Spin and pseudospin symmetric Dirac particles in the field of Tietz–Hua potential including Coulomb tensor interaction*

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(Received 11 December 2012; revised manuscript received 24 February 2013)

Approximate analytical solutions of the Dirac equation for Tietz–Hua (TH) potential including Coulomb-like tensor (CLT) potential with arbitrary spin–orbit quantum number κ are obtained within the Pekeris approximation scheme to deal with the spin–orbit coupling terms $\kappa(\kappa\pm1)r^{-2}$. Under the exact spin and pseudospin symmetric limitation, bound state energy eigenvalues and associated unnormalized two-component wave functions of the Dirac particle in the field of both attractive and repulsive TH potential with tensor potential are found using the parametric Nikiforov–Uvarov (NU) method. The cases of the Morse oscillator with tensor potential, the generalized Morse oscillator with tensor potential, and the non-relativistic limits have been investigated.

Keywords: Dirac equation, Tietz–Hua (TH) potential, Coulomb-like tensor (CLT) potential, generalized Morse potential

PACS: 03.65.Ge, 03.65.Fd, 03.65.Pm, 02.30.Gp **DOI:** 10.1088/1674-1056/22/9/090305

1. Introduction

The spin and the pseudospin (p-spin) symmetries of the Dirac Hamiltonian were discovered many years ago. However, these symmetries have recently been recognized empirically in nuclear and hadronic spectroscopes.[1] In the framework of the Dirac equation, the p-spin symmetry is used to characterize the deformed nuclei and superdeformation to establish an effective shell model.[2–4] On the other hand, the spin symmetry is relevant for mesons.[5] With spin symmetry, the radial scalar potential $S(r)$ and vector potential $V(r)$ are nearly equal, $S(r)\approx V(r)$, whereas the p-spin symmetry occurs when $S(r)\approx -V(r)$.[6,7] The p-spin symmetry refers to a quasi-degeneracy of single nucleon doublets with non-relativistic quantum number $(n, l, j=l+1/2)$ and $(n-1, l+2, j=l+3/2)$, where n, l, and j are single nucleon radial, orbital and total angular quantum numbers, respectively.[8,9] The total angular momentum is $j=\tilde{l}+\tilde{s}$, where $\tilde{l}=l+1$ is the pseudo-angular momentum, and $\tilde{s}$ is the p-spin angular momentum.[10]

Lisboa *et al.*[11] have studied a generalized relativistic harmonic oscillator for spin-1/2 fermions by solving the Dirac equation with quadratic vector and scalar potentials including a linear tensor potential with spin and p-spin symmetry. Furthermore, Akcay[12] has shown that the Dirac equation for scalar and vector quadratic potentials and Coulomb-like tensor potential with spin and p-spin symmetry can be solved exactly. In these studies, it has been found out that the tensor interaction removes the degeneracy between two states in the p-spin doublets. The tensor coupling under the spin and p-spin symmetry has also been studied in Refs. [13] and [14]. Furthermore, the nuclear properties have been studied by using tensor couplings.[15,16] Very recently, various types of potentials like Hulthén[10] and Woods-Saxon[17] including Coulomb-like potential have been studied with the conditions of spin and p-spin symmetry.

The Tietz–Hua (TH) oscillatory potential is one of the best molecular potentials to describe the vibration energy spectra of diatomic molecules.[18,19] It is much more realistic than the Morse oscillator in the description of molecular dynamics at moderate and high rotation–vibration quantum numbers.[20–22] Kunc and Gordillo-Vázquez[23] derived the analytical expressions for the rotation–vibration energy levels of diatomic molecules represented by the TH rotating oscillator potential using the Hamilton–Jacobi theory and the Bohr–Sommerfeld quantization rule. This potential takes the following form:

$$V_{\rm TH}(r)=D\left[\frac{1-{\rm e}^{-b_{\rm h}(r-r_{\rm e})}}{1-c_{\rm h}\,{\rm e}^{-b_{\rm h}(r-r_{\rm e})}}\right]^2,\quad b_{\rm h}=a(1-c_{\rm h}),\qquad(1)$$

where the parameters r, $r_{\rm e}$, $b_{\rm h}$, D, and $c_{\rm h}$ are the internuclear distance, the molecular bond length, the Morse constant, the potential well depth, and the potential constant, respectively.[23] In the limit when the potential constant $c_{\rm h}$ approaches zero, the TH oscillatory potential turns into the Morse potential.[24] In Fig. 1, we draw the potential given by Eq. (1) for three different types of molecular potentials,

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namely, the TH, the Morse, and the generalized Morse potentials. The following set of parameter values are used: $r_e = 0.4$ fm, $c_h = 0.1$, $D = 15.0$ fm^{-1}, and $b_h = 0.8$ fm^{-1}.

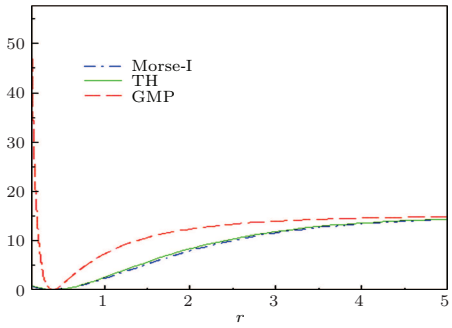


Fig. 1. (color online) Shapes of potentials discussed in this work.

Over the past few years, the Nikiforov–Uvarov (NU) method[25] has been proved to be a powerful tool in solving the second-order differential equation. It has been applied successfully to a large number of potential models.[26–32] This method has also been used to solve the spinless (spin-0) Schrödinger[33–37] and Klein–Gordon (KG)[38–42] equations and also relativistic spin-1/2 Dirac equations[43–47] with different potential models.

Since the relativistic solution is indispensable, we need to solve the Dirac equation with a flexible parameter TH molecular potential model including a Coulomb-like tensor (CLT) potential coupling. However, the Dirac–TH–CLT problem can no longer be solved in a closed form due to the existence of spin–orbit coupling centrifugal term $\kappa(\kappa\pm1)r^{-2}$, so it is necessary to resort to approximate methods. Therefore, we use the Pekeris approximation scheme[48] to deal with this term and solve approximately the Dirac equation with the TH rotational potential and CLT coupling potential for arbitrary spin–orbit quantum number κ.

The aim of the present work is to obtain the approximate relativistic bound state energy eigenvalue equation and the corresponding unnormalized two-component wave function of the spin-1/2 Dirac particle moving in the field of the TH rotating oscillatory potential with tensor coupling potential using the concepts of parametric NU method.[49] The spin and p-spin symmetric limitations are studied. Furthermore, we consider the Morse oscillator case,[50–52] the generalized Morse oscillator case,[49] and the non-relativistic solution.[53]

The rest of this paper is organized as follows. In Section 2, we briefly introduce the Dirac equation with radial scalar and vector potentials coupled with a radial tensor potential. In Section 3, with spin and p-spin symmetries, we solve the Dirac equation for the scalar-vector TH potential coupled with CLT potential to obtain the approximate relativistic energy eigenvalue equation and the corresponding two component wave function for arbitrary spin–orbit quantum number κ using a short version of the NU method. A new Pekeris approximation scheme is used to deal with the spin–orbit centrifugal and pseudo-centrifugal terms. In Section 4, we consider some particular cases of much interest from our solution. Finally, our final remarks and conclusion are given in Section 5.

2. Dirac equation with a tensor coupling

The Dirac equation for a particle of mass $m$moving in the field of attractive radial scalar $S(r)$, repulsive vector $V(r)$, and tensor $U(r)$ potentials (in the relativistic units $\hbar = c = 1$) takes the following form:[11]

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\boldsymbol{\beta}(m+S(r))+V(r)-\mathrm{i}\boldsymbol{\beta}\boldsymbol{\alpha}\cdot\boldsymbol{r}U(r)]\,\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}), \tag{2}$$

where E is the relativistic energy of the system, $\boldsymbol{p} = -\mathrm{i}\boldsymbol{\nabla}$ is the three-dimensional (3D) momentum operator, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ represent the usual 4×4 usual Dirac matrices which are composed of the three 2×2 Pauli matrices and the 2×2 unit matrix. For spherical nuclei, the Dirac Hamiltonian commutes with the total angular momentum operator $\boldsymbol{J} = \boldsymbol{L}+\boldsymbol{S}$ and the spin–orbit coupling operator $\boldsymbol{K} = -\boldsymbol{\beta}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, where $\boldsymbol{L}$ and $\boldsymbol{S}$ are the orbital and spin momentum, respectively. The eigenvalues of the spin–orbit coupling operator are $\kappa = l > 0$ and $\kappa = -(l+1) < 0$ for unaligned spin $(j = l-1/2)$ and aligned spin $(j = l+1/2)$, respectively. Thus, the Dirac wave function takes the following form:

$$\psi_{n\kappa}(\boldsymbol{r}) = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r)\mathrm{Y}^{l}_{jm}(\theta,\phi) \\ \mathrm{i}G_{n\kappa}(r)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)\end{pmatrix}, \tag{3}$$

where $\mathrm{Y}^{l}_{jm}(\theta,\phi)$ and $\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)$ are the spin and p-spin spherical harmonics, respectively, $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the upper- and lower-spinor radial functions, respectively. Inserting Eq. (3) into Eq. (2) and making use of the following relations:[54]

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}+\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}), \tag{4a}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = \boldsymbol{\sigma}\cdot\boldsymbol{r}\left(\boldsymbol{r}\cdot\boldsymbol{p}+\mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}\right), \tag{4b}$$

and properties

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi) = (\kappa-1)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi), \tag{5a}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{l}_{jm}(\theta,\phi) = -(\kappa+1)\mathrm{Y}^{l}_{jm}(\theta,\phi), \tag{5b}$$

$$\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}}{r}\right)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi) = -\mathrm{Y}^{l}_{jm}(\theta,\phi), \tag{5c}$$

$$\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}}{r}\right)\mathrm{Y}^{l}_{jm}(\theta,\phi) = -\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi), \tag{5d}$$

we obtain the following two coupled differential equations satisfying the upper and lower radial functions $F_{n\kappa}(r)$and $G_{n\kappa}(r)$:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)=(m+E-\Delta(r))\,G_{n\kappa}(r), \quad (6a)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r)=(m-E+\Sigma(r))\,F_{n\kappa}(r), \quad (6b)$$

where the sum and difference potentials, respectively, are defined by

$$\Sigma(r)=V(r)+S(r), \quad \Delta(r)=V(r)-S(r). \quad (7)$$

Combining Eqs. (6a) and (6b), we obtain the second-order differential equations satisfying the radial functions, $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$, as

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\left(\frac{2\kappa}{r}-U(r)-\frac{\mathrm{d}}{\mathrm{d}r}\right)U(r)\right.\\&-[M+E_{n\kappa}-\Delta(r)][M-E_{n\kappa}+\Sigma(r)]\\&\left.+\frac{1}{M+E_{n\kappa}-\Delta(r)}\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)\right\}\\&\times F_{n\kappa}(r)=0, \quad (8)\\&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\left(\frac{2\kappa}{r}-U(r)+\frac{\mathrm{d}}{\mathrm{d}r}\right)U(r)\right.\\&-[M+E_{n\kappa}-\Delta(r)][M-E_{n\kappa}+\Sigma(r)]\\&\left.-\frac{1}{M-E_{n\kappa}+\Sigma(r)}\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)\right\}\\&\times G_{n\kappa}(r)=0, \quad (9)\end{aligned}$$

where $\kappa(\kappa+1)=l(l+1)$ and $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$. These radial wave equations are required to satisfy the necessary boundary conditions in the interval $r\in(0,\infty)$, i.e., $F_{n\kappa}(r=0)=G_{n\kappa}(0)=0$, $F_{n\kappa}(r\to\infty)\simeq 0$, $G_{n\kappa}(r\to\infty)\simeq 0$.

3. Dirac bound states

3.1. Spin symmetric limit

Under the exact spin symmetric condition, we take the sum potential $\Sigma(r)$ as the TH potential, the difference potential $\Delta(r)$ as a constant, and the tensor potential $U(r)$ as Coulomb-like potential. Then, we have the following forms:

$$\Sigma(r)=D\left(\frac{1-\mathrm{e}^{-b_{\mathrm{h}}(r-r_{\mathrm{e}})}}{1-c_{\mathrm{h}}\,\mathrm{e}^{-b_{\mathrm{h}}(r-r_{\mathrm{e}})}}\right)^2,$$

$$\Delta(r)=C_{\mathrm{s}}, \quad U(r)=-\frac{H}{r}, \quad (10)$$

where C_{s} and $H=Ze^2/(4\pi\varepsilon_0)$ are two constants. Inserting Eq. (10) into Eq. (8), we can obtain an equation satisfying the upper component as

$$\begin{aligned}&\frac{\mathrm{d}^2F_{n\kappa}(r)}{\mathrm{d}r^2}-\left[\frac{\gamma}{r^2}-\tilde{E}_{n\kappa}^2\left(\frac{1-\mathrm{e}^{-b_{\mathrm{h}}(r-r_{\mathrm{e}})}}{1-c_{\mathrm{h}}\,\mathrm{e}^{-b_{\mathrm{h}}(r-r_{\mathrm{e}})}}\right)^2\right]\\&\times F_{n\kappa}(r)=0, \quad (11a)\\&\gamma=(\kappa+H)(\kappa+H+1), \quad (11b)\\&\tilde{D}=D(E_{n\kappa}+m-C_{\mathrm{s}}), \quad (11c)\\&\tilde{E}_{n\kappa}^2=(E_{n\kappa}+m-C_{\mathrm{s}})(E_{n\kappa}-m). \quad (11d)\end{aligned}$$

As seen in Eq. (11b), the introduction of the tensor potential has an effect on changing the spin–orbit quantum number into a new spin–orbit quantum number (i.e., $\kappa\to\kappa+H$). Furthermore, equation (11a) has an exact rigorous solution only for the state with $\kappa=-(1+H)$ due to the existence of the strong singular new spin–orbit coupling term γr^{-2}.[55] However, when this term is taken into account, the corresponding radial Dirac equation can no longer be solved in a closed form and it is necessary to resort to approximate methods. The most widely known approximation method is the Pekeris approximation,[48] in which the new spin–orbit coupling term γr^{-2} is expanded in terms of singular exponential functions $\mathrm{e}^{-r/a}$ which are compatible with the solvability of the problem for $r\ll a$. Because of this singularity, the validity of such an approximation is limited to only a few of the lowest energy states.[47]

Now we need to perform a new approximation for this spin–orbit coupling term as a function of the relevant TH rotating potential parameters as shown in Appendix A. This approximation in the limit where $c_{\mathrm{h}}=0$ is reduced to the usual approximation used for the Morse potential case.[50,51,56] Thus, employing the approximation scheme derived in Appendix A and making an appropriate change of variables: $x=(r-r_{\mathrm{e}})/r_{\mathrm{e}}\in(-1,\infty)$, we can then rewrite Eq. (11a) as

$$\begin{aligned}&\frac{\mathrm{d}^2F_{n\kappa}(x)}{\mathrm{d}x^2}+\left[\omega^2-\gamma\left(D_0+D_1\frac{\mathrm{e}^{-\alpha x}}{1-c_{\mathrm{h}}\,\mathrm{e}^{-\alpha x}}\right.\right.\\&\left.+D_2\frac{\mathrm{e}^{-2\alpha x}}{(1-c_{\mathrm{h}}\,\mathrm{e}^{-\alpha x})^2}\right)-\nu^2\left(\frac{1-\mathrm{e}^{-\alpha x}}{1-c_{\mathrm{h}}\,\mathrm{e}^{-\alpha x}}\right)^2\Bigg]\\&\times F_{n\kappa}(x)=0, \quad (12a)\\&\nu^2=\tilde{D}r_{\mathrm{e}}^2, \quad \omega^2=\tilde{E}_{n\kappa}^2r_{\mathrm{e}}^2, \quad \alpha=b_{\mathrm{h}}r_{\mathrm{e}}, \quad (12b)\end{aligned}$$

where the explicit forms of the constants D_{i} $(i=1,2,3)$ are defined in Appendix A and are expressed in terms of the potential parameters $(c_{\mathrm{h}}, b_{\mathrm{h}}, r_{\mathrm{e}})$. Setting a new variable $s(x)=\mathrm{e}^{-\alpha x}\in(\mathrm{e}^{b_{\mathrm{h}}r_{\mathrm{e}}},0)$, we can decompose the spin-symmetric Dirac equation (12a) into the Schrödinger-type equation satisfying the upper-spinor component $F_{n,\kappa}(s)$,

$$\begin{aligned}&\frac{\mathrm{d}^2F_{n,\kappa}(s)}{\mathrm{d}s^2}+\frac{(1-c_{\mathrm{h}}s)}{s(1-c_{\mathrm{h}}s)}\frac{\mathrm{d}F_{n,\kappa}(s)}{\mathrm{d}s}\\&+\frac{-As^2+Bs-C}{s^2(1-c_{\mathrm{h}}s)^2}F_{n,\kappa}(s)=0, \quad F_{n,\kappa}(0)=0, \quad (13)\end{aligned}$$

where $F_{n\kappa}(x)\equiv F_{n,\kappa}(s)$ has been used. Equation (13) is now amendable to the application of the Nikiforov–Uvarov (NU) method.[25] In Appendix B, we will introduce a shortcut to the NU method which has recently been derived in Ref. [57] to make the solution of the hypergeometric second-order differential equation simple, short, and elegant. This parametric NU

method[46,49,58–61] avoids the difficulty in choosing the essential parameters and the root which are necessary to obtain the energy eigenvalues and wave functions. Now, if one compares Eq. (13) with its counterpart (B2), the following constant parameters are identified as:

$$A = \frac{1}{\alpha^2}\left[\left(\gamma D_0 - \omega^2\right)c_{\rm h}^2 - \gamma D_1 c_{\rm h} + \gamma D_2 + \nu^2\right], \quad (14a)$$

$$B = \frac{1}{\alpha^2}\left[2\left(\gamma D_0 - \omega^2\right)c_{\rm h} - \gamma D_1 + 2\nu^2\right], \quad (14b)$$

$$C = \frac{1}{\alpha^2}\left[\gamma D_0 - \omega^2 + \nu^2\right], \quad (14c)$$

and also the specific values for the coefficients,

$$c_1 = 1, \;\; c_2 = c_{\rm h}, \;\; c_3 = c_{\rm h}. \quad (15)$$

Furthermore, in order to obtain the bound state solutions of Eq. (13), it is necessary to calculate the remaining parametric coefficients, that is, $c_{\rm i}$ $(i = 4, 5, ..., 13)$ by means of relation (B5). Thus, their specific values are displayed in Table 1 for the relativistic TH potential with a tensor coupling potential. The essential polynomial functions and root k can be calculated through the relations (B6)–(B9) as

$$\begin{aligned}
\pi(s) &= \frac{1}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2} \\
&\quad - \left[\frac{c_{\rm h}}{2} + \frac{1}{2}\sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]}\right. \\
&\quad \left. + \frac{c_{\rm h}}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2}\right] s, \\
k &= -\frac{1}{\alpha}\left[\gamma D_1 - 2\nu^2(1-c_{\rm h})\right] \\
&\quad - \frac{1}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2} \\
&\quad \times \sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]}, \\
\tau(s) &= 1 + \frac{1}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2} \\
&\quad - 2\left[c_{\rm h} + \frac{1}{2}\sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]}\right. \\
&\quad \left. + \frac{c_{\rm h}}{2}\sqrt{\gamma D_0 + \nu^2 - \omega^2}\right] s, \\
\tau'(s) &= -2\left[c_{\rm h} + \frac{1}{2}\sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]}\right. \\
&\quad \left. + \frac{c_{\rm h}}{2}\sqrt{\gamma D_0 + \nu^2 - \omega^2}\right] < 0. \quad (16)
\end{aligned}$$

Next, employing these parametric coefficients along with the relation (B10), we can readily obtain the analytic form of the energy equation for the Dirac–TH–CLT problem as

$$\begin{aligned}
&\left(n + \frac{1}{2}\right)^2 c_{\rm h} + \frac{c_{\rm h}}{4} + \left(n + \frac{1}{2}\right) \\
&\times \left[\sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]}\right. \\
&\left. + \frac{2c_{\rm h}}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2}\right] + \frac{1}{\alpha^2}\left[\gamma D_1 - 2\nu^2(1-c_{\rm h})\right] \\
&+ \frac{1}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2}\sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]} \\
&= 0. \quad (17)
\end{aligned}$$

It can also be rearranged in a more convenient form for the $c_{\rm h} \neq 0$ case as

$$\begin{aligned}
&c_{\rm h}\left(n + \frac{1}{2}\right) + \frac{c_{\rm h}}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2} \\
&+ \frac{1}{2}\sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]} \\
&= \frac{\sigma}{\alpha^2}\sqrt{\left(\gamma D_0 - \omega^2\right)c_{\rm h}^2 - \gamma D_1 c_{\rm h} + \gamma D_2 + \nu^2}, \quad (18)
\end{aligned}$$

where $\sigma = \pm 1$. Recalling

$$\begin{aligned}
\gamma &= (\kappa + H)(\kappa + H + 1), \\
\nu^2 &= r_{\rm e}^2 D\left(E_{n\kappa} + m - C_{\rm s}\right), \\
\omega^2 &= r_{\rm e}^2\left(E_{n\kappa} + m - C_{\rm s}\right)\left(E_{n\kappa} - m\right),
\end{aligned}$$

one can obtain the implicit dependence of the above energy equation formula on energy $E_{n\kappa}$.

Table 1. Specific values of the coefficients for the spin-symmetric Dirac–TH problem with a Coulomb tensor potential.

Constant	Analytic value
c_4	0
c_5	$-\frac{c_{\rm h}}{2}$
c_6	$\frac{c_{\rm h}^2}{4} + \frac{1}{\alpha^2}\left[\left(\gamma D_0 - \omega^2\right)c_{\rm h}^2 - \gamma D_1 c_{\rm h} + \gamma D_2 + \nu^2\right]$
c_7	$-\frac{1}{\alpha^2}\left[2\left(\gamma D_0 - \omega^2\right)c_{\rm h} - \gamma D_1 + 2\nu^2\right]$
c_8	$\frac{1}{\alpha^2}\left[\gamma D_0 - \omega^2 + \nu^2\right]$
c_9	$\frac{c_{\rm h}^2}{4} + \frac{1}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]$
c_{10}	$\frac{2}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2}$
c_{11}	$\frac{1}{c_{\rm h}}\sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]}$
c_{12}	$\frac{1}{\alpha}\sqrt{\gamma D_0 + \nu^2 - \omega^2}$
c_{13}	$\frac{1}{2} + \frac{1}{2c_{\rm h}}\sqrt{c_{\rm h}^2 + \frac{4}{\alpha^2}\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]}$

On the other hand, in order to establish the upper-spinor component of the wave functions $F_{n,\kappa}(r)$, namely, Eq. (8), the relations (B11)–(B14) are used. Firstly, we find the first part of the wave function to be

$$\begin{aligned}
\phi(s) &= s^{\sqrt{\gamma D_0 + \nu^2 - \omega^2}/\alpha} \\
&\quad \times (1 - c_{\rm h} s)^{1/2 + \sqrt{c_{\rm h}^2 + (4/\alpha^2)\left[\gamma D_2 + \nu^2(1-c_{\rm h})^2\right]}/2c_{\rm h}}. \quad (19)
\end{aligned}$$

Secondly, we calculate the weight function as

$$\rho(s) = s^{2\sqrt{\gamma D_0 + \nu^2 - \omega^2}/\alpha}$$

$$\times (1-c_{\rm h}s)^{\sqrt{c_{\rm h}^2+(4/\alpha^2)\left[\gamma D_2+\nu^2(1-c_{\rm h})^2\right]}/c_{\rm h}}, \quad (20)$$

which gives the second part of the wave function as

$$y_n(s) = P_n^{\left(2\sqrt{\gamma D_0+\nu^2-\omega^2}/\alpha,\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\gamma D_2+\nu^2(1-c_{\rm h})^2\right]}/c_{\rm h}\right)} \times (1-2c_{\rm h}s), \quad (21)$$

where $P_n^{(a,b)}(1-2c_{\rm h}s)$ are the orthogonal Jacobi polynomial. Finally, the upper spinor component for arbitrary spin–orbit quantum number κ can be found through the relation (B14) as

$$\begin{aligned} F_{n\kappa}(r) &= N_{n\kappa}\,{\rm e}^{-\sqrt{\gamma D_0+\nu^2-\omega^2}(r-r_{\rm e})/r_{\rm e}} \\ &\times \left(1-c_{\rm h}\,{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right)^{\frac{1}{2}+\frac{1}{2c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\gamma D_2+\nu^2(1-c_{\rm h})^2\right]}} \\ &\times P_n^{\left(\frac{2}{\alpha}\sqrt{\gamma D_0+\nu^2-\omega^2},\frac{1}{c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\gamma D_2+\nu^2(1-c_{\rm h})^2\right]}\right)} \\ &\times \left(1-2c_{\rm h}\,{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right), \end{aligned} \quad (22)$$

where $N_{n\kappa}$ is the normalization constant. From the relation (B15), the upper spinor can be expressed in terms of the hypergeometric function as

$$\begin{aligned} F_{n\kappa}(r) &= N_{n\kappa}\frac{(a_0+1)_n}{n!}\,{\rm e}^{-\sqrt{\gamma D_0+\nu^2-\omega^2}(r-r_{\rm e})/r_{\rm e}} \\ &\times \left(1-c_{\rm h}\,{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right)^{\frac{1}{2}+\frac{1}{2c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\gamma D_2+\nu^2(1-c_{\rm h})^2\right]}} \\ &\times {}_2{\rm F}_1\left(-n,1+a_0+b_0+n;a_0+1;c_{\rm h}\,{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right), \end{aligned}$$

where

$$a_0 = \frac{2}{\alpha}\sqrt{\gamma D_0+\nu^2-\omega^2},$$
$$b_0 = \frac{1}{c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\gamma D_2+\nu^2(1-c_{\rm h})^2\right]}.$$

The lower-spinor component of the wave function can also be calculated by using Eq. (6a). Introducing the following recurrence relation:

$$\frac{\rm d}{{\rm d}z}\left({}_2{\rm F}_1\left(a,b;c;z\right)\right) = \left(\frac{ab}{c}\right){}_2{\rm F}_1\left(a+1,b+1;c+1;z\right),$$

one can easily obtain the lower spinor $G_{n\kappa}(r)$ as

$$\begin{aligned} &G_{n\kappa}(r) \\ &= N_{n\kappa}\frac{(a_0+1)_n}{n!}\frac{1}{(m+E_{n\kappa}-C_{\rm s})} \\ &\times \left\{\left(\frac{(\kappa+A)}{r}-\frac{1}{r_r}\sqrt{\gamma D_0+\nu^2-\omega^2}\right.\right. \\ &\left.+\frac{\alpha\left(c_{\rm h}+\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\gamma D_2+\nu^2(1-c_{\rm h})^2\right]}\right)\,{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}}{2r_{\rm e}(1-c_{\rm h}\,{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}})}\right) \\ &\times F_{n\kappa}(r)+\frac{\alpha c_{\rm h}n}{r_{\rm e}}\left(\frac{1+a_0+b_0+n}{a_0+1}\right) \\ &\times {\rm e}^{-\sqrt{\gamma D_0+\nu^2-\omega^2}(r-r_{\rm e})/r_{\rm e}} \\ &\times \left(1-c_{\rm h}\,{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right)^{\frac{1}{2}+\frac{1}{2c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\gamma D_2+\nu^2(1-c_{\rm h})^2\right]}} \\ &\times {}_2{\rm F}_1\left(-n+1,a_0+b_0+n+2;a_0+2;c_{\rm h}\,{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right)\Bigg\}, \end{aligned} \quad (23)$$

where $E \neq -m+C_{\rm s}$. In the presence of the exact spin symmetry ($C_{\rm s}=0$), there are no bound negative energy states.

In addition, to calculate the bound state solutions of the Dirac equation with the Morse potential including the CLT potential, we follow the NU method and the procedures of solution explained in Refs. [50], [51], and [56], where the nonrelativistic and relativistic solutions of the Morse potential were found. The bound state energy eigenvalue equation and the wave function with spin symmetry are shortly derived in Appendix C because of its importance.

3.2. Pseudospin symmetric limit

In this part, we will consider the exact p-spin symmetric solution of the Dirac equation for the TH potential and CLT potential. We take the following potentials:

$$\Sigma(r) = C_{\rm ps},\ \ \Delta(r) = D\left(\frac{1-{\rm e}^{-b_{\rm h}(r-r_{\rm e})}}{1-c_{\rm h}\,{\rm e}^{-b_{\rm h}(r-r_{\rm e})}}\right)^2,$$
$$U(r) = -\frac{H}{r}, \quad (24)$$

where $C_{\rm ps}$ and H are constants. Inserting the above potentials into Eq. (9), we obtain the equation satisfying the lower radial component of the wave function in the following form:

$$\frac{{\rm d}^2G_{n\kappa}(r)}{{\rm d}r^2}-\left[\frac{\tilde\gamma}{r^2}-\tilde{\tilde E}_{n\kappa}^2+\tilde{\tilde D}\left(\frac{1-{\rm e}^{b_{\rm h}(r-r_{\rm e})}}{1-c_{\rm h}\,{\rm e}^{b_{\rm h}(r-r_{\rm e})}}\right)\right] \times G_{n\kappa}(r)=0, \quad (25a)$$
$$\tilde\gamma = (\kappa+H)(\kappa+H-1),\ \ \tilde{\tilde D} = D\left(E_{n\kappa}-m-C_{\rm ps}\right), \quad (25b)$$
$$\tilde{\tilde E}_{n\kappa}^2 = (E_{n\kappa}-m-C_{\rm s})(E_{n\kappa}+m). \quad (25c)$$

Furthermore, using the Pekeris approximation of the strong singularity r^{-2} derived in Appendix A, we can then rewrite the above equation as

$$\frac{{\rm d}^2F_{n\kappa}(r)}{{\rm d}r^2}+\left[\tilde\omega^2-\tilde\gamma\left(D_0+D_1\frac{-{\rm e}^{\alpha x}}{1-c_{\rm h}\,{\rm e}^{-\alpha x}}\right.\right. + D_2\frac{-2\,{\rm e}^{\alpha x}}{(1-c_{\rm h}\,{\rm e}^{-\alpha x})^2}\Bigg)-\tilde\nu^2\left(\frac{1-{\rm e}^{-\alpha x}}{1-c_{\rm h}\,{\rm e}^{-\alpha x}}\right)^2\Bigg] \times G_{n\kappa}(r)=0, \quad (26a)$$
$$\tilde\nu^2 = \tilde D r_{\rm e}^2,\ \ \tilde\omega^2 = \tilde{\tilde E}_{n\kappa}^2 r_{\rm e}^2,\ \ \alpha = b_{\rm h}r_{\rm e}, \quad (26b)$$

where the explicit forms of the constants $D_{\rm i}$ $(i=1,2,3)$ are defined in Eqs. (A3a) and (A3b). This equation has the same form as Eq. (12a) obtained previously in the spin symmetric case. Thus, the energy eigenvalue equation and the associated radial lower component of the wave function can be obtained following the same solution procedure as that used in the previous subsection. In this manner, we will not give detailed calculations but present the final results only. We may also use

the parametric mappings from spin symmetry to p-spin symmetry explained in Refs. [46], [49], and [58]–[61].

Therefore, using Eq. (17), we may obtain the energy eigenvalue equation for the nuclei in the field of TH rotating potential and CLT potential with the exact p-spin symmetry as

$$\begin{aligned}
&\left(n+\frac{1}{2}\right)^2 c_{\rm h}+\frac{c_{\rm h}}{4}+\left(n+\frac{1}{2}\right)\\
&\times\left[\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}\right.\\
&\left.+\frac{2c_{\rm h}}{\alpha}\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2}\right]+\frac{1}{\alpha^2}\left[\tilde{\gamma}D_1-2\tilde{\nu}^2(1-c_{\rm h})\right]\\
&+\frac{1}{\alpha}\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}\\
&=0, \qquad (27)
\end{aligned}$$

from which we can obtain energy equation for the $c_{\rm h}\neq 0$ case as

$$\begin{aligned}
&c_{\rm h}\left(n+\frac{1}{2}\right)+\frac{c_{\rm h}}{\alpha}\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2}\\
&+\frac{1}{2}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}\\
&=\frac{\sigma}{\alpha^2}\sqrt{\left(\tilde{\gamma}D_0-\tilde{\omega}^2\right)c_{\rm h}^2-\tilde{\gamma}D_1c_{\rm h}+\tilde{\gamma}D_2+\tilde{\nu}^2},\\
&\sigma=\pm 1. \qquad (28)
\end{aligned}$$

Recalling

$$\begin{aligned}
\tilde{\gamma}&=(\kappa+H)(\kappa+H-1),\\
\tilde{\nu}^2&=r_{\rm e}^2D\left(E_{n\kappa}-m-C_{\rm ps}\right),\\
\tilde{\omega}^2&=r_{\rm e}^2\left(E_{n\kappa}-m-C_{\rm ps}\right)\left(E_{n\kappa}+m\right),
\end{aligned}$$

one can obtain the implicit dependence of the above energy equation on energy $E_{n\kappa}$. In order to establish the lower-spinor components of the wave functions $G_{n,\kappa}(r)$, Eq. (9), we only quote the final result from Eq. (22) as

$$\begin{aligned}
G_{n\kappa}(r)&=\tilde{N}_{n\kappa}{\rm e}^{-\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2}(r-r_{\rm e})/r_{\rm e}}\\
&\times\left(1-c_{\rm h}{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right)^{\frac{1}{2}+\frac{1}{2c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}}\\
&\times P_n^{\left(\frac{2}{\alpha}\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2},\frac{1}{c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}\right)}\\
&\times\left(1-2c_{\rm h}{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right), \qquad (29)
\end{aligned}$$

where $\tilde{N}_{n\kappa}$ is the normalization constant. From Eq. (B15), the lower spinor can be expressed in terms of the hypergeometric function as

$$\begin{aligned}
G_{n\kappa}(r)&=\tilde{N}_{n\kappa}\frac{(\tilde{a}_0+1)_n}{n!}{\rm e}^{-\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2}(r-r_{\rm e})/r_{\rm e}}\\
&\times\left(1-c_{\rm h}{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right)^{\frac{1}{2}+\frac{1}{2c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}}\\
&\times{}_2{\rm F}_1(-n,1+\tilde{a}_0+\tilde{b}_0+n;\tilde{a}_0+1;c_{\rm h}{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{a}_0&=\frac{2}{\alpha}\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2},\\
\tilde{b}_0&=\frac{1}{c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}.
\end{aligned}$$

On the other hand, the upper-spinor $F_{n\kappa}(r)$ component of the wave function can be calculated as

$$\begin{aligned}
&F_{n\kappa}(r)\\
&=\tilde{N}_{n\kappa}\frac{(\tilde{a}_0+1)_n}{n!}\frac{1}{\left(m-E_{n\kappa}+C_{\rm ps}\right)}\\
&\times\left\{\left(-\frac{(\kappa+A)}{r}-\frac{1}{r_r}\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2}\right.\right.\\
&\left.+\frac{\alpha\left(c_{\rm h}+\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}\right){\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}}{2r_{\rm e}(1-c_{\rm h}{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}})}\right)\\
&\times G_{n\kappa}(r)\\
&+\frac{\alpha c_{\rm h}n}{r_{\rm e}}\left(\frac{1+\tilde{a}_0+\tilde{b}_0+n}{\tilde{a}_0+1}\right){\rm e}^{-\sqrt{\tilde{\gamma}D_0+\tilde{\nu}^2-\tilde{\omega}^2}(r-r_{\rm e})/r_{\rm e}}\\
&\times\left(1-c_{\rm h}{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right)^{\frac{1}{2}+\frac{1}{2c_{\rm h}}\sqrt{c_{\rm h}^2+\frac{4}{\alpha^2}\left[\tilde{\gamma}D_2+\tilde{\nu}^2(1-c_{\rm h})^2\right]}}\\
&\times{}_2{\rm F}_1\left(-n+1,\tilde{a}_0+\tilde{b}_0+n+2;\tilde{a}_0+2;c_{\rm h}{\rm e}^{-\alpha(r-r_{\rm e})/r_{\rm e}}\right)\Big\}, \qquad (30)
\end{aligned}$$

where $E\neq m+C_{\rm ps}$. In the presence of the exact pseudospin symmetry ($C_{\rm ps}=0$), there are no bound positive energy states.

4. Some special cases

In this section, we consider some special cases of interest from the TH rotating potential and CLT potential as follows.

Firstly, when $c_{\rm h}=0$ and $b_{\rm h}=a$, the TH rotating potential is reduced to the Morse oscillator (version I), i.e.,

$$\lim_{c_{\rm h}\to 0}V(r)=V_{\rm M}^{\rm (I)}(r)=D\left({\rm e}^{-2a(r-r_{\rm e})}-2{\rm e}^{-a(r-r_{\rm e})}\right)+D. \qquad (31)$$

Also, when we neglect the last term in Eq. (31), we obtain the second version of the Morse potential (version II) as

$$V_{\rm M}^{\rm (II)}=D\left({\rm e}^{-2a(r-r_{\rm e})}-2{\rm e}^{-a(r-r_{\rm e})}\right). \qquad (32)$$

Recently, Berkdemir[50,51] and Aydoğdu and Sever[52] have studied the above potential in the context of the relativistic theory.

4.1. Dirac–Morse–CLT problem ($H\neq 0$ case)

To find the solution of the Dirac equation with Morse oscillator potential and CLT potential, we insert $c_{\rm h}=0$ into Eq. (17) and the values of ν^2, ω^2, and γ followed by a little algebra, we obtain the energy equation as

$$\begin{aligned}
&\left[(\kappa+H)(\kappa+H+1)D_0+r_{\rm e}^2\left(E_{n\kappa}+m-C_{\rm s}\right)\right.\\
&\left.\times\left(D-E_{n\kappa}+m\right)\right]^{1/2}
\end{aligned}$$

$$= -\left(n+\frac{1}{2}\right)\alpha + \frac{\left[r_e^2 D\left(E_{n\kappa}+m-C_s\right)-(\kappa+H)(\kappa+H+1)D_1/2\right]}{\sqrt{(\kappa+H)(\kappa+H+1)D_2+r_e^2 D\left(E_{n\kappa}+m-C_s\right)}}, \quad (33)$$

where the constants D_i $(i=1,2,3)$ in the limit of $c_h \to 0$ are given by Eq. (A4).

4.2. Dirac–Morse problem ($H=0$ case)

When $H=0$, equation (33) is reduced to the Dirac–Morse solution,

$$\sqrt{\kappa(\kappa+1)D_0+r_e^2\left(E_{n\kappa}+m-C_s\right)\left(D-E_{n\kappa}+m\right)} = -\left(n+\frac{1}{2}\right)\alpha + \frac{\left[r_e^2 D\left(E_{n\kappa}+m-C_s\right)-\kappa(\kappa+1)D_1/2\right]}{\sqrt{\kappa(\kappa+1)D_2+r_e^2 D\left(E_{n\kappa}+m-C_s\right)}}. \quad (34)$$

4.3. Schrödinger–Morse problem

The nonrelativistic limit solution is obtained when $C_s=0$, $E_{n\kappa}+m \approx 2\mu$, $E_{n\kappa}-m \approx E_{nl}$, and $\kappa(\kappa+1) \to l(l+1)$ from Eq. (34) as

$$E_{nl} = D + \frac{l(l+1)D_0}{2\mu r_e^2} - \frac{\alpha^2}{2\mu r_e^2} \times \left[\frac{r_e^2\left(2\mu D - l(l+1)D_1/2r_e^2\right)}{\alpha\sqrt{l(l+1)D_2+2\mu r_e^2 D}} - \left(n+\frac{1}{2}\right)\right]^2. \quad (35)$$

Very recently, we have solved the Schrödinger equation for the rotating Morse potential taking a new position-dependent mass ansatz.[63]

Secondly, when we set the potential parameters to be $b_h=a$ and $c_h=e^{-ar_e}$, the TH potential is reduced into the generalized Morse potential (GMP) proposed by Deng and Fan,[64] i.e.,

$$V_{GM}(r) = D\left(1-\frac{b}{e^{ar}-1}\right)^2, \quad b = e^{ar_e}-1. \quad (36)$$

4.3.1. Dirac–GMP–CLT problem ($H \neq 0$ case)

The energy equation can be found from Eq. (18) to be

$$c_h\left(n+\frac{1}{2}\right) + \frac{c_h}{\alpha}\left[(\kappa+H)(\kappa+H+1)D_0 + r_e^2\left(E_{n\kappa}+m-C_s\right)\left(D-E_{n\kappa}+m\right)\right]^{1/2} + \frac{1}{2}\left\{c_h^2 + \frac{4}{\alpha^2}\left[(\kappa+H)(\kappa+H+1)D_2 + r_e^2 D\left(E_{n\kappa}+m-C_s\right)(1-c_h)^2\right]\right\}^{1/2} = \frac{\sigma}{\alpha^2}\left(\lambda_0 c_h^2 - (\kappa+H)(\kappa+H+1)D_1 c_h + (\kappa+H)(\kappa+H+1)D_2 + r_e^2\left(E_{n\kappa}+m-C_s\right)D\right)^{1/2},$$

$$\lambda_0 = (\kappa+H)(\kappa+H+1)D_0 - r_e^2\left(E_{n\kappa}+m-C_s\right)\left(E_{n\kappa}-m\right), \quad (37)$$

where $\sigma=\pm 1$, $\alpha=ar_e$, $c_h=e^{-ar_e}$, and the constants D_i $(i=1,2,3)$ for this case are given by (A3).

4.3.2. Dirac–GMP problem ($H=0$ case)

The energy equation can be found from Eq. (37) to be

$$c_h\left(n+\frac{1}{2}\right) + \frac{c_h}{\alpha}\left[\kappa(\kappa+1)D_0 + r_e^2\left(E_{n\kappa}+m-C_s\right) \times \left(D-E_{n\kappa}+m\right)\right]^{1/2} + \frac{1}{2}\left\{c_h^2 + \frac{4}{\alpha^2}\left[\kappa(\kappa+1)D_2 + r_e^2 D\left(E_{n\kappa}+m-C_s\right)(1-c_h)^2\right]\right\}^{1/2} = \frac{\sigma}{\alpha^2}\left[\lambda_1 c_h^2 - \kappa(\kappa+1)D_1 c_h + \kappa(\kappa+1)D_2 + r_e^2\left(E_{n\kappa}+m-C_s\right)D\right]^{1/2},$$

$$\lambda_1 = \kappa(\kappa+1)D_0 - r_e^2\left(E_{n\kappa}+m-C_s\right)\left(E_{n\kappa}-m\right). \quad (38)$$

Very recently, we have studied the approximate bound state solutions of the Dirac equation for GM potential with any value of κ in the spin and p-spin symmetric limitations using the parametric generalization of the NU method and employing a new improved approximation scheme to deal with the spin–orbit barrier term.[49]

To show the procedure of determining the energy eigenvalues from Eqs. (17) and (27), we take a set of physical parameter values: $r_e=2.40873$ fm, $b_h=0.988879$ fm^{-1}, $D=5.0$ fm^{-1}, $m=10.0$ fm^{-1}, and $C_s=10.0$.[52] In Tables 2 and 3 listed are the calculated energy eigenvalues of the spin-1/2 Dirac particle for the TH-potential in the presence and the absence of the Coulomb tensor potential. In Table 2, we present the energy spectrum for the spin symmetric case. Obviously, the pairs $(np_{1/2}, np_{3/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, $(ng_{7/2}, ng_{9/2})$, etc, are degenerate states. Thus, each pair is considered to be a spin doublet and has positive energy.[49] In Table 3, we give the numerical results for the pseudo-spin symmetric case. In this case, we take the set of parameter values as follows: $r_e=2.40873$ fm, $b_h=0.988879$ fm^{-1}, $D=5.0$ fm^{-1}, $m=10.0$ fm^{-1}, and $C_{ps}=-10.0$.[44] We observe the degeneracy in the following doublets: $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$, etc. Thus, each pair is considered to be p-spin doublet and has negative energy.[49]

Finally, we plot the relativistic energy eigenvalues of the TH potential and CLT potential with spin and pseudospin symmetry limitations in Figs. 2 and 3. In these figures, we plot the energy eigenvalues of spin and p-spin symmetry limits versus potential parameters H, r_e, b_h, D, and c_h.

Table 2. Bound state energy eigenvalues (in units of fm^{-1}) of the spin-symmetry TH with tensor potential for several values of n and κ with $c_{\mathrm{h}} = 0.01$.

l	$n, \kappa < 0$	$(l, j = l+1/2)$	$E_{n,\kappa<0}$, $H=1$	$E_{n,\kappa<0}$, $H=0$	$n, \kappa > 0$	$(l, j = l-1/2)$	$E_{n,\kappa>0}$, $H=1$	$E_{n,\kappa>0}$, $H=0$
1	0, –2	$0p_{3/2}$	0.005960053369	0.01564453620	0, 1	$0p_{1/2}$	0.02928502468	0.01564453620
2	0, –3	$0d_{5/2}$	0.01564453620	0.02928502468	0, 2	$0d_{3/2}$	0.04685683826	0.02928502468
3	0, –4	$0f_{7/2}$	0.02928502468	0.04685683826	0, 3	$0f_{5/2}$	0.06836569674	0.04685683826
4	0, –5	$0g_{9/2}$	0.04685683826	0.06836569674	0, 4	$0g_{7/2}$	0.09382202943	0.06836569674
1	1, –2	$1p_{3/2}$	0.05813716344	0.07117324667	1, 1	$1p_{1/2}$	0.09266344719	0.07117324667
2	1, –3	$1d_{5/2}$	0.07117324667	0.09266344719	1, 2	$1d_{3/2}$	0.1199395479	0.09266344719
3	1, –4	$1f_{7/2}$	0.09266344719	0.1199395479	1, 3	$1f_{5/2}$	0.1520130533	0.1199395479
4	1, –5	$1g_{9/2}$	0.1199395479	0.1520130533	1, 4	$1g_{7/2}$	0.1885051316	0.1520130533

Table 3. Bound state energy eigenvalues (in units of fm^{-1}) of the pseudospin symmetry TH with tensor potential for several values of n and κ with $c_{\mathrm{h}} = -0.01$.

$\tilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}$ $H=8$	$E_{n,\kappa<0}$ $H=1$	$E_{n,\kappa<0}$ $H=0$	$n-1$, $\kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}$ $H=8$	$E_{n-1,\kappa>0}$ $H=1$	$E_{n-1,\kappa>0}$ $H=0$
1	1, –1	$1s_{1/2}$	−0.04825373660	–	−0.007823542732	0, 2	$0d_{3/2}$	−0.02191791117	−0.01923909438	−0.007823542732
2	1, –2	$1p_{3/2}$	−0.04646766226	−0.007823542732	−0.01923909438	0, 3	$0f_{5/2}$	−0.001318795237	−0.03080439455	−0.01923909438
3	1, –3	$1d_{5/2}$	−0.04034304847	−0.01923909438	−0.03080439455	0, 4	$0g_{7/2}$	0.02565405271	−0.04034304847	−0.03080439455
4	1, –4	$1f_{7/2}$	−0.03080439455	−0.03080439455	−0.04034304847	0, 5	$0h_{9/2}$	0.05923329078	−0.04646766226	−0.04034304847
1	2, –1	$2s_{1/2}$	−0.09770038174	–	−0.008528478379	1, 2	$1d_{3/2}$	−0.1311748422	−0.02328053998	−0.008528478379
2	2, –2	$2p_{3/2}$	−0.08025089987	−0.008528478379	−0.02328053998	1, 3	$1f_{5/2}$	−0.1340439229	−0.04154662334	−0.02328053998
3	2, –3	$2d_{5/2}$	−0.06110448219	−0.02328053998	−0.04154662334	1, 4	$1g_{7/2}$	−0.1320606648	−0.06110448219	−0.04154662334
4	2, –4	$2f_{7/2}$	−0.04154662334	−0.04154662334	−0.06110448219	1, 5	$1h_{9/2}$	−0.1249128778	−0.08025089987	−0.06110448219

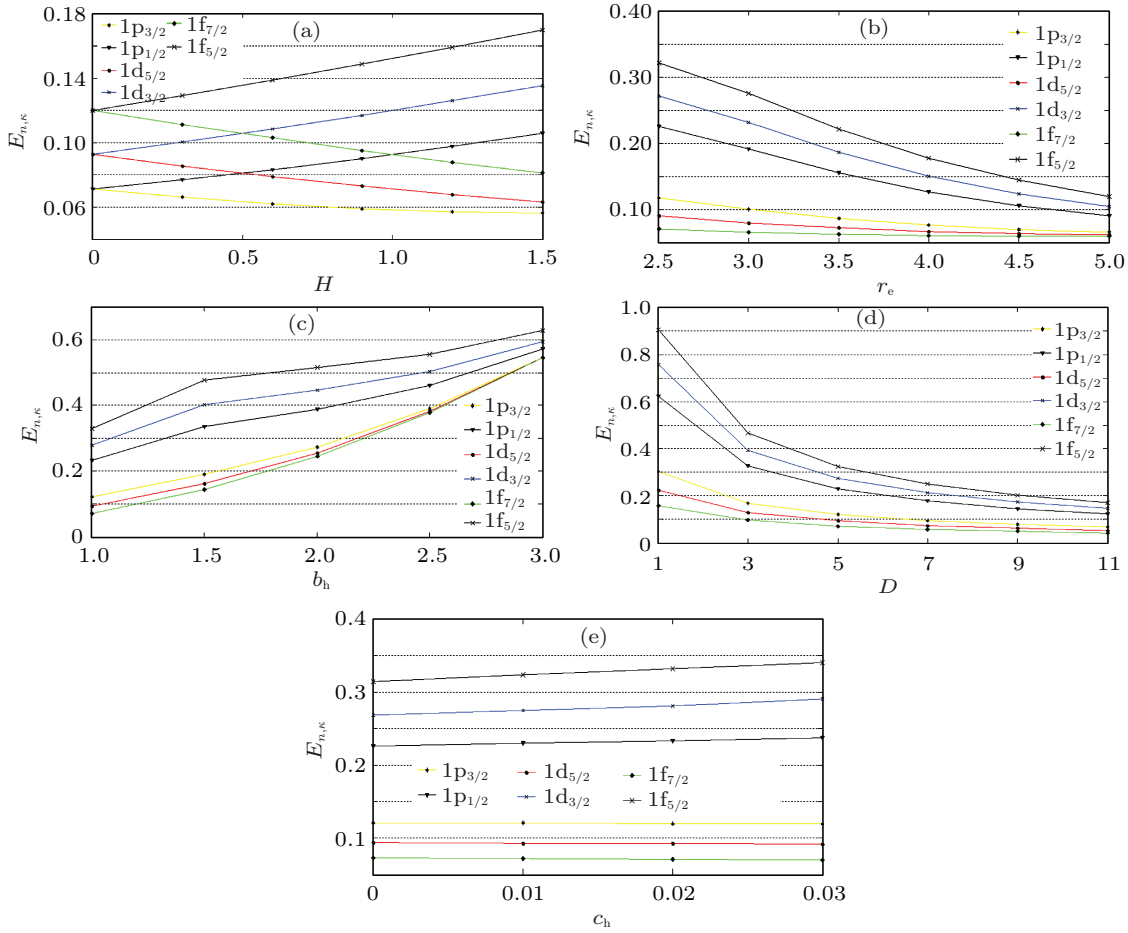


Fig. 2. (color online) Contributions of the potential parameters (a) H, (b) r_{e}, (c) b_{h}, (d) D, and (e) c_{h} to the energy levels with spin symmetry.

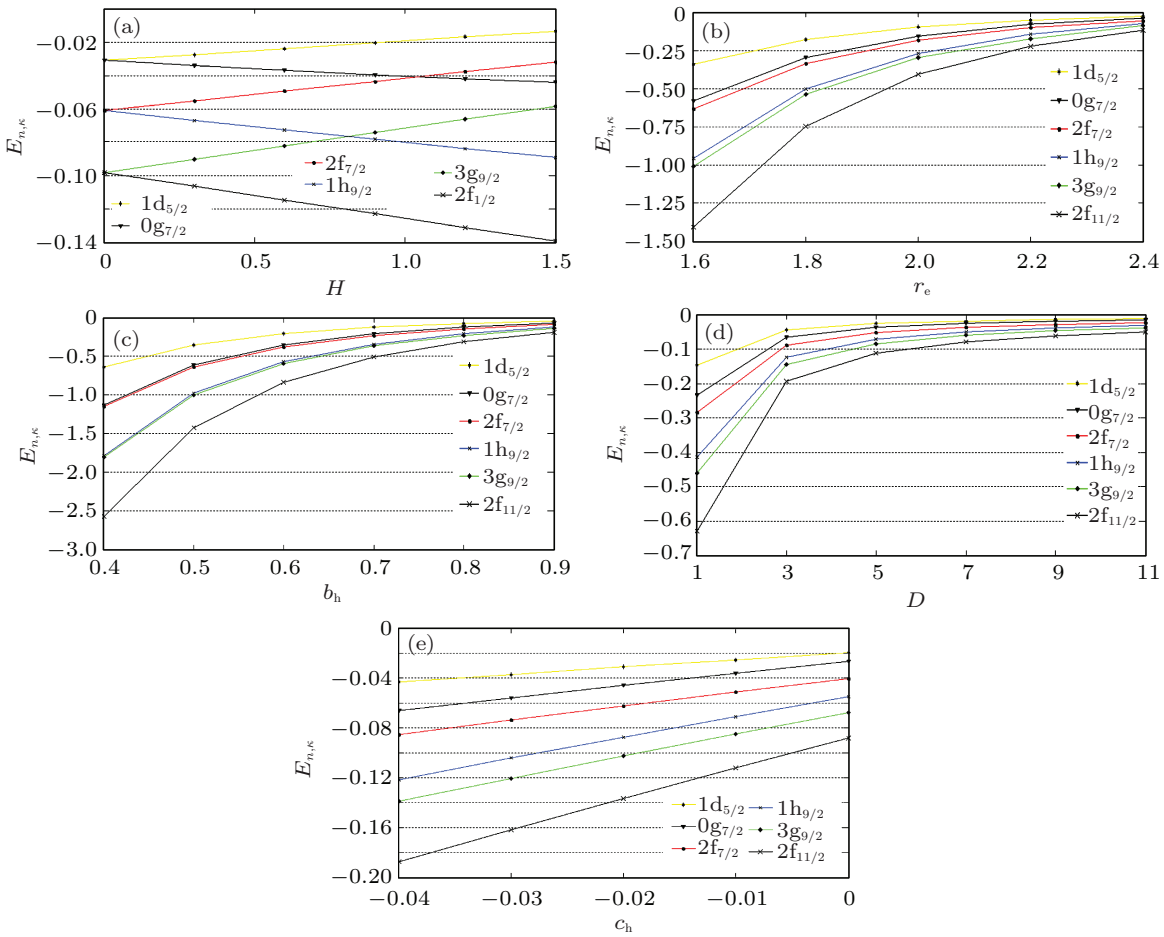


Fig. 3. (color online) Contributions of the potential parameters (a) H, (b) r_e, (c) b_h, and (d) c_h to the energy levels with the pseudospin symmetry.

In Fig. 2, we investigate the effects of the potential parameters on the p-spin doublet splitting by considering the following pairs of orbital states: $(1p_{1/2}, 1p_{3/2})$, $(1d_{3/2}, 1d_{5/2})$, and $(1f_{5/2}, 1f_{7/2})$. From Fig. 2(a), we observe that in the case of $H = 0$ (no tensor interaction), members of spin doublets have the same energy. However, in the presence of the tensor potential, i.e., $H \neq 0$, these degeneracies are removed. We can also see in Fig. 2(a) that the spin doublet splitting increases with increasing H. The reason is that the term $2\kappa H$ makes different contributions to different levels in the spin doublet, because H takes different values for different states in the spin doublet. From Fig. 2(b), we observe that spin doublet splitting increases as r_e decreases. The reason can be found by considering the term $(\kappa+H)(\kappa+H+1)/r_e^2$ in the energy eigenvalue equation. As mentioned, the term $2\kappa H$ is responsible for the spin doublet splitting. Therefore, contribution of the $2\kappa H$ to the energy levels decreases with increasing r_e due to $2\kappa H/r_e^2$. In Fig. 2(c), the contribution of the potential parameter b_h to the spin doublet splitting is present. From Fig. 2(c), it can be seen that magnitude of the energy difference between members of the spin doublet decreases as the value of b_h increases. The effect of the parameter D on the energy eigenvalues of the members of the p-spin doublet can be seen in Fig. 2(d). One can see in Fig. 2(d) that splitting between the pair of orbitals decreases as the value of D increases. Finally, we study the effect of the parameter c_h on the energy splitting of spin doublet in Fig. 2(e) and we can see that splitting between the pair of orbitals increases as the value of c_h increases.

In Fig. 3, we investigate the effect of the potential parameters on the pseudospin doublet splitting by considering the following pairs of orbitals: $(1d_{5/2}, 0g_{7/2})$, $(2f_{7/2}, 1h_{9/2})$, $(3g_{9/2}, 3i_{11/2})$, and one can observe that the results obtained in the p-spin symmetric limit resemble the ones observed in the spin symmetric limit.

5. Final remarks and conclusions

In this work, we have investigated the bound state solutions of the Dirac equation with the rotating TH potential with Coulomb coupling potential for any spin–orbit quantum number κ. By making a Pekeris approximation to deal with the spin–orbit coupling term, we obtain the energy eigenvalue equation and the unnormalized two spinor components of the radial wave function expressed in terms of the Jacobi polynomial and hypergeometric function. The problem is studied within the spin and p-spin symmetric limitations. In the presence of CLT potential, our solution can be reduced to

the relativistic rotating generalized Morse solution by simply making the proper transformation of parameters: $b_{\rm h}=a$ and $c_{\rm h}=\mathrm{e}^{-ar_{\rm e}}$ and to the rotating Morse solution by taking $c_{\rm h}=0$ and $b_{\rm h}=a$. Also, the relativistic solution can be reduced into the Schrödinger solution under the nonrelativistic limit. Our numerical results are compared with the existing energy spectra for the particular case where $c_{\rm h}=0$ (the Morse case). The main idea of the introduction of the tensor potential is to remove the degeneracy of the doublet states by converting the spin–orbit quantum number into another shifted spin–orbit quantum number (i.e., $\kappa\to\kappa+H$) in both spin and pseudospin symmetric limitations.

Appendix A: Pekeris approximation to the spin–orbit centrifugal term

The spin–orbit centrifugal (pseudo centrifugal) term in Eq. (11a) (Eq. 25a) can be expanded around $r=r_{\rm e}$ into a series of powers of $x=(r-r_{\rm e})/r_{\rm e}\in(-1,\infty)$ as

$$V_{\rm so}(r)=\frac{\gamma}{r^2}=\frac{\gamma}{r_{\rm e}^2(1+x)^2}=\frac{\gamma}{r_{\rm e}^2}(1-2x+3x^2-4x^3+\cdots),\quad x\ll 1, \tag{A1}$$

where $\gamma=(\kappa+H)(\kappa+H\pm1)$. It is sufficient to keep expansion terms only up to the second order x^2. The above spin–orbit potential (A1) can be substituted into the original TH potential to keep the factorizability of the corresponding Schrödinger-like equation. Taking the centrifugal (pseudo centrifugal) term as

$$\tilde{V}_{\rm so}(r)=\frac{\gamma}{r_{\rm e}^2}\left[D_0+D_1\frac{\mathrm{e}^{-\alpha x}}{1-c_{\rm h}\mathrm{e}^{-\alpha x}}+D_2\frac{\mathrm{e}^{-2\alpha x}}{(1-c_{\rm h}\mathrm{e}^{-\alpha x})^2}\right],\quad x\ll 1/\alpha,\quad \alpha x\ll 1,\quad \gamma=(\kappa+H)(\kappa+H\pm1), \tag{A2}$$

where $\alpha=b_{\rm h}r_{\rm e}$ and $D_{\rm i}$ are the parameters of coefficients $(i=0,1,2)$. After making a Taylor expansion of expression (A2) up to the second-order term x^2, and then comparing equal powers with those in expression (A1), we can readily determine $D_{\rm i}$ parameters as a function of specific potential parameters $b_{\rm h}$, $c_{\rm h}$, and $r_{\rm e}$ as follows:

$$D_0=1-\frac{1}{\alpha}(1-c_{\rm h})(3+c_{\rm h})+\frac{3}{\alpha^2}(1-c_{\rm h})^2, \tag{A3a}$$

$$D_1=\frac{2}{\alpha}(1-c_{\rm h})^2(2+c_{\rm h})-\frac{6}{\alpha^2}(1-c_{\rm h})^3, \tag{A3b}$$

$$D_2=-\frac{1}{\alpha}(1-c_{\rm h})^3(1+c_{\rm h})+\frac{3}{\alpha^2}(1-c_{\rm h})^4, \tag{A3c}$$

where $\alpha=b_{\rm h}r_{\rm e}$. In the limit of $c_{\rm h}=0$, we recover the approximations obtained for the Morse potential case,[50]

$$\lim_{c_{\rm h}\to0}D_0=1-\frac{3}{\alpha}+\frac{3}{\alpha^2},\quad \lim_{c_{\rm h}\to0}D_1=\frac{4}{\alpha}-\frac{6}{\alpha^2},\quad \lim_{c_{\rm h}\to0}D_2=-\frac{1}{\alpha}+\frac{3}{\alpha^2}, \tag{A4}$$

where $\alpha=ar_{\rm e}$.

Appendix B: Parametric NU method

The NU method is used to solve second-order differential equations with the appropriate coordinate transformation $s=s(r)$[25]

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{B1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, with at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution. We present a shortcut for the method. Thus, at first we write the general form of the Schrödinger-like equation (B1) in a more general form applicable to any potential as follows:[57–59]

$$\psi_n''(s)+\left(\frac{c_1-c_2s}{s(1-c_3s)}\right)\psi_n'(s)+\left(\frac{-As^2+Bs-C}{s^2(1-c_3s)^2}\right)\psi_n(s)=0, \tag{B2}$$

satisfying the wave functions

$$\psi_n(s)=\phi(s)y_n(s). \tag{B3}$$

Comparing (B2) with its counterpart (B1), we obtain the following identifications:

$$\tilde{\tau}(s)=c_1-c_2s,\quad \sigma(s)=s(1-c_3s),\quad \tilde{\sigma}(s)=-As^2+Bs-C. \tag{B4}$$

Following the NU method,[25] we obtain the following necessary parameters.[57]

(i) Relevant constants:

$$c_4=\frac{1}{2}(1-c_1),\quad c_5=\frac{1}{2}(c_2-2c_3),\quad c_6=c_5^2+A,$$
$$c_7=2c_4c_5-B,\quad c_8=c_4^2+C,\quad c_9=c_3(c_7+c_3c_8)+c_6,$$
$$c_{10}=c_1+2c_4+2\sqrt{c_8}-1>-1,$$
$$c_{11}=1-c_1-2c_4+\frac{2}{c_3}\sqrt{c_9}>-1,\quad c_3\neq0,$$
$$c_{13}=-c_4+\frac{1}{c_3}(\sqrt{c_3}-c_5)>0,\quad c_3\neq0. \tag{B5}$$

(ii) Essential polynomial functions:

$$\pi(s)=c_4+c_5s-[(\sqrt{c_9}+c_3\sqrt{c_8})s-\sqrt{c_8}], \tag{B6}$$

$$k=-(c_7+2c_3c_8)-2\sqrt{c_8c_9}, \tag{B7}$$

$$\tau(s)=c_1+2c_4-(c_2-2c_5)s-2[(\sqrt{c_9}+c_3\sqrt{c_8})s-\sqrt{c_8}], \tag{B8}$$

$$\tau'(s)=-2c_3-2(\sqrt{c_9}+c_3\sqrt{c_8})<0. \tag{B9}$$

(iii) Energy equation:

$$c_2n-(2n+1)c_5+(2n+1)(\sqrt{c_9}+c_3\sqrt{c_8})+n(n-1)c_3$$

$$+ c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0. \tag{B10}$$

(iv) Wave functions:

$$\rho(s) = s^{c_{10}}(1-c_3s)^{c_{11}}, \tag{B11}$$

$$\phi(s) = s^{c_{12}}(1-c_3s)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0, \tag{B12}$$

$$y_n(s) = P_n^{(c_{10},c_{11})}(1-2c_3s), \quad c_{10} > -1, \quad c_{11} > -1, \tag{B13}$$

$$\psi_{n\kappa}(s) = N_{n\kappa}s^{c_{12}}(1-c_3s)^{c_{13}}P_n^{(c_{10},c_{11})}(1-2c_3s), \tag{B14}$$

where $P_n^{(\mu,\nu)}(x)$ ($\mu > -1$, $\nu > -1$, and $x \in [-1,1]$) are Jacobi polynomials with

$$P_n^{(a_0,b_0)}(1-2s) = \frac{(a_0+1)_n}{n!}{}_2\mathrm{F}_1(-n, 1+a_0+b_0+n; a_0+1; s), \tag{B15}$$

and $N_{n\kappa}$ is a normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function as

$$\psi_{n\kappa}(s) = N_{n\kappa}s^{c_{12}}(1-c_3s)^{c_{13}} \times {}_2\mathrm{F}_1(-n, 1+c_{10}+c_{11}+n; c_{10}+1; c_3s), \tag{B16}$$

where $c_{12} > 0$, $c_{13} > 0$, and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

Appendix C: Dirac–Morse problem and Coulomb-like tensor potential

This case should be treated separately especially for the radial wave function. Here, we extend the relativistic bound state solution of Refs. [50] and [51] to include the Coulomb tensor coupling potential as follows. The Schrödinger-type second-order equation (12a) with spin-symmetry for the case where $c_\mathrm{h} = 0$ and $b_\mathrm{h} = a$, using the transformation $s = \mathrm{e}^{-\alpha x}$, $x = (r - r_\mathrm{e})/r_\mathrm{e}$, can be rewritten as

$$\frac{\mathrm{d}^2F_{n\kappa}(s)}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}F_{n\kappa}(s)}{\mathrm{d}s} + \frac{1}{s^2}[\varepsilon_3 s^2 - \varepsilon_2 s + \varepsilon_1]F_{n\kappa}(s),$$

$$\varepsilon_1^2 = \frac{1}{\alpha^2}\left(\gamma D_0 + \nu^2 - \omega^2\right), \quad \varepsilon_2 = \frac{1}{\alpha^2}\left(2\nu^2 - \gamma D_1\right),$$

$$\varepsilon_3 = \frac{1}{\alpha^2}\left(\nu^2 + \gamma D_2\right), \tag{C1}$$

where

$$\nu^2 = r_\mathrm{e}^2 D\left(E_{n\kappa} + m - C_\mathrm{s}\right),$$

$$\omega^2 = r_\mathrm{e}^2\left(E_{n\kappa} + m - C_\mathrm{s}\right)\left(E_{n\kappa} - m\right),$$

$$\gamma = (\kappa + H)(\kappa + H + 1), \alpha = ar_\mathrm{e}. \tag{C2}$$

Furthermore, from Eq. (29) in Ref. [50], $-\varepsilon_1^2 = -[\varepsilon_2/2\sqrt{\varepsilon_3} - (n+1/2)]^2$, we obtain the relativistic energy equation for the Morse potential including CLT potential as

$$\begin{aligned}\left(E_{n\kappa} - m\right)\left(E_{n\kappa} + m - C_\mathrm{s}\right) &= D + \frac{(\kappa+H)(\kappa+H+1)D_0}{r_\mathrm{e}^2} \\ &- a^2\left[\frac{r_\mathrm{e}\left((E_{n\kappa}+m-C_\mathrm{s})D - (\kappa+H)(\kappa+H+1)D_1/2r_\mathrm{e}^2\right)}{a\sqrt{(\kappa+H)(\kappa+H+1)D_2 + r_\mathrm{e}^2(E_{n\kappa}+m-C_\mathrm{s})D}}\right. \\ &- \left.\left(n+\frac{1}{2}\right)\right]^2.\end{aligned} \tag{C3}$$

Next, we calculate the first part of the radial two-component wave function following the way in Ref. [50] as

$$\begin{aligned}\rho(s) &= s^{1+\frac{2}{\alpha}\sqrt{\gamma D_0+\nu^2-\omega^2}}\,\mathrm{e}^{-\frac{2}{\alpha}\sqrt{\nu^2+\gamma D_2}s}, \\ y_{n\kappa}(s) &= B_{n\kappa}s^{-\left(1+\frac{2}{\alpha}\sqrt{\gamma D_0+\nu^2-\omega^2}\right)}\,\mathrm{e}^{\frac{2}{\alpha}\sqrt{\nu^2+\gamma D_2}s} \\ &\times \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[s^{\left(1+n+\frac{2}{\alpha}\sqrt{\gamma D_0+\nu^2-\omega^2}\right)}\,\mathrm{e}^{-\frac{2}{\alpha}\sqrt{\nu^2+\gamma D_2}s}\right] \\ &= L_n^{\left(1+\frac{2}{\alpha}\sqrt{\gamma D_0+\nu^2-\omega^2}\right)}(w), \\ w &= \frac{2}{\alpha}\sqrt{\nu^2+\gamma D_2}s,\end{aligned} \tag{C4}$$

and the second part

$$\phi(w) = \left(\frac{2}{\alpha}\sqrt{\nu^2+\gamma D_2}\right)^{-\frac{1}{\alpha}\sqrt{\gamma D_0+\nu^2-\omega^2}} \times w^{\frac{1}{\alpha}\sqrt{\gamma D_0+\nu^2-\omega^2}}\,\mathrm{e}^{-w/2}.$$

Thus, the upper component of the radial wave function becomes

$$\begin{aligned}&F_{n\kappa}(r) \\ &= A_{n\kappa}\,\mathrm{e}^{-\frac{1}{r_\mathrm{e}}\sqrt{(\kappa+H)(\kappa+H+1)D_0+\nu^2-\omega^2}(r-r_\mathrm{e})} \\ &\times \mathrm{e}^{-\frac{1}{\alpha}\sqrt{\nu^2+(\kappa+H)(\kappa+H+1)D_2}\,\mathrm{e}^{-a(r-r_\mathrm{e})}} \\ &\times L_n^{\left(1+\frac{2}{\alpha}\sqrt{(\kappa+H)(\kappa+H+1)D_0+\nu^2-\omega^2}\right)} \\ &\times \frac{2}{\alpha}\sqrt{\nu^2+(\kappa+H)(\kappa+H+1)D_2}\,\mathrm{e}^{-a(r-r_\mathrm{e})},\end{aligned} \tag{C5}$$

where

$$A_{n\kappa} = \frac{4an!\left(1+n+\frac{2}{\alpha}\left((\kappa+H)(\kappa+H+1)D_0+\nu^2-\omega^2\right)\right)^2}{\left(1+n+\frac{2}{\alpha}\left((\kappa+H)(\kappa+H+1)D_0+\nu^2-\omega^2\right)\right)!} \times \left(\frac{2}{\alpha}\sqrt{\gamma D_2+\nu^2}\right)^{\frac{2}{\alpha}\sqrt{(\kappa+H)(\kappa+H+1)D_2+\nu^2-\omega^2}}. \tag{C6}$$

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