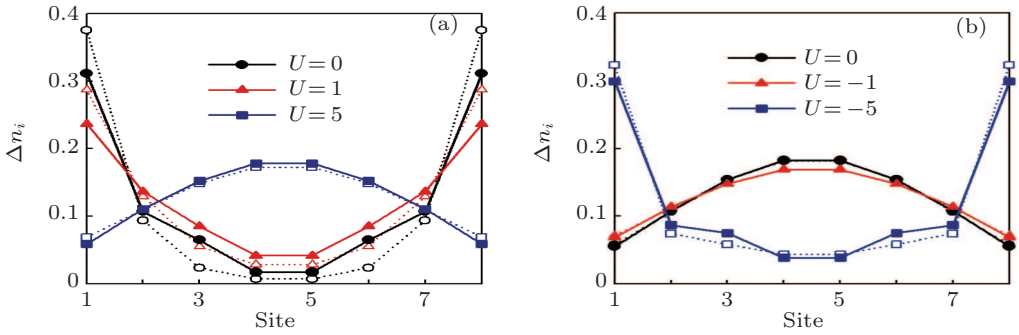


Fig. 2. (color online) The distribution of the quasi-particle when adding a boson to the half-filling fermionic system with OBC. (a) $M = -1$; (b) $M = 1$. For comparison, the corresponding ones of adding a fermion are shown in dotted lines.

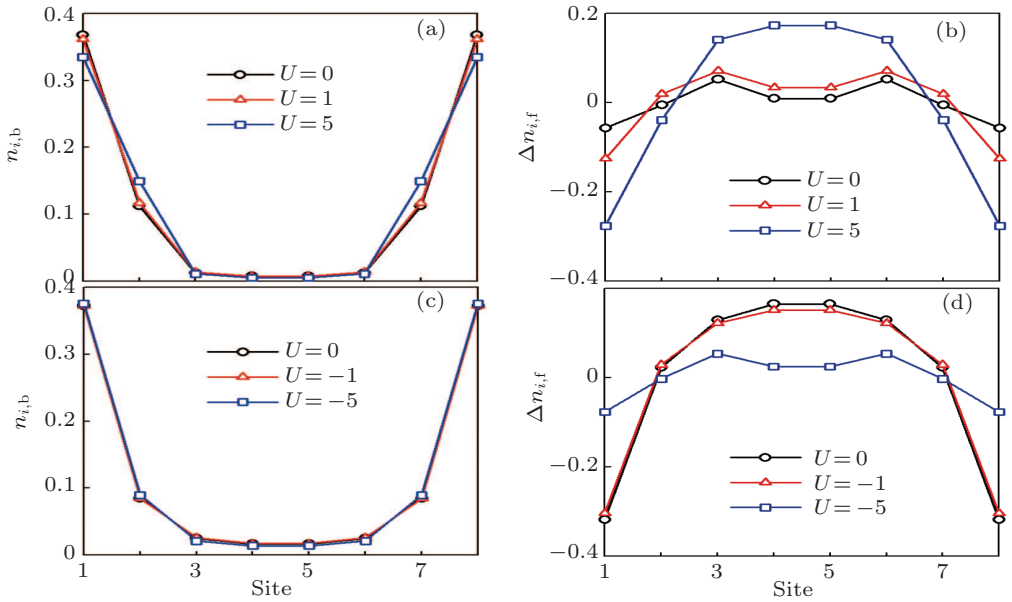


Fig. 3. (color online) The distribution of the added boson (a) $M = -1$ and (c) $M = 1$; the variation of the distribution of the fermionic system due to adding the boson (b) $M = -1$ and (d) $M = 1$. For each line in (b) and (d), the sum of the values is zero.

Till now, we show that when adding a boson to a half-filling fermionic topological system with OBC, the mixture still has topological properties. To further understand the nature of the edge states, it is helpful to study the distribution of the added boson and the induced variation of the distribution of the fermionic system. The results at $M = -1$ are shown in Figs. 3(a) and 3(b). As shown in Fig. 3, the added boson mainly distributes near the edges no matter whether the system is topological or trivial. However, the remaining fermionic system will be affected dramatically by the added boson depending on the topological properties of the system.

The above calculations are carried on a system of $N = 8$ sites with 8 fermions and 1 boson. For PBC, both the fermions and boson are uniformly distributed on the chain. When the chain is broken and edges appear, the boson is pushed to the edges, which is not due to the topological properties but the tendency for phase separation in the mixture. The response of the fermionic system depends on the topological properties. For the cases of $U = 0$ and $U = 1$ in Fig. 3(b), when the system is in a topological phase, the distribution of the 8 fermions is weakly perturbed and thus combined with the hardcore boson at the edges, and edges states appear. While for the case of $U = 5$ shown in Fig. 3(b), when the system is in a trivial phase, the distribution of the 8 fermions is strongly perturbed both in the bulk and at the edges, which makes the whole distribution of the quasi-particle becomes bulk like. We also calculate the

case of $M = 1$ and the results are similar, which are shown in Figs. 3(c) and 3(d).

4. Topological properties of a hardcore Bose–Fermi mixture at half-filling

Next, we study the topological properties of a hardcore Bose–Fermi mixture. Only the half-filling case of 7 fermions and 1 boson on a $N = 8$ chain is considered since the ground state becomes degenerate if there are more bosons. We first study the non-interacting case. Figure 4 shows the lowest two eigen-energies of the mixture for PBC and the Berry phase of the ground-state. For $-3.94 < M < -0.06$, the ground-state is gapped and its Berry phase has a quantized value π. Therefore, the system is in a topological phase for $-3.94 < M < -0.06$, which is slightly different from that of a fermionic system. Compared with the pure fermionic system, we notice that: the gap separating the ground-state is greatly reduced in the topological region; the ground-state becomes degenerate outside the topological region. The difference between the topological and trivial regions is the value of M. For the trivial case, the term $(M + 2B)\sigma_z$ in Eq. (1) is larger, and thus makes the orbits c and d more different from each other, which may be the reason for the vanishing gap here.

Then, we study the effect of interactions on the topological properties of the mixture. We consider two typical cases: (i) $M = -1$ when the mixture is in a topological phase at $U = 0$ and (ii) $M = 1$ when the mixture is in a trivial phase. By calculating the Berry phase of the mixture, it is found that for the former case the topological phase is broken by a repulsive interaction with the critical value $U'_{c_1} = 3.65$ while for the latter case it can be generated by an attractive one and the critical value is $U'_{c_2} = -3.21$. We calculate the QPES in (E, U) plane. For the mixture, the added or removed particle can be a fermion or a hardcore boson. For both cases, in-gap states appear at U where the Berry phase is π (see Fig. 5). The above results are further confirmed by the distribution of the quasi-particle on the system with OBC.

These results show that replacing a fermion with a hardcore boson almost does not change the topological properties of a 1D system. A qualitative explanation can be given by mapping the hardcore boson to the spinless fermion with Matsubara–Matsuda and Jordan–Wigner transformations.[48,49] Since there is only one boson in the system, the phases in the transformation will vanish. However, the constraint Eq. (2) will modify the transformations. Since there is one particle averagely on each site in the mixture, the modification is small. Thus, the hardcore boson is similar to a fermion and the mixture behaves like a pure fermionic system.

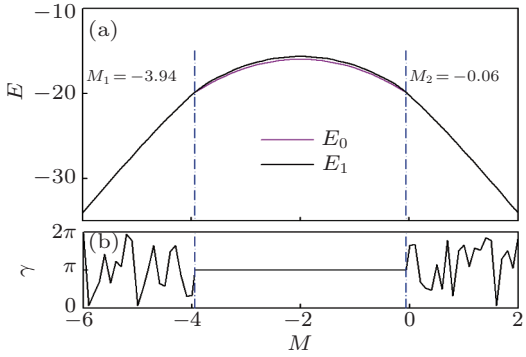


Fig. 4. (color online) (a) The lowest two energies of the mixture for PBC versus M. (b) The Berry phase versus M. The vertical dashed lines denote the critical values. Here, the interaction $U = 0$. The randomness of the Berry phase outside the topological phase is due to the degeneracy of the ground-state.

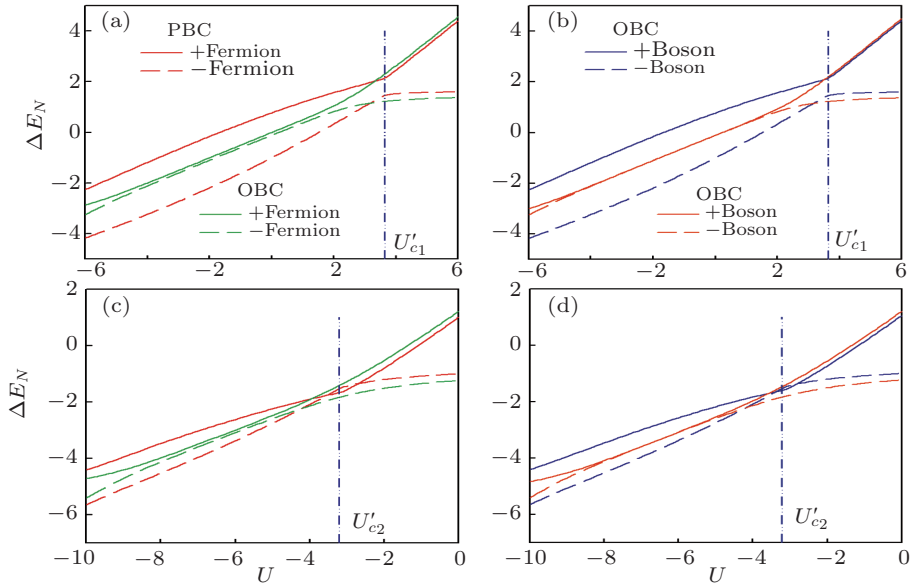


Fig. 5. (color online) The energies of the quasi-particle for PBC and OBC. (a) and (b) $M = -1$; (c) and (d) $M = 1$; (a) and (c) the added or removed particle is fermion; (b) and (d) it is a hardcore boson.

5. Conclusion

In conclusion, we study the topological properties of the 1D hardcore Bose–Fermi mixture by calculating the QPES, the distributions of the quasi-particle, and the Berry phase using ED method. Firstly, we probe the edge states using a hardcore boson. When it is added to a topological fermion system, the quasi-particle shows the features of edge states. While for the case of trivial fermionic system, the quasi-particle is bulk like. The results show that a system manifests its topological properties even in the presence of a different kind of particle. Then, we study a mixture with 7 fermions and 1 boson. It is found that the mixture also has topological properties and its behaviors are similar to that of the corresponding fermionic system except for different critical values. We present a qualitative explanation to understand such behaviors using the mapping between the hardcore boson and the spinless fermion.

These results show the existence of topological properties in a 1D hardcore Bose–Fermi mixture. The cases in higher dimensions are also interesting, which will be further researched later. Due to the rapid development in the field of cold atoms, a Bose–Fermi mixture in an optical lattice has been realized experimentally. It is very hopeful that the model Eq. (1) is engineered and the results are realized experimentally.

References

[1] Kane C L and Mele E J 2005 *Phys. Rev. Lett.* **95** 146802
[2] Kane C L and Mele E J 2005 *Phys. Rev. Lett.* **95** 226801
[3] Moore J E 2010 *Nature* **464** 194
[4] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
[5] Qi X L and Zhang S C 2011 *Rev. Mod. Phys.* **83** 1057
[6] Fu L and Kane C L 2008 *Phys. Rev. Lett.* **100** 096407
[7] Seradjeh B, Moore J E and Franz M 2009 *Phys. Rev. Lett.* **103** 066402
[8] Wang Z M 2011 *Acta Phys. Sin.* **60** 094205 (in Chinese)
[9] Ren Ji Rong, Zhu T and Duan Y S 2008 *Chin. Phys. Lett.* **25** 367
[10] Li K, Sayipjamal D and Wang J H 2008 *Chin. Phys. B* **17** 1716
[11] Varney C N, Sun K, Rigol M and Galitski V 2010 *Phys. Rev. B* **82** 115125
[12] Hohenadler M, Lang T C and Assaad F F 2011 *Phys. Rev. Lett.* **106** 100403
[13] Yu S L, Xie X C and Li J X 2011 *Phys. Rev. Lett.* **107** 010401
[14] Zheng D, Zhang G M and Wu C J 2011 *Phys. Rev. B* **84** 205121
[15] Yamaji Y and Imada M 2011 *Phys. Rev. B* **83** 205122
[16] Liu J B, Li J H, Song P J and Li W B 2008 *Chin. Phys. B* **17** 38
[17] Wang Z, Qi X L and Zhang S C 2010 *Phys. Rev. Lett.* **105** 256803
[18] Wang Z, Qi X L and Zhang S C 2012 *Phys. Rev. B* **85** 165126
[19] Wang Z and Zhang S C 2012 *Phys. Rev. X* **2** 031008
[20] Gurarie V 2011 *Phys. Rev. B* **83** 085426
[21] Fidkowski L and Kitaev A 2010 *Phys. Rev. B* **81** 134509
[22] Tang E and Wen X G 2012 *Phys. Rev. Lett.* **109** 096403
[23] Tang E, Mei J W and Wen X G 2011 *Phys. Rev. Lett.* **106** 236802
[24] Neupert T, Santos L, Chamon C and Mudry C 2011 *Phys. Rev. Lett.* **106** 236804
[25] Sun K, Gu Z, Katsura H and Das Sarma S 2011 *Phys. Rev. Lett.* **106** 236803
[26] Wang F and Ran Y 2011 *Phys. Rev. B* **84** 241103
[27] Hu X, Kargarian M and Fiete G A 2011 *Phys. Rev. B* **84** 155116
[28] Neupert T, Santos L, Ryu S, Chamon C and Mudry C 2011 *Phys. Rev. B* **84** 165107
[29] Weeks C and Franz M 2012 *Phys. Rev. B* **85** 041104
[30] Sheng D N, Gu Z C, Sun K and Sheng L 2011 *Nature Commun.* **2** 389
[31] Regnault N and Bernevig B A 2011 *Phys. Rev. X* **1** 021014
[32] Wu Y L, Bernevig B A and Regnault N 2012 *Phys. Rev. B* **85** 075116
[33] Wang Y F, Gu Z C, Gong C D and Sheng D N 2011 *Phys. Rev. Lett.* **107** 146803
[34] Bloch I, Dalibard J and Zwerger W 2008 *Rev. Mod. Phys.* **80** 885
[35] Goldman N, Urban D F and Bercioux D 2011 *Phys. Rev. A* **83** 063601
[36] Stanescu T D, Galitski V and Das Sarma S 2010 *Phys. Rev. A* **82** 013608
[37] Goldman N, Satija I, Nikolic P, Bermudez A, Martin-Delgado M A, Lewenstein M and Spielman I B 2010 *Phys. Rev. Lett.* **105** 255302
[38] Liu X J, Liu X, Wu C and Sinova J 2010 *Phys. Rev. A* **81** 033622
[39] Günter K, Stöferle T, Moritz H, Köhl M and Esslinger T 2006 *Phys. Rev. Lett.* **96** 180402
[40] Ospelkaus S, Ospelkaus C, Humbert L, Sengstock K and Bongs K 2006 *Phys. Rev. Lett.* **97** 120403
[41] Guo H M, Shen S Q and Feng S P 2012 *Phys. Rev. B* **86** 085124
[42] Guo H M and Shen S Q 2011 *Phys. Rev. B* **84** 195107
[43] Resta R 1994 *Rev. Mod. Phys.* **66** 899
[44] Xiao D, Chang M C and Niu Q 2010 *Rev. Mod. Phys.* **82** 1959
[45] Niu Q, Thouless D J and Wu Y S 1985 *Phys. Rev. B* **31** 3372
[46] Alavi S A 2003 *Chin. Phys. Lett.* **20** 605
[47] Yan X B 2007 *Chin. Phys. Lett.* **24** 2170
[48] Matsubara T and Matsuda H 1956 *Prog. Theor. Phys.* **16** 569
[49] Jordan P and Wigner E 1928 *Z. Phys.* **47** 631