Detaching two single-mode squeezing operators from the two-mode squeezing operator*

Fan Hong-Yi(范洪义) and Da Cheng(笪 诚)†

Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, *China*

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It has been common knowledge that the single-mode squeezing operator and the two-mode squeezing operator are independent of each other. However, in this work we find that after using the technique of integration within $\mathfrak{Q}$-ordering and $\mathfrak{P}$-ordering, we can detach two single-mode squeezing operators from the two-mode squeezing operator. In other words, we show that the two-mode squeezing operator can be split into a $\mathfrak{P}$-ordered two-mode squeezing operator (with a new squeezing parameter) and two single-mode squeezing operators (with another squeezing parameter). This tells us that the two-mode squeezing mechanism also involves some single-mode squeezing.

Keywords: $\mathfrak{Q}$-ordering and $\mathfrak{P}$-ordering, single-mode and two-mode squeezing operators, detaching

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1. Introduction

The birth of modern quantum mechanics was characterized by the basic commutative relation $[Q,P]=\mathrm{i}\hbar$ ($\hbar=1$), where Q and P are the coordinate and momentum operators, respectively.[1] Operator ordering problems often arise in dealing with quantization and in calculating various matrix elements of operators. In quantum optics theory, operator ordering problems often arise in converting operator master equations into classical c-number equations in phase space. The basic operator ordering problem is to rearrange operators into $Q-P$ or $P-Q$ ordering. Even though this kind of ordering problem has been paid attention for many years by theoreticians, it is not solved satisfactorily due to the lack of an efficient method. In a preceding paper,[2] we have introduced both the technique of integration within operators' $Q-P$ ordering (let the symbol $\mathfrak{Q}$ denote the ordering when all Q_{s} stand on the left of all P_{s}) and the technique of integration within operators' $P-Q$ ordering (let the symbol $\mathfrak{P}$ denote the ordering when all P_{s} stand on the left of all Q_{s}), which provide us with a new method of arranging operators into either $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering, and we can derive some new fundamental operator identities about their mutual reordering. The $\mathfrak{Q}$- and $\mathfrak{P}$-ordered formulas of the Wigner operator are also deduced in Ref. [2], which together with the Weyl quantization rule gives a convenient method for arranging operators into either $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering. In this work, we shall employ the $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering method to detach two single-mode squeezing operators from a two-mode squeezing operator. At first glance, this seems absurd since it is common knowledge that the two-mode squeezing operator is independent of the single-mode squeezing operator, though the latter may be considered as the degenerate case of the former. Nevertheless, as one will see later, we can really detach some single-mode squeezing operators from a given two-mode squeezing operator.

Squeezed state is an important concept in quantum mechanics and quantum optics,[3,4] because one of its quadratures bears little noise than the coherent state and it can be used in gravitational detection and precise quantum communication. Theoretically, squeezed states are constructed by operating the squeezing operators on a vacuum state (one-mode or two-mode). As is common knowledge, squeezing is classified into two types: the single-mode squeezing operator S_1

$$S_{1\mathrm{a}}=\mathrm{e}^{\lambda\left(a^2-a^{\dagger 2}\right)/2},\quad S_{1\mathrm{b}}=\mathrm{e}^{\lambda\left(b^2-b^{\dagger 2}\right)/2}, \tag{1}$$

and the two-mode squeezing operator S_2

$$S_2=\mathrm{e}^{\lambda\left(ab-a^{\dagger}b^{\dagger}\right)}, \tag{2}$$

where $\left[a,a^{\dagger}\right]=\left[b,b^{\dagger}\right]=1$. By introducing

$$a=\frac{Q_1+\mathrm{i}P_1}{\sqrt{2}},\quad b=\frac{Q_2+\mathrm{i}P_2}{\sqrt{2}}, \tag{3}$$

we have

$$S_{1\mathrm{a}}=\mathrm{e}^{\mathrm{i}\lambda(P_1Q_1+Q_1P_1)/2},\quad S_{1\mathrm{b}}=\mathrm{e}^{\mathrm{i}\lambda(P_2Q_2+Q_2P_2)/2}, \tag{4}$$

$$S_2=\exp\left[\mathrm{i}\lambda\left(P_1Q_2+P_2Q_1\right)\right]. \tag{5}$$

Usually, people think that S_2 is of no mathematical relevance to $S_{1\mathrm{a}}S_{1\mathrm{b}}$. Physically, this tells us that the two-mode squeezing mechanism is of no relevance to the single-mode squeezing mechanism. A new question now arises: can we detach some single-mode squeezing operator from the two-mode squeezing operator? If yes, this will imply that two-mode squeezing mechanism also involves some single-mode squeezing. Before answering this question, we briefly review the technique of integration within $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering of operators.

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†Corresponding author. E-mail: dacheng@ustc.edu

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2. Technique of integration within $\mathfrak{Q}$-ordering or $\mathfrak{P}$-ordering

Operators' $\mathfrak{Q}$-ordering means putting all $Q_{\rm s}$ on the left of all $P_{\rm s}$ after using $[Q,P]={\rm i}$. Thus,

$$\mathfrak{Q}\left(P^mQ^n\right)=Q^nP^m. \tag{6}$$

When a function $f(Q,{\rm i}P)$ is rearranged as $g(Q,{\rm i}P)$ (Q on the left of P), it is in $\mathfrak{Q}$-ordering and denoted as

$$f(Q,{\rm i}P)=\mathfrak{Q}\left[g(Q,{\rm i}P)\right]. \tag{7}$$

In reference to the fundamental bosonic commutative relation $\left[a,a^\dagger\right]=1$, we have the following correspondence:

$$\left[a,a^\dagger\right]=1\rightarrow[{\rm i}P,Q]=1,\quad \hbar=1. \tag{8}$$

In reference to the operator identity

$${\rm e}^{-\lambda a^\dagger a}=:\exp\left\{\left({\rm e}^{-\lambda}-1\right)a^\dagger a\right\}:, \tag{9}$$

we have the $\mathfrak{Q}$-ordering of ${\rm e}^{-\lambda Q({\rm i}P)}$

$${\rm e}^{-\lambda Q({\rm i}P)}=\mathfrak{Q}\left[\exp\left\{\left({\rm e}^{-\lambda}-1\right)Q({\rm i}P)\right\}\right]. \tag{10}$$

Like that a is commutable with $a^\dagger$ within the normal ordering symbol, within the $\mathfrak{Q}$-ordering symbol Q is commutable with P. Namely, though $[Q,P]={\rm i}$, we have $\mathfrak{Q}[PQ]=\mathfrak{Q}[QP]=QP$, so we can perform integration over c-number within $\mathfrak{Q}(\cdots)$. This is called the technique of integration within $\mathfrak{Q}$-ordering. For example, the $\mathfrak{Q}$-ordering form of $\delta(p-P)\,\delta(q-Q)$ is

$$\begin{aligned}&\delta(p-P)\,\delta(q-Q)\\ =&\iint_{-\infty}^{\infty}\frac{{\rm d}u{\rm d}v}{4\pi^2}{\rm e}^{{\rm i}(p-P)v}{\rm e}^{{\rm i}(q-Q)u}\\ =&\iint_{-\infty}^{\infty}\frac{{\rm d}u{\rm d}v}{4\pi^2}{\rm e}^{{\rm i}(q-Q)u+{\rm i}uv}{\rm e}^{{\rm i}(p-P)v}\\ =&\int_{-\infty}^{\infty}\frac{{\rm d}v}{2\pi}\delta(q-Q+v)\,{\rm e}^{{\rm i}(p-P)v}\\ =&\mathfrak{Q}\left[\int_{-\infty}^{\infty}\frac{{\rm d}v}{2\pi}\delta(q-Q+v)\,{\rm e}^{{\rm i}v(p-P)}\right]\\ =&\frac{1}{2\pi}\mathfrak{Q}\left[{\rm e}^{-{\rm i}(q-Q)(p-P)}\right].\end{aligned} \tag{11}$$

On the other hand, $\mathfrak{P}$-ordering means that

$$\mathfrak{P}\left(Q^nP^m\right)=P^mQ^n. \tag{12}$$

Due to $[Q,-{\rm i}P]=1$,

$$\left[a,a^\dagger\right]=1\rightarrow[Q,-{\rm i}P]=1,\quad \hbar=1. \tag{13}$$

Therefore, the $\mathfrak{P}$-ordering of ${\rm e}^{-\lambda(-{\rm i}P)Q}$ is

$${\rm e}^{-\lambda(-{\rm i}P)Q}=\mathfrak{P}\left[\exp\left\{\left({\rm e}^{-\lambda}-1\right)(-{\rm i}P)Q\right\}\right]. \tag{14}$$

Within the $\mathfrak{P}$-ordering, Q is commutable with P. In other words, though $[Q,P]={\rm i}$, we have $\mathfrak{P}[QP]=\mathfrak{P}[PQ]=PQ$, so we can perform integration over c-number within $\mathfrak{P}(\cdots)$. For example, the $\mathfrak{P}$-ordering form of $\delta(q-Q)\,\delta(p-P)$ is

$$\begin{aligned}\delta(q-Q)\,\delta(p-P)=&\iint_{-\infty}^{\infty}\frac{{\rm d}u{\rm d}v}{4\pi^2}{\rm e}^{{\rm i}(q-Q)u}{\rm e}^{{\rm i}(p-P)v}\\ =&\iint_{-\infty}^{\infty}\frac{{\rm d}u{\rm d}v}{4\pi^2}{\rm e}^{{\rm i}(p-P)v-{\rm i}uv}{\rm e}^{{\rm i}u(q-Q)}\\ =&\int_{-\infty}^{\infty}\frac{{\rm d}u}{2\pi}\delta(p-P-u)\,{\rm e}^{{\rm i}u(q-Q)}\\ =&\mathfrak{P}\left[\int_{-\infty}^{\infty}\frac{{\rm d}u}{2\pi}\delta(p-P-u)\,{\rm e}^{{\rm i}u(q-Q)}\right]\\ =&\frac{1}{2\pi}\mathfrak{P}\left[{\rm e}^{{\rm i}(p-P)(q-Q)}\right].\end{aligned} \tag{15}$$

In Ref. [5], we have noticed that in coordinate representation the single-mode squeezing operator is

$$S_{1{\rm a}}=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}{\rm d}q\,|q/\mu\rangle\langle q|. \tag{16}$$

This integration has been carried out by using the technique of integration within normal ordering, and now we can use the technique of integration within $\mathfrak{P}$-ordering to tackle it. Using the completeness of momentum representation

$$\int_{-\infty}^{\infty}{\rm d}p\,|p\rangle\langle p|=1,\ P|p\rangle=p|p\rangle, \tag{17}$$

$\langle q|\ p\rangle=\exp({\rm i}pq)/\sqrt{2\pi}$, and

$$\delta(p-P)\,\delta(q-Q)=|p\rangle\langle p|\,q\rangle\langle q|=\frac{1}{\sqrt{2\pi}}|p\rangle\langle q|\,{\rm e}^{-{\rm i}pq}, \tag{18}$$

we have

$$\begin{aligned}S_1=&\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}{\rm d}q\left(\int_{-\infty}^{\infty}{\rm d}p\,|p\rangle\langle p|\right)|q/\mu\rangle\langle q|\\ =&\frac{1}{\sqrt{2\pi\mu}}\iint_{-\infty}^{\infty}{\rm d}q{\rm d}p{\rm e}^{-{\rm i}pq/\mu}\,|p\rangle\langle q|\\ =&\frac{1}{\sqrt{2\pi\mu}}\iint_{-\infty}^{\infty}{\rm d}q{\rm d}p{\rm e}^{{\rm i}pq(1-1/\mu)}\,|p\rangle\langle q|\,{\rm e}^{-{\rm i}pq}\\ =&\frac{1}{\sqrt{\mu}}\iint_{-\infty}^{\infty}{\rm d}q{\rm d}p{\rm e}^{{\rm i}pq(1-1/\mu)}\delta(p-P)\,\delta(q-Q)\\ =&\frac{1}{\sqrt{\mu}}\iint_{-\infty}^{\infty}{\rm d}q{\rm d}p{\rm e}^{{\rm i}pq(1-1/\mu)}\times\mathfrak{P}\left[\delta(p-P)\,\delta(q-Q)\right]\\ =&\frac{1}{\sqrt{\mu}}\mathfrak{P}\left[{\rm e}^{-{\rm i}PQ(1/\mu-1)}\right].\end{aligned} \tag{19}$$

This is the $\mathfrak{P}$-ordering of S_1. Then, using Eq. (14), we obtain

$$\begin{aligned}S_1=&\frac{1}{\sqrt{\mu}}\mathfrak{P}\left[\exp\left\{\left({\rm e}^{-\lambda}-1\right)(-{\rm i}P)Q\right\}\right]\\ =&\frac{1}{\sqrt{\mu}}{\rm e}^{-{\rm i}PQ\ln(1/\mu)}={\rm e}^{{\rm i}\lambda(PQ+QP)/2},\quad \mu={\rm e}^{\lambda}.\end{aligned} \tag{20}$$

Thus, we have used the $\mathfrak{P}$-ordering integration to reach

$$\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}{\rm d}q\,|q/\mu\rangle\langle q|={\rm e}^{{\rm i}\lambda(PQ+QP)/2},\ \mu={\rm e}^{\lambda}. \tag{21}$$

3. Detaching the single-mode squeezing operator from the two-mode squeezing operator

In Ref. [5], we have employed the technique of integration within normal ordering to show the coordinate representation of the two-mode squeezing operator

$$\begin{aligned} S_2 &= \mathrm{e}^{\lambda\left(ab-a^{\dagger}b^{\dagger}\right)} \\ &= \iint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2 |q_1\cosh\lambda - q_2\sinh\lambda, \\ &\quad q_2\cosh\lambda - q_1\sinh\lambda\rangle\langle q_1, q_2|. \end{aligned} \tag{22}$$

Now we consider performing ket-bra integration on the right of Eq. (22) by the $\mathfrak{P}$-ordering method. Based on Eq. (18), we have

$$\begin{aligned} &\iint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2 \left(\iint_{-\infty}^{\infty} \mathrm{d}p_1\,\mathrm{d}p_2\,|p_1,p_2\rangle\langle p_1,p_2|\right) \\ &\times |q_1\cosh\lambda - q_2\sinh\lambda, q_2\cosh\lambda - q_1\sinh\lambda\rangle\langle q_1,q_2| \\ &= \frac{1}{2\pi}\iint \mathrm{d}q_1\,\mathrm{d}q_2 \iint \mathrm{d}p_1\,\mathrm{d}p_2 \\ &\times \mathrm{e}^{-\mathrm{i}p_1(q_1\cosh\lambda - q_2\sinh\lambda) - \mathrm{i}p_2(q_2\cosh\lambda - q_1\sinh\lambda)} \\ &\times |p_1,p_2\rangle\langle q_1,q_2| \\ &= \frac{1}{2\pi}\iint \mathrm{d}q_1\,\mathrm{d}q_2 \iint \mathrm{d}p_1\,\mathrm{d}p_2 \\ &\times \mathrm{e}^{-\mathrm{i}(p_1q_1+p_2q_2)(\cosh\lambda-1)+\mathrm{i}(p_2q_1+p_1q_2)\sinh\lambda} \\ &\times |p_1,p_2\rangle\langle q_1,q_2|\,\mathrm{e}^{-\mathrm{i}p_1q_1-\mathrm{i}p_2q_2} \\ &= \iint \mathrm{d}q_1\,\mathrm{d}q_2 \iint \mathrm{d}p_1\,\mathrm{d}p_2 \\ &\times \mathrm{e}^{-\mathrm{i}(p_1q_1+p_2q_2)(\cosh\lambda-1)+\mathrm{i}(p_2q_1+p_1q_2)\sinh\lambda} \\ &\times \delta(p_1-P_1)\,\delta(p_2-P_2)\,\delta(q_1-Q_1)\,\delta(q_2-Q_2) \\ &= \mathfrak{P}\left\{\mathrm{e}^{-\mathrm{i}(P_1Q_1+P_2Q_2)(\cosh\lambda-1)+\mathrm{i}(P_2Q_1+P_1Q_2)\sinh\lambda}\right\}, \end{aligned}$$

which is in $\mathfrak{P}$-ordering. Furthermore, using the property that Q is commutable with P within $\mathfrak{P}$-ordering, we derive

$$\begin{aligned} &\mathfrak{P}\left\{\mathrm{e}^{-\mathrm{i}(P_1Q_1+P_2Q_2)(\cosh\lambda-1)+\mathrm{i}(P_2Q_1+P_1Q_2)\sinh\lambda}\right\} \\ &= \mathfrak{P}\left\{\sum_{n=0}^{\infty}\frac{\mathrm{i}^n\sinh^n\lambda}{n!}\left[(P_2Q_1+P_1Q_2)^n\right]\right. \\ &\quad \left.\times\,\mathrm{e}^{-\mathrm{i}(P_1Q_1+P_2Q_2)(\cosh\lambda-1)}\right\} \\ &= \mathfrak{P}\left\{\sum_{n=0}^{\infty}\frac{\mathrm{i}^n\sinh^n\lambda}{n!}\sum_{l=0}^{n}\binom{n}{l}P_2^lP_1^{n-l}\right. \\ &\quad \left.\times\,\mathrm{e}^{-\mathrm{i}(P_1Q_1+P_2Q_2)(\cosh\lambda-1)}Q_1^lQ_2^{n-l}\right\} \\ &= \sum_{n=0}^{\infty}\frac{\mathrm{i}^n\sinh^n\lambda}{n!}\sum_{l=0}^{n}\binom{n}{l}P_2^lP_1^{n-l} \\ &\quad \times\mathfrak{P}\{\mathrm{e}^{-\mathrm{i}(P_1Q_1+P_2Q_2)(\cosh\lambda-1)}\}Q_1^lQ_2^{n-l}. \end{aligned} \tag{23}$$

Then, using the identity (14), we have

$$\begin{aligned} S_2 &= \sum_{n=0}^{\infty}\frac{\mathrm{i}^n\sinh^n\lambda}{n!}\sum_{l=0}^{n}\binom{n}{l}P_2^lP_1^{n-l} \\ &\quad \times\mathrm{e}^{-\mathrm{i}(P_1Q_1+P_2Q_2)\ln\cosh\lambda}Q_1^lQ_2^{n-l}. \end{aligned} \tag{24}$$

Then, using the Baker–Hausdorff formula, we have

$$\begin{aligned} &\mathrm{e}^{-\mathrm{i}P_1Q_1\ln\cosh\lambda}Q_1\,\mathrm{e}^{\mathrm{i}P_1Q_1\ln\cosh\lambda} \\ &= Q_1 + [-\mathrm{i}P_1Q_1\ln\cosh\lambda, Q_1] \\ &\quad + \frac{1}{2!}[-\mathrm{i}P_1Q_1\ln\cosh\lambda, [-\mathrm{i}P_1Q_1\ln\cosh\lambda, Q_1]] + \cdots \\ &= Q_1 - Q_1\ln\cosh\lambda + \frac{1}{2!}Q_1(\ln\cosh\lambda)^2 + \cdots \\ &= Q_1\,\mathrm{e}^{-\ln\cosh\lambda} \\ &= Q_1/\cosh\lambda, \end{aligned} \tag{25}$$

so equation (24) becomes

$$\begin{aligned} S_2 &= \sum_{n=0}^{\infty}\frac{\mathrm{i}^n\sinh^n\lambda}{n!}\sum_{l=0}^{n}\binom{n}{l}P_2^lP_1^{n-l}Q_1^lQ_2^{n-l} \\ &\quad \times\cosh^{-n}\lambda\,\mathrm{e}^{-\mathrm{i}(P_1Q_1+P_2Q_2)\ln\cosh\lambda}. \end{aligned} \tag{26}$$

Due to

$$\begin{aligned} \sum_{l=0}^{n}\binom{n}{l}P_2^lP_1^{n-l}Q_1^lQ_2^{n-l} &= \mathfrak{P}\left[\sum_{l=0}^{n}\binom{n}{l}Q_1^lP_2^lP_1^{n-l}Q_2^{n-l}\right] \\ &= \mathfrak{P}[(Q_1P_2+P_1Q_2)^n], \end{aligned} \tag{27}$$

we have

$$\begin{aligned} &\left[\sum_{n=0}^{\infty}\frac{\mathrm{i}^n\sinh^n\lambda\cosh^{-n}\lambda}{n!}\mathfrak{P}(Q_1P_2+P_1Q_2)^n\right] \\ &= \mathfrak{P}\,\mathrm{e}^{\mathrm{i}\tanh\lambda(Q_1P_2+P_1Q_2)}. \end{aligned} \tag{28}$$

Thus,

$$\begin{aligned} S_2 &= \exp[\mathrm{i}\lambda(P_1Q_2+P_2Q_1)] \\ &= \left(\mathfrak{P}\,\mathrm{e}^{\mathrm{i}\tanh\lambda(Q_1P_2+P_1Q_2)}\right)\mathrm{e}^{-\mathrm{i}(P_1Q_1+P_2Q_2)\ln\cosh\lambda}. \end{aligned} \tag{29}$$

Here, the first term is the $\mathfrak{P}$-ordered two-mode squeezing operator with the parameter $\tanh\lambda$, while the second exponential operator, which is detached from S_2, is just the two single-mode squeezing operators with the new squeezing parameter $\ln\cosh\lambda$.

4. Conclusion

In summary, we have found that the two-mode squeezing operator can be further split into a $\mathfrak{P}$-ordered two-mode squeezing operator and two single-mode squeezing operators. This reveals a new point of view that the two-mode squeezing mechanism also involves some single-mode squeezing.

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