Thermal vacuum state for the two-coupled-oscillator model at finite temperature: Derivation and application*

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Following the spirit of thermo field dynamics initiated by Takahashi and Umezawa, we employ the technique of integration within an ordered product of operators to derive the thermal vacuum state (TVS) for the Hamiltonian H of the two-coupled-oscillator model. The ensemble averages of the system are derived conveniently by using the TVS. In addition, the entropy for this system is discussed based on the relation between the generalized Hellmann–Feynman theorem and the entroy variation in the context of the TVS.

Keywords: partial trace method, thermo field dynamics, integration within an ordered product of operators, generalized Hellmann–Feynman theorem

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1. Introduction

In mesoscopic physics, Louisell is the first to quantize a mesoscopic L–C circuit as a quantum harmonic oscillator. However, his discussion is for the case of zero-temperature.[1] In fact, physical systems are never perfectly isolated from their environments. Discussion restricted to the case of zero temperature is unrealistic. Fan and Liang[2] pointed out that since the electric current generates Joule heat, one should take the thermo effect into account, thus every physical observable should be evaluated in the context of ensemble average. Besides, since the entropy increases with the generation of Joule heat, one should consider how temperature affects the variation of entropy.

Currently, enormous effort is put to understanding the rich dynamics of interacting Boson systems.[3] For conveniently expressing the ensemble averages at nonzero temperature T as equivalent pure state expectation values, Takahashi and Umezawa proposed the formalism of thermo field dynamics (TFD).[4,5] This convenience is obtained at the expanse of doubling the degrees of freedom of the quantum system considered, i.e., introducing a fictitious field or called a tilde-conjugate field. We believe that one can analyze the properties of various Boson systems at finite temperature by discussing the corresponding thermal equilibrium state and its applications.[6,7] Since complicated dynamic Bose systems description is based on a model of two coupled harmonical oscillators,[8,9] in this paper, we begin with tackling the two-coupled-oscillator model, which can be produced in a mesoscopic L–C circuit. The Hamiltonian ($\hbar = 1$) reads

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \lambda a_1^\dagger a_2 + \lambda^* a_1 a_2^\dagger, \tag{1}$$

where a_i ($i = 1, 2$) are the annihilation boson operators of the fields with energy $\hbar\omega_{1,2}$, and λ is the coupling constant. We will find out the corresponding thermo vacuum state for this system at finite temperature.

Following the spirit of TFD, we employ the integration within an ordered product of operators (IWOP) technique[10] to derive a thermal vacuum state (TVS) for the Hamiltonian (a pure state in the four-dimensional extended Hilbert space). Taking the partial trace of the TVS, we obtain the density operator $\mathrm{e}^{-\beta H}/\mathrm{Tr}(\mathrm{e}^{-\beta H})$, where $\beta = (kT)^{-1}$, and k is the Boltzmann constant. With the TVS, we present the analytical expression of the Wigner function[11,12] for this system. In addition, the purity of quantum states and the ensemble average of this system in quantum statistical mechanics[13] are obtained by using the TVS. In particular, entropy is discussed according to the relation between the generalized Hellmann–Feynman theorem (GHFT) and the entropy variation in the context of the TVS.[14–17]

Our paper is arranged as follows. In Section 2, we briefly review some basic theories, such as thermal field dynamics and the partial trace method, as well as the relation between the GHFT and the entropy variation. In Section 3, we deduce the TVS and consequently the partition function for Hamiltonian

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H of the two-coupled-oscillator model. As an application, in Section 4 we employ the TVS to calculate the ensemble average for the system in terms of the Wigner function and purity. Finally, the entropy variation is obtained in Section 5.

2. Brief review of basic theories

In this section, we make a simple review on the thermal field dynamics and the partial method as well as the GHFT. Historically, Umezawa and Takahashi were the first to introduce the vacuum (a pure state) expectation values to coincide with the statistical average. Recently Fan has expressed the GHFT in the formalism of pure state which is defined in the enlarged eigenvector space of the Hamiltonian.

2.1. Thermal field dynamics and partial trace method

For a density operator $\rho = \mathrm{e}^{-\beta H}/Z(\beta)$ of Hamiltonian H, following Umezawa–Takahashi' idea, one can introduce the thermal vacuum state $|\phi(\beta)\rangle$ corresponding to the pure state in the extended Hilbert space as follows:

$$|\phi(\beta)\rangle = Z^{-1/2}(\beta)\sum_n \mathrm{e}^{-\beta E_n/2}|\phi_n,\phi_{\tilde{n}}\rangle, \tag{2}$$

with

$$\begin{aligned} &H|\phi_n\rangle = E_n|\phi_n\rangle,\ \tilde{H}|\phi_{\tilde{n}}\rangle = E_n|\phi_{\tilde{n}}\rangle,\\ &\langle\phi_{n'}|\phi_n\rangle = \langle\phi_{\tilde{n}'}|\phi_{\tilde{n}}\rangle = \delta_{n',n},\\ &\langle\phi(\beta)|\phi(\beta)\rangle = 1. \end{aligned} \tag{3}$$

Here a fictitious dynamical system $\tilde{H}$, denoted with the tilde, has been introduced, and $|\phi_n,\phi_{\tilde{n}}\rangle$ is defined in the enlarged eigenvector space of the Hamiltonian. Let Tr denote the trace operation over both the system freedom (expressed by tr) and the environment freedom (expressed by $\widetilde{\mathrm{tr}}$), i.e. Tr = tr$\widetilde{\mathrm{tr}}$, then we have

$$\begin{aligned} \langle\phi(\beta)|A|\phi(\beta)\rangle &= \mathrm{Tr}[A|\phi(\beta)\rangle\langle\phi(\beta)|]\\ &= \mathrm{tr}\left[A\widetilde{\mathrm{tr}}|\phi(\beta)\rangle\langle\phi(\beta)|\right]. \end{aligned} \tag{4}$$

Since $|\phi(\beta)\rangle$ involves both real and fictitious modes, we should note that $\widetilde{\mathrm{tr}}[|\phi(\beta)\rangle\langle\phi(\beta)|] \neq \langle\phi(\beta)|\phi(\beta)\rangle$. Thus for a given Hamiltonian H, we have

$$\widetilde{\mathrm{tr}}[|\phi(\beta)\rangle\langle\phi(\beta)|] = \mathrm{e}^{-\beta H}/Z(\beta), \tag{5}$$

which indicates that if we can find a density operator of pure state $|\phi(\beta)\rangle$ in the doubled Hilbert space, its partial trace over the tilde freedom may lead to the density operator $\mathrm{e}^{-\beta H}/Z(\beta)$ of the system. In addition, the average value of operator A can be calculated as an equivalent expectation value with pure state $|\phi(\beta)\rangle$, i.e.,

$$\langle A\rangle = \mathrm{tr}\left[A\,\mathrm{e}^{-\beta H}/Z(\beta)\right] = \langle\phi(\beta)|A|\phi(\beta)\rangle. \tag{6}$$

2.2. GHFT in the formalism of pure state expectation and entropy variation by GHFT in the context of $|\phi(\beta)\rangle$

In Ref. [18], Fan and Chen have derived the GHFT regarding the mixed state ensemble average (statistical average) in quantum statistics

$$\begin{aligned} &\frac{\partial}{\partial\chi}\langle H(\chi)\rangle_{\mathrm{e}}\\ &= \left\langle\left(1+\beta\langle H(\chi)\rangle_{\mathrm{e}} - \beta H(\chi)\right)\frac{\partial H(\chi)}{\partial\chi}\right\rangle_{\mathrm{e}}, \end{aligned} \tag{7}$$

where $\langle A\rangle_{\mathrm{e}} \equiv z^{-1}(\beta,\chi)\,\mathrm{Tr}\left(\mathrm{e}^{-\beta H(\chi)}A\right)$ with $z(\beta,\chi) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta H(\chi)}\right)$, and χ is a parameter involved in the system. Their derivation is entirely based on the ensemble average formula $\langle H(\lambda)\rangle_{\mathrm{e}}$. Recently, they also obtained the GHFT in the pure state expectation value of $|\phi(\beta)\rangle$[16]

$$\begin{aligned} &\frac{\partial}{\partial\chi}\langle\phi(\beta)|H(\chi)|\phi(\beta)\rangle\\ &= \langle\phi(\beta)|\left[1+\beta\langle\phi(\beta)|H(\chi)|\phi(\beta)\rangle - \beta H(\chi)\right]\\ &\quad\times\frac{\partial H(\chi)}{\partial\chi}|\phi(\beta)\rangle. \end{aligned} \tag{8}$$

The essence of these two different formalisms of the GHFT in Eqs. (7) and (8) is the same. Assuming that $H(\chi)$ is independent of β (notice that $\partial H(\chi)/\partial\beta = 0$), we can obtain the energy fluctuation

$$(\Delta H)^2 = -\frac{\partial\langle\phi(\beta)|H(\chi)|\phi(\beta)\rangle}{\partial\beta}. \tag{9}$$

On the other hand, the quantum-mechanical entropy (the von Neumann entropy) is defined as

$$S = -k\mathrm{Tr}(\rho\ln\rho), \tag{10}$$

which can be rewritten as

$$\begin{aligned} S &= \beta k\mathrm{Tr}[\rho H(\chi)] + k\mathrm{Tr}[\rho\ln Z(\beta,\chi)]\\ &= \frac{1}{T}\langle\phi(\beta)|H(\chi)|\phi(\beta)\rangle + k\ln Z(\beta,\chi). \end{aligned} \tag{11}$$

It then follows that

$$\begin{aligned} \frac{\partial S(\chi)}{\partial\chi} &= \frac{1}{T}\frac{\partial\langle\phi(\beta)|H(\chi)|\phi(\beta)\rangle}{\partial\chi}\\ &\quad -\frac{1}{T}\left\langle\phi(\beta)\left|\frac{\partial H(\chi)}{\partial\chi}\right|\phi(\beta)\right\rangle, \end{aligned} \tag{12}$$

which reveals the relation between the GHFT and the entropy variation of the von Neumann entropy.

3. Thermal vacuum state and partition function for the two-coupled-oscillator model

For the interesting system with Hamiltonian (1), the normalized density operator $\rho = \mathrm{e}^{-\beta H}/Z$ is

$$\rho = Z^{-1}\exp\left[\begin{pmatrix} a_1^\dagger & a_2^\dagger\end{pmatrix}\begin{pmatrix} -\beta\omega_1 & -\beta\lambda\\ -\beta\lambda^* & -\beta\omega_2\end{pmatrix}\begin{pmatrix} a_1\\ a_2\end{pmatrix}\right], \tag{13}$$

with the partition function $Z = \mathrm{Tr}\,\mathrm{e}^{-\beta H}$. According to the following normally ordered expansion formula (: : denotes the normal ordering)[19,20]

$$\exp\left[\left(a_1^\dagger\ a_2^\dagger\right)\Lambda\begin{pmatrix}a_1\\a_2\end{pmatrix}\right] = :\exp\left[\left(a_1^\dagger\ a_2^\dagger\right)\left(\mathrm{e}^{\Lambda}-1\right)\begin{pmatrix}a_1\\a_2\end{pmatrix}\right]:, \tag{14}$$

$$\exp\begin{pmatrix}-\beta\omega_1 & -\beta\lambda\\ -\beta\lambda^* & -\beta\omega_2\end{pmatrix} = \begin{pmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{21} & \Pi_{22}\end{pmatrix}, \tag{15}$$

with

$$\begin{aligned}\Pi_{11} &= \mathrm{e}^{-\frac{\beta(\omega_1+\omega_2)}{2}}\left(\cosh\frac{\beta\Xi}{2}+\Omega\sinh\frac{\beta\Xi}{2}\right),\\ \Pi_{22} &= \mathrm{e}^{-\frac{\beta(\omega_1+\omega_2)}{2}}\left(\cosh\frac{\beta\Xi}{2}-\Omega\sinh\frac{\beta\Xi}{2}\right),\\ \Pi_{12} &= -\frac{2\lambda}{\Xi}\mathrm{e}^{-\frac{\beta(\omega_1+\omega_2)}{2}}\sinh\frac{\beta\Xi}{2}=\Pi_{21}^*,\end{aligned} \tag{16}$$

where $\Xi = \sqrt{(\omega_1-\omega_2)^2+4\lambda\lambda^*}$ and $\Omega = (\omega_2-\omega_1)/\Xi$, we have the following normal ordering of $\mathrm{e}^{-\beta H}$:

$$\begin{aligned}\mathrm{e}^{-\beta H} = :\exp&\left[(\Pi_{11}-1)a_1^\dagger a_1+(\Pi_{22}-1)a_2^\dagger a_2\right]\\ &\times\exp\left(\Pi_{12}a_1^\dagger a_2+\Pi_{21}a_1a_2^\dagger\right):.\end{aligned} \tag{17}$$

Noticing $|00\rangle\langle 00| =: \exp\left(-a_1^\dagger a_1-a_2^\dagger a_2\right):$ and using the integration formula[21]

$$\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{\zeta|z|^2+\xi z+\eta z^*} = -\frac{1}{\zeta}\mathrm{e}^{-\frac{\xi\eta}{\zeta}},\quad \mathrm{Re}(\xi)<0, \tag{18}$$

we can rewrite Eq. (17) as

$$\begin{aligned}\mathrm{e}^{-\beta H} &= \int\frac{\mathrm{d}^2\alpha_1\mathrm{d}^2\alpha_2}{\pi^2}\left|\tilde{\alpha}_1\tilde{\alpha}_2\right\rangle\\ &\quad\times\exp\left[\left(a_1^\dagger\ a_2^\dagger\right)\Theta\begin{pmatrix}\alpha_1^*\\ \alpha_2^*\end{pmatrix}\right]\\ &\quad\times\left|0\tilde{0},0\tilde{0}\right\rangle\left\langle 0\tilde{0},0\tilde{0}\right|\\ &\quad\times\exp\left[\left(\alpha_1\ \alpha_2\right)\Theta\begin{pmatrix}a_1\\a_2\end{pmatrix}\right]\left|\tilde{\alpha}_1\tilde{\alpha}_2\right\rangle\\ &=\widetilde{\mathrm{tr}}\left\{\int\frac{\mathrm{d}^2\alpha_1\mathrm{d}^2\alpha_2}{\pi^2}\left|\tilde{\alpha}_1\tilde{\alpha}_2\right\rangle\left\langle\tilde{\alpha}_1\tilde{\alpha}_2\right|\right.\\ &\quad\times\left[\exp\left[\left(a_1^\dagger\ a_2^\dagger\right)\Theta\begin{pmatrix}\alpha_1^*\\ \alpha_2^*\end{pmatrix}\right]\left|0\tilde{0},0\tilde{0}\right\rangle\left\langle 0\tilde{0},0\tilde{0}\right|\right.\\ &\quad\left.\left.\times\exp\left[\left(\alpha_1\ \alpha_2\right)\Theta\begin{pmatrix}a_1\\a_2\end{pmatrix}\right]\right]\right\},\end{aligned} \tag{19}$$

where $\Theta = \begin{pmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{21} & \Theta_{22}\end{pmatrix}$, and $|\tilde{\alpha}_1\tilde{\alpha}_2\rangle$ is the coherent state[22,23] in the fictitious mode with

$$\begin{aligned}\left|\tilde{\alpha}_i\right\rangle &= \exp\left(\alpha_i\tilde{a}_i^\dagger-\alpha_i^*\tilde{a}_i\right)\left|\tilde{0}\right\rangle,\\ \tilde{a}_i\left|\tilde{\alpha}_i\right\rangle &= \alpha_i\left|\tilde{\alpha}_i\right\rangle,\ \left\langle\tilde{0}\right|\left.\tilde{\alpha}_i\right\rangle = \mathrm{e}^{-|\alpha_i|^2/2}.\end{aligned}$$

And the condition

$$\begin{aligned}\Pi_{11} &= \Theta_{11}^2+\Theta_{12}\Theta_{21},\\ \Pi_{12} &= \Theta_{12}(\Theta_{11}+\Theta_{22}),\\ \Pi_{22} &= \Theta_{22}^2+\Theta_{12}\Theta_{21},\\ \Pi_{21} &= \Theta_{21}(\Theta_{11}+\Theta_{22})\end{aligned} \tag{20}$$

is satisfied. On the other hand, noting $\rho Z(\beta) = \mathrm{e}^{-\beta H}$ and setting $\rho = \widetilde{\mathrm{tr}}\left[\left|\phi(\beta)\right\rangle\left\langle\phi(\beta)\right|\right]$ in Eq. (19), we obtain

$$\begin{aligned}\left|\phi(\beta)\right\rangle &= \sqrt{\frac{1}{Z(\beta)}}\exp\left(\Theta_{11}a_1^\dagger\tilde{a}_1^\dagger+\Theta_{22}a_2^\dagger\tilde{a}_2^\dagger\right)\\ &\quad\times\exp\left(\Theta_{12}a_1^\dagger\tilde{a}_2^\dagger+\Theta_{21}\tilde{a}_1^\dagger a_2^\dagger\right)\left|0\tilde{0},0\tilde{0}\right\rangle,\end{aligned} \tag{21}$$

where

$$\begin{aligned}\Theta_{11} &= \mathrm{e}^{-\frac{\beta(\omega_1+\omega_2)}{4}}\left(\cosh\frac{\beta\Xi}{4}+\Omega\sinh\frac{\beta\Xi}{4}\right),\\ \Theta_{22} &= \mathrm{e}^{-\frac{\beta(\omega_1+\omega_2)}{4}}\left(\cosh\frac{\beta\Xi}{4}-\Omega\sinh\frac{\beta\Xi}{4}\right),\\ \Theta_{12} &= -\frac{2\lambda}{\Xi}\mathrm{e}^{-\frac{\beta(\omega_1+\omega_2)}{4}}\sinh\frac{\beta\Xi}{4}=\Theta_{21}^*.\end{aligned} \tag{22}$$

Equation (21) is the TVS, which is a pure state in the pair-doubled Fock space for the Hamiltonian (1) at finite temperature and is important for the following calculations.

The partition function $Z(\beta)$ is determined by $Z(\beta) = \mathrm{tr}\left(\mathrm{e}^{-\beta H}\right)$. Using the normal ordering of $\mathrm{e}^{-\beta H}$ in Eq. (17) and inserting the coherent state completeness relation $\int(\mathrm{d}^2z_1\mathrm{d}^2z_2/\pi^2)\left|z_1,z_2\right\rangle\left\langle z_1,z_2\right| = 1$ as well as using the integral formula in Eq. (18), we have

$$Z(\beta) = \left[\left(1-\mathrm{e}^{-\beta\varepsilon_1}\right)\left(1-\mathrm{e}^{-\beta\varepsilon_2}\right)\right]^{-1}, \tag{23}$$

where we have set $\varepsilon_1 = (\omega_1+\omega_2+\Xi)/2$ and $\varepsilon_2 = (\omega_1+\omega_2-\Xi)/2$. Knowing the partition function $Z(\beta)$ and using the thermodynamical relation $\langle H\rangle_{\mathrm{e}} = -(\partial/\partial\beta)\ln Z(\beta)$, we obtain the internal energy of the system

$$\langle H\rangle_{\mathrm{e}} = \frac{\varepsilon_1}{\mathrm{e}^{\beta\varepsilon_1}-1}+\frac{\varepsilon_2}{\mathrm{e}^{\beta\varepsilon_2}-1}. \tag{24}$$

In particular, when $\lambda = 0$, leading to $\varepsilon_1\to\omega_1, \varepsilon_2\to\omega_2$, then $H\to\omega_1a_1^\dagger a_1+\omega_2a_2^\dagger a_2$, we can verify that $Z_{\lambda=0}(\beta) = Z_1(\beta)Z_2(\beta)$ in Eq. (23), where

$$Z_i(\beta) = \left(1-\mathrm{e}^{-\beta\omega_i}\right)^{-1},\quad (i=1,2) \tag{25}$$

are the partition functions for the systems with Hamiltonians $\omega_ia_i^\dagger a_i$. Similarly, equation (24) reduces to $\langle H\rangle_{\mathrm{e}(\lambda=0)} = U = U_1+U_2$ as expected, where

$$U_i = \frac{\omega_i}{\mathrm{e}^{\beta\omega_i}-1},\quad (i=1,2) \tag{26}$$

are the internal energies of simple harmonic oscillators with Hamiltonians $\omega_ia_i^\dagger a_i$.

4. The Wigner function and purity of the two-coupled-oscillator system at finite temperature

Knowing the TVS $|\phi(\beta)\rangle$, any average value of operator A corresponding to the system can be calculated as an equivalent expectation value by using Eq. (6). The Wigner function is a powerful tool used to investigate the nonclassicality of a quantum state.[24] As an application, we calculate the Wigner function of the system and then discuss the purity in this section.

For the TVS $|\phi(\beta)\rangle$, the Wigner function is given by

$$W(\alpha_1,\alpha_2)=\langle\phi(\beta)|\Delta(\alpha_1)\Delta(\alpha_2)|\phi(\beta)\rangle, \tag{27}$$

where $\Delta(\alpha_i)$ is the Wigner operator

$$\Delta(\alpha_i)=\frac{2\,\mathrm{e}^{2|\alpha_i|^2}}{\pi}\int\frac{\mathrm{d}^2 z_i}{\pi}|z_i\rangle\langle -z_i|\,\mathrm{e}^{-2\left(z_i\alpha_i^*-z_i^*\alpha_i\right)}, \tag{28}$$

with $|z_i\rangle$ being the coherent state and $\alpha_i=(q_i+\mathrm{i}p_i)/\sqrt{2}$. Substituting Eq. (28) into Eq. (27), we obtain

$$\begin{aligned}W(\alpha_1,\alpha_2) &= \frac{4M}{\pi^2}\exp(J_1|\alpha_1|^2+J_2|\alpha_2|^2)\\ &\quad\times\exp(+J_3\alpha_1\alpha_2^*+J_3^*\alpha_1^*\alpha_2),\end{aligned} \tag{29}$$

where we have set $M=\tanh(\beta\varepsilon_1/2)\tanh(\beta\varepsilon_2/2)$ and

$$\begin{aligned}J_1 &= -2\frac{\sinh[\beta(\omega_1+\omega_2)/2]-\Omega\sinh(\beta\Xi/2)}{\cosh[\beta(\omega_1+\omega_2)/2]+\cosh(\beta\Xi/2)},\\ J_2 &= -2\frac{\sinh[\beta(\omega_1+\omega_2)/2]+\Omega\sinh(\beta\Xi/2)}{\cosh[\beta(\omega_1+\omega_2)/2]+\cosh(\beta\Xi/2)},\\ J_3 &= -\frac{4\lambda^*}{\Xi}\frac{\sinh(\beta\Xi/2)}{\cosh[\beta(\omega_1+\omega_2)/2]+\cosh(\beta\Xi/2)}.\end{aligned} \tag{30}$$

Equation (29) is just the Wigner function of the TVS showing the Gaussian form. In particular, when $\lambda=0$, leading to $M\to\tanh(\beta\omega_1/2)\tanh(\beta\omega_2/2)$, $J_1\to-2\tanh(\beta\omega_1/2$, $J_2\to-2\tanh(\beta\omega_2/2)$, $J_3\to0$, equation (29) reduce to $W(\alpha_1,\alpha_2)=W_1(\alpha_1)W_2(\alpha_2)$, where

$$W_i(\alpha_i)=\frac{2}{\pi}\tanh\frac{\beta\omega_i}{2}\exp\left(-2\tanh\frac{\beta\omega_i}{2}|\alpha_i|^2\right) \tag{31}$$

is the Wigner function of the system with the Hamiltonian $H=\omega_i a^\dagger a$ $(i=1,2)$. From Eq. (27), we can see that the calculation of the Wigner function for the two-coupled-oscillator model at finite temperature is converted into the expectation value of the Wigner operator in the TVS.

The purity is fundamental in distinguishing a pure state from a mixed one. As we know, the purity P of a quantum state ρ is defined as $P\equiv\mathrm{Tr}[\rho^2]$ with $0<P\leq1$. We can calculate the purity by using the Wigner function instead of the density operator[25] as follows:

$$P=\pi^2\int\mathrm{d}^2\alpha_1\,\mathrm{d}^2\alpha_2\,(W(\alpha_1,\alpha_2))^2. \tag{32}$$

Moreover, we find that the purity $P=\mathrm{Tr}\left(\rho^2\right)$ is just

$$P=M=\tanh\frac{\beta\varepsilon_1}{2}\tanh\frac{\beta\varepsilon_2}{2}. \tag{33}$$

Based on the the relation in Eq. (33), we depict the variation of the purity as a function of temperature T for some given $\omega=\omega_2$ and λ with $k=1$ in Fig. 1. The figure illustrates that the purity has a monotonically decreasing with the increasing temperature T. When T goes to zero, the purity tends to unity (characteristic of a pure state). While T goes to a high temperature, the purity tends to zero.

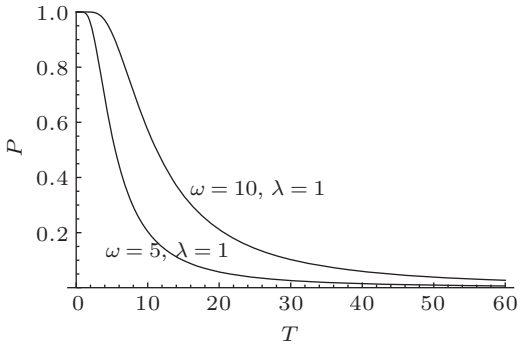


Fig. 1. Purity P as a function of temperature T for different $\omega_1=\omega_2=\omega$ and λ.

5. Entropy variation with respect to the temperature

In this section, based on the relation between the GHFT and the entropy variation, we investigate the influence of the temperature on the entropy of the two-coupled-oscillator model. Choosing ω_1 as the differential parameter, we then calculate $\langle\phi(\beta)|(\partial H/\partial\omega_1)|\phi(\beta)\rangle$ and $\langle\phi(\beta)|H|\phi(\beta)\rangle$ first.

According to the Weyl–Wigner quantization rule, the ensemble average for operator $H(\chi)$ reads

$$\langle\phi(\beta)|H(\chi)|\phi(\beta)\rangle=\int\mathrm{d}^2\alpha_1\,\mathrm{d}^2\alpha_2 h(\chi)W(\alpha_1,\alpha_2), \tag{34}$$

where $h(\chi)$ is the classical Weyl corresponding function of operator $H(\chi)$. Taking $a_1^\dagger a_1=\partial H/\partial\omega_1$ as an example, due to $\alpha_1^*\alpha_1-1/2$ is the classical Weyl correspondence of $a_1^\dagger a_1$, using Eqs. (29) and (34), we have

$$\begin{aligned}\left\langle a_1^\dagger a_1\right\rangle &= \langle\phi(\beta)|\frac{\partial H}{\partial\omega_1}|\phi(\beta)\rangle\\ &= \int\mathrm{d}^2\alpha_1\,\mathrm{d}^2\alpha_2(\alpha_1^*\alpha_1-1/2)W(\alpha_1,\alpha_2)\\ &= Z(\beta)-\frac{\Xi-\omega_1+\omega_2}{2\Xi}Z(\beta)\,\mathrm{e}^{-\beta\varepsilon_1}\\ &\quad-\frac{\Xi+\omega_1-\omega_2}{2\Xi}Z(\beta)\,\mathrm{e}^{-\beta\varepsilon_2}-1.\end{aligned} \tag{35}$$

In particular, when $\lambda=0$, $\left\langle a_1^\dagger a_1\right\rangle=\left(\mathrm{e}^{\beta\omega_1}-1\right)^{-1}$ as expected. In the same way, we obtain $\langle\phi(\beta)|H|\phi(\beta)\rangle=\langle H\rangle_{\mathrm{e}}$, which is in consistent with Eq. (24).

As long as we know the internal energy $\langle\phi(\beta)|H|\phi(\beta)\rangle$ and $\langle\phi(\beta)|(\partial H/\partial\omega_1)|\phi(\beta)\rangle$, we can find other thermodynamic functions. By differentiating $\langle\phi(\beta)|H|\phi(\beta)\rangle$ with respect to β, we obtain the energy fluctuation of the system H as follows:

$$(\Delta H)^2 = \frac{\varepsilon_1^2 \mathrm{e}^{\beta\varepsilon_1}}{\left(\mathrm{e}^{\beta\varepsilon_1}-1\right)^2} + \frac{\varepsilon_2^2 \mathrm{e}^{\beta\varepsilon_2}}{\left(\mathrm{e}^{\beta\varepsilon_2}-1\right)^2}. \tag{36}$$

Substituting Eqs. (24) and (35) to Eq. (12), we readily get the distribution of entropy as follows:

$$\begin{aligned} S = & -k\ln\left[\left(1-\mathrm{e}^{-\beta\varepsilon_1}\right)\left(1-\mathrm{e}^{-\beta\varepsilon_2}\right)\right] \\ & -\frac{k\beta\varepsilon_1}{1-\mathrm{e}^{\beta\varepsilon_1}} - \frac{k\beta\varepsilon_2}{1-\mathrm{e}^{\beta\varepsilon_2}}. \end{aligned} \tag{37}$$

Obviously when $\lambda = 0$, the entropy in Eq. (37) corresponds to two independent oscillators. Based on the the relation in Eq. (37), we depict the variation of the entropy as a function of temperature T for some given $\omega = \omega_2$ and λ with $k = 1$ in Fig. 2. The figure illustrates that the entropy has a monotonically increasing with the increasing temperature T. In particular when T goes to zero, the entropy tends to zero.

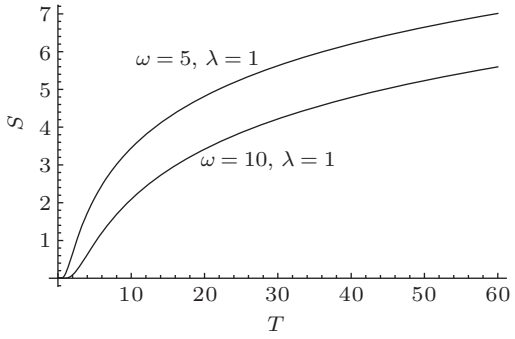


Fig. 2. Entropy S as a function of temperature T for different $\omega_1 = \omega_2 = \omega$ and λ.

6. Conclusion

In conclusion, using the partial trace method and the IWOP technique, we have obtained an equivalent pure state for the two-coupled-oscillator model at finite temperature. It is convenient to calculate ensemble averages at nonzero temperature T as equivalent pure state expectation values. After obtaining the TVS, we present the analytical expression of the Wigner function, which is a Gaussian function related with the temperature. In addition, the purity of quantum states and the ensemble average of this system in quantum statistical mechanics is obtained. The entropy is also discussed according to the relation between the GHFT and the entropy variation in the context of the TVS. Solid state circuit quantum electrodynamics systems have been widely explored in quantum information processing in recent years, and the *L–C* circuit is one of them. These results may be useful for exploring the applications of the continuous variable mode[26–28] produced in the *L–C* circuit system in quantum information processing.

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