Symmetries and conserved quantities of discrete wave equation associated with the Ablowitz–Ladik–Lattice system*

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In this paper, we present a new method to obtain the Lie symmetries and conserved quantities of the discrete wave equation with the Ablowitz–Ladik–Lattice equations. Firstly, the wave equation is transformed into a simple difference equation with the Ablowitz–Ladik–Lattice method. Secondly, according to the invariance of the discrete wave equation and the Ablowitz–Ladik–Lattice equations under infinitesimal transformation of dependent and independent variables, we derive the discrete determining equation and the discrete restricted equations. Thirdly, a series of the discrete analogs of conserved quantities, the discrete analogs of Lie groups, and the characteristic equations are obtained for the wave equation. Finally, we study a model of a biological macromolecule chain of mechanical behaviors, the Lie symmetry theory of discrete wave equation with the Ablowitz–Ladik–Lattice method is verified.

Keywords: symmetry, invariant, Ablowitz–Ladik–Lattice system, wave equation

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1. Introduction

Conserved laws play an important role in theoretical physics, analytical mechanics, biology, etc. To study the conserved quantities of a wave equation by using the symmetry theory is a direction developed recently in mathematical physics. The symmetry theory mainly covers the Noether symmetry and the Lie symmetry of the systems.[1–4] The Noether symmetry is an invariance of the Hamilton's action of the system under the infinitesimal transformations of groups on time and coordinates. The Lie symmetry is an invariance of the differential equations under the infinitesimal transformations of groups on time and coordinates.[5–7]

Recently, the symmetry of a wave equation was studied in Ref. [8]. Studying the wave equation with the Lie and the Noether symmetries to solve the complicated determining equation or the Killing equations is a difficult task.[9–13] So it is difficult to obtain the conserved quantities.[14–19] In this paper, we present a new method for studying the Lie symmetries of a nonlinear wave equation. Firstly, to reduce the wave equation with the method of Ablowitz–Ladik–Lattice, the continuous wave equation is transformed into a simple discrete net equation. Secondly, we give the discrete vector operator of infinitesimal transformation of dependent and independent variables. Thirdly, according to the invariance of the discrete wave equation and the restricted equations under the infinitesimal transformation of dependent and independent variables, we derive the determining equation of the Lie symmetry. Finally, we study a model of a biological macromolecule chain of mechanical behaviors by integrating the characteristic equations; a series of conserved quantities is obtained.

2. Wave equation

We consider a nonlinear wave equation as follows:[20]

$$u_{tt} = [f(u)\,u_x]_x . \tag{1}$$

In the last two decades, the search for the symmetries of one-dimensional nonlinear wave equations has been carried out. Torrisi and Valenti[21] have investigated the symmetries of the following equations:

$$u_{tt} = [f_1(u)\,u_x]_x , \tag{2}$$

$$u_{tt} = [f_2(u)\,u_x + g_1(x,u)]_x , \tag{3}$$

where f_1, f_2, and g_1 are arbitrary functions of their arguments, t is the time, and x is the one-space coordinate. An attempt was made to collect many types of equations previously discussed by considering a wide class of nonlinear wave equation of the form

$$u_{tt} = f_3(x, v_x)\,v_{xx} + g_2(x, v_x) , \tag{4}$$

where f_3 and g_2 are arbitrary functions of x, t, u_x, and u_t. Here, we consider the discrete case of Eq. (4).

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3. Reduction of discrete wave equation with the Ablowitz–Ladik–Lattice method

In order to obtain a discrete wave equation, we use the Ablowitz–Ladik–Lattice system

$$\begin{aligned}\dot{u}_n(t) &= (\alpha + u_n x_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n,\\ \dot{x}_n(t) &= -(\alpha + u_n x_n)(x_{n+1} + x_{n-1}) + 2\alpha x_n.\end{aligned} \tag{5}$$

Differentiate Eq. (5) with respect to t, which is equivalent to

$$\begin{aligned}\ddot{u}_n(t) &= (\dot{u}_n(t)x_n + \dot{x}_n(t)u_n)(u_{n+1} + u_{n-1})\\ &\quad + (\alpha + x_n u_n)(\dot{u}_{n+1} + \dot{u}_{n-1}) - 2\alpha\dot{u}_n(t),\\ \ddot{x}_n(t) &= -(\dot{u}_n(t)x_n + \dot{x}_n(t)u_n)(x_{n+1} + x_{n-1})\\ &\quad - (\alpha + x_n u_n)(\dot{x}_{n+1} + \dot{x}_{n-1}) + 2\alpha\dot{x}_n(t),\end{aligned} \tag{6}$$

where

$$\begin{aligned}\dot{u}_{n+1}(t) &= (\alpha + u_{n+1}x_{n+1})(u_{n+2} + u_n) - 2\alpha u_{n+1},\\ \dot{u}_{n-1}(t) &= (\alpha + u_{n-1}x_{n-1})(u_{n-2} + u_n) - 2\alpha u_{n-1},\end{aligned}$$

and make $\alpha = 0$, then we can obtain

$$\begin{aligned}\ddot{u}_n(t) &= (u_{n+1} + u_{n-1})\left[u_n x_n^2(u_{n+1} + u_{n-1})\right.\\ &\quad \left. - u_n^2 x_n(x_{n+1} + x_{n-1})\right] + u_n x_n[x_{n+1}u_{n+1}(u_{n+2} + u_n)\\ &\quad + u_{n-1}x_{n-1}(u_n + u_{n-2})],\\ \ddot{x}_n(t) &= -\left[u_n x_n^2(u_{n+1} + u_{n-1}) - u_n^2 x_n(x_{n+1} + x_{n-1})\right]\\ &\quad \times (x_{n+1} + x_{n-1}) + u_n x_n\left[x_{n+1}u_{n+1}(x_{n+2} + x_n)\right.\\ &\quad \left. + u_{n-1}x_{n-1}(x_n + x_{n-2})\right].\end{aligned} \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (4), we can reduce the discrete wave equation to

$$\begin{aligned}&\left[u_n x_n^2(u_{n+1} + u_{n-1}) - u_n{}^2 x_n(x_{n+1} + x_{n-1})\right](u_{n+1} + u_{n-1})\\ &\quad + u_n x_n\left[x_{n+1}u_{n+1}(u_n + u_{n+2}) + x_{n-1}u_{n-1}(u_n + u_{n-2})\right]\\ &= f_3(u_{n+1} + u_{n-1})[x_{n+1}u_{n+1}(x_{n+2} + x_n)\\ &\quad + x_{n-1}u_{n-1}(x_{n-2} + x_n) - x_n(x_{n+1} + x_{n-1})^2]^{-1} + g_2.\end{aligned} \tag{8}$$

If we fix

$$\begin{aligned}f_3 &= u_n x_n[x_{n+1}u_{n+1}(x_{n+2} + x_n) + x_{n-1}u_{n-1}(x_{n-2} + x_n)\\ &\quad - x_n(x_{n+1} + x_{n-1})^2](u_{n+1} + u_{n-1})^{-1},\\ g_2 &= u_n x_n x_{n+1}u_{n+1}(u_n + u_{n+2}) + u_n x_n^2(u_{n+1} + u_{n-1})^2,\end{aligned} \tag{9}$$

then equation (8) can be rewritten as

$$\begin{aligned}&x_{n-1}u_{n-1}(u_n + u_{n-2})\\ &\quad - u_n(u_{n-1} + u_{n+1})(x_{n-1} + x_{n+1}) - 1 = 0.\end{aligned} \tag{10}$$

The Ablowitz–Ladik–Lattice system can be written as

$$\begin{aligned}u_{n+1} - u_n &= (\alpha + u_n x_n)(u_{n+1} + u_{n-1})(t_{n+1} - t_n)\\ &\quad - 2\alpha u_n(t_{n+1} - t_n),\\ x_{n+1} - x_n &= -(\alpha + u_n x_n)(x_{n+1} - x_n)(t_{n+1} - t_n)\\ &\quad + 2\alpha x_n(t_{n+1} - t_n).\end{aligned} \tag{11}$$

Introduce the infnitesimal transformation with respect to the following variables:

$$\begin{aligned}t_n^* &= t_n + \varepsilon\lambda_1(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\\ t_{n+1}^* &= t_{n+1} + \varepsilon\lambda_2(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\\ x_{n-1}^* &= x_{n-1} + \varepsilon\xi_1(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\\ x_n^* &= x_n + \varepsilon\xi_2(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\\ x_{n+1}^* &= x_{n+1} + \varepsilon\xi_3(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\\ u_{n-2}^* &= u_{n-2} + \varepsilon\eta_1(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\\ u_{n-1}^* &= u_{n-1} + \varepsilon\eta_2(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\\ u_n^* &= u_n + \varepsilon\eta_3(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\\ u_{n+1}^* &= u_{n+1} + \varepsilon\eta_4(t_n, t_{n+1}, x_{n-1}, x_n, x_{n+1}, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}),\end{aligned} \tag{12}$$

where λ_i, ξ_j, and η_k ($i = 1, 2$; $j = 1, 2, 3$; $k = 1, 2, 3$) constitute the Lie group of infnitesimal transformations. Furthermore, we introduce the operator of the infinitesimal transformation

$$\begin{aligned}\mathrm{X} &= \lambda_1\frac{\partial}{\partial t_n} + \lambda_2\frac{\partial}{\partial t_{n+1}} + \xi_1\frac{\partial}{\partial x_{n-1}} + \xi_2\frac{\partial}{\partial x_n} + \xi_3\frac{\partial}{\partial x_{n+1}}\\ &\quad + \eta_1\frac{\partial}{\partial u_{n-2}} + \eta_2\frac{\partial}{\partial u_{n-1}} + \eta_3\frac{\partial}{\partial u_n} + \eta_4\frac{\partial}{\partial u_{n+1}}.\end{aligned} \tag{13}$$

The invariants of Eqs. (10) and (11) under the infinitesimal transformation lead to the determining equation

$$\begin{aligned}&\xi_1[u_{n-1}(u_n + u_{n-2}) - u_n(u_{n-1} + u_{n+1})]\\ &- \xi_3[u_n(u_{n-1} + u_{n+1})] + \eta_1(x_{n-1}u_{n-1})\\ &+ \eta_2[x_{n-1}(u_n + u_{n-2}) - u_n(x_{n-1} + x_{n+1})]\\ &+ \eta_3[x_{n-1}u_{n-1} - (u_{n-1} + u_{n+1})(x_{n-1} + x_{n+1})]\\ &- \eta_4[u_n(x_{n-1} + x_{n+1})] = 0,\end{aligned} \tag{14}$$

and the restrained equations

$$\begin{aligned}
&\eta_4[1-(\alpha+u_nx_n)(t_{n+1}-t_n)]\\
&-\eta_3[1+x_n(u_{n+1}+u_{n-1})(t_{n+1}-t_n)\\
&-2\alpha(t_{n+1}-t_n)]+(\alpha+u_nx_n)\eta_2(t_{n+1}-t_n)\\
=\;&\xi_2u_n(u_{n+1}+u_{n-1})(t_{n+1}-t_n)\\
&+(\alpha+u_nx_n)(u_{n+1}+u_{n-1})(\lambda_2-\lambda_1),
\end{aligned}\tag{15}$$

$$\begin{aligned}
\xi_3-\xi_2 =\;&-\eta_3x_n(x_{n+1}-x_n)(t_{n+1}-t_n)\\
&-\xi_2u_n(x_{n+1}-x_n)(t_{n+1}-t_n)\\
&-(\alpha+u_nx_n)(\xi_3-\xi_2)(t_{n+1}-t_n)\\
&-(\alpha+u_nx_n)(x_{n+1}-x_n)(\lambda_2-\lambda_1)\\
&-2\alpha\xi_2(t_{n+1}-t_n)+2\alpha x_n(\lambda_2-\lambda_1).
\end{aligned}\tag{16}$$

Equations (14)–(16) have a series of solutions as follows. Let

$$\begin{aligned}
\lambda_1 &= \xi_1=\xi_3=\eta_3=0,\;\; \eta_2=-1,\;\eta_4=1,\\
\lambda_2 &= \frac{1-2(\alpha+u_nx_n)}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})}(t_{n+1}-t_n),\\
\xi_2 &= \frac{[1-2(\alpha+u_nx_n)][2\alpha x_n-(\alpha+u_nx_n)(x_{n+1}-x_n)]}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})(1+(\alpha+u_nx_n))},\\
\eta_1 &= \frac{u_n+u_{n-2}}{u_{n-1}},
\end{aligned}\tag{17}$$

then the characteristic equation with infinitesimal transformation (13) will be

$$\begin{aligned}
&\frac{\mathrm{d}t_{n+1}}{\dfrac{1-2(\alpha+u_nx_n)}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})}(t_{n+1}-t_n)}\\
=\;&\frac{\mathrm{d}x_{n+1}}{\dfrac{[1-2(\alpha+u_nx_n)][2\alpha x_n-(\alpha+u_nx_n)(x_{n+1}-x_n)]}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})[1+(\alpha+u_nx_n)]}}\\
=\;&\frac{\mathrm{d}u_{n-2}}{(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})/u_{n-1}}\\
=\;&\frac{\mathrm{d}u_{n-1}}{-1}=\frac{\mathrm{d}u_{n+1}}{1}.
\end{aligned}\tag{18}$$

From Eq. (18), we can obtain the following conserved quantities:

$$\begin{aligned}
I_1 &= \frac{1}{u_nx_{n-1}+u_nx_{n+1}}\ln\frac{x_{n-1}u_{n-2}-u_nx_{n+1}}{u_n+u_{n-2}}\\
&\quad+\frac{1}{2}u_{n-1}^2,
\end{aligned}\tag{19}$$

$$\begin{aligned}
I_2 &= \frac{1}{u_nx_{n-1}+u_nx_{n+1}}\ln\frac{x_{n-1}u_{n-2}-u_nx_{n+1}}{u_n+u_{n-2}}\\
&\quad-u_{n-1}u_{n+1},
\end{aligned}\tag{20}$$

$$I_3 = \frac{1-2(\alpha+u_nx_n)}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})}(t_{n+1}-t_n),\tag{21}$$

$$\begin{aligned}
I_4 &= (u_{n+1}+u_{n-1})\\
&\quad\times[2\alpha x_n-(\alpha+u_nx_n)(x_{n+1}-x_n)]^{\frac{(1-2(\alpha+u_nx_n))}{(1+(\alpha+u_nx_n))}}.
\end{aligned}\tag{22}$$

When equations (14)–(16) have the following solutions:

$$\begin{aligned}
\lambda_1 &= \xi_1=\xi_2=\eta_3=0,\;\; \eta_1=-1,\;\eta_2=1,\\
\eta_4 &= [x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})\\
&\quad-u_n(x_{n-1}+x_{n+1})][u_n(x_{n-1}+x_{n+1})]^{-1},\\
\lambda_2 &= \frac{x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})}{u_n(\alpha+u_nx_n)(u_{n+1}+u_{n-1})(x_{n+1}+x_{n-1})}\\
&\quad-\frac{u_n^2}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})},\\
\xi_3 &= \frac{2\alpha x_n-(\alpha+u_nx_n)(x_{n+1}-x_n)}{1+(\alpha+u_nx_n)(t_{n+1}-t_n)}\\
&\quad\times\left[\frac{x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})}{u_n(\alpha+u_nx_n)(u_{n+1}+u_{n-1})(x_{n+1}+x_{n-1})}\right.\\
&\quad\left.-\frac{u_n^2}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})}\right],
\end{aligned}\tag{23}$$

and the characteristic equation is written as

$$\begin{aligned}
&\frac{\mathrm{d}u_{n-2}}{-1}=\frac{\mathrm{d}u_{n-1}}{1}\\
=\;&\mathrm{d}u_{n+1}\{[x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})\\
&-u_n(x_{n-1}+x_{n+1})][u_n(x_{n-1}+x_{n+1})]^{-1}\}^{-1}\\
=\;&\mathrm{d}x_{n+1}\left\{\frac{2\alpha x_n-(\alpha+u_nx_n)(x_{n+1}-x_n)}{1+(\alpha+u_nx_n)(t_{n+1}-t_n)}\right.\\
&\times\left[\frac{x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})}{u_n(\alpha+u_nx_n)(u_{n+1}+u_{n-1})(x_{n+1}+x_{n-1})}\right.\\
&\left.\left.-\frac{u_n^2}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})}\right]\right\}^{-1}\\
=\;&\mathrm{d}t_{n+1}\left[\frac{x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})}{u_n(\alpha+u_nx_n)(u_{n+1}+u_{n-1})(x_{n+1}+x_{n-1})}\right.\\
&\left.-\frac{u_n^2}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})}\right]^{-1}.
\end{aligned}\tag{24}$$

From Eq. (24), we obtain the following conserved quantities:

$$\begin{aligned}
I_5 &= -x_{n-1}u_{n-1}u_{n-2}+\frac{1}{2}x_{n-1}^2u_nu_{n-2}^2-x_{n-1}u_n^2u_{n-2}x_{n+1}\\
&\quad+\frac{1}{3}x_{n-1}^2u_{n-2}^3-\frac{1}{2}x_{n-1}u_nu_{n-2}^2x_{n+1}\\
&\quad-u_nu_{n-2}(x_{n-1}+x_{n+1})+u_nu_{n+1}(x_{n-1}+x_{n+1}),
\end{aligned}\tag{25}$$

$$\begin{aligned}
I_6 &= u_n(u_{n+1}-u_{n-1})(x_{n-1}+x_{n+1})\\
&\quad-[x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})]u_{n-1},
\end{aligned}\tag{26}$$

$$\begin{aligned}
I_7 &= \frac{u_n^2u_{n-2}}{(u_{n+1}+u_{n-1})}\\
&\quad-\frac{x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}^2/2-u_nu_{n-2}x_{n+1})}{u_n(u_{n+1}+u_{n-1})(x_{n+1}+x_{n-1})}\\
&\quad+[1+(\alpha+u_nx_n)(t_{n+1}-t_n)]\\
&\quad\times\ln(2\alpha x_n-(\alpha+u_nx_n)(x_{n+1}-x_n)),
\end{aligned}\tag{27}$$

$$\begin{aligned}
I_8 &= \frac{x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})}{u_n(\alpha+u_nx_n)(u_{n+1}+u_{n-1})(x_{n+1}+x_{n-1})}u_{n-1}\\
&\quad-\frac{u_n^2}{(\alpha+u_nx_n)}\ln(u_{n+1}+u_{n-1})-t_{n+1}.
\end{aligned}\tag{28}$$

When equations (14)–(16) have the following solutions:

$$\begin{aligned}
\lambda_1 &= \xi_1 = \xi_2 = \eta_3 = 0, \quad \eta_1 = 1, \quad \eta_4 = -1,\\
\lambda_2 &= -\frac{(1-(\alpha+u_nx_n))(t_{n+1}-t_n)}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})}\\
&\quad + \frac{(x_{n-1}u_{n-2}-x_{n-1}u_{n-1}-u_nx_{n+1})(t_{n+1}-t_n)}{u_n(u_{n+1}+u_{n-1})(x_{n+1}+x_{n-1})},\\
\eta_2 &= \frac{x_{n-1}u_{n-2}-x_{n-1}u_{n-1}-u_nx_{n+1}}{u_n(x_{n+1}+x_{n-1})},\\
\xi_3 &= \frac{(\alpha+u_nx_n)(x_{n-1}u_{n-2}-x_{n-1}u_{n-1}-u_nx_{n+1})(x_{n+1}-x_n)}{u_n(u_{n+1}-u_{n-1})(1+(\alpha+u_nx_n))(x_{n+1}+x_{n-1})}\\
&\quad - \frac{(\alpha+u_nx_n)(1-(\alpha+u_nx_n))}{(1+(\alpha+u_nx_n))},
\end{aligned} \tag{29}$$

the characteristic equations on the Lie group of transformation (29) are written as

$$\begin{aligned}
&\frac{\mathrm{d}u_{n-2}}{1}\\
&= \frac{\mathrm{d}u_{n-1}}{\dfrac{-u_n(x_{n-1}+x_{n+1})-x_{n-1}u_{n-1}}{x_{n-1}(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})}}\\
&= \frac{\mathrm{d}u_{n+1}}{-1}\\
&= \mathrm{d}x_{n+1}\Bigg\{[(\alpha+u_nx_n)(x_{n-1}u_{n-2}-x_{n-1}u_{n-1}-u_nx_{n+1})\\
&\quad \times (x_{n+1}-x_n)][u_n(u_{n+1}-u_{n-1})(1+(\alpha+u_nx_n))\\
&\quad \times (x_{n+1}+x_{n-1})]^{-1}\\
&\quad - \frac{(\alpha+u_nx_n)(1-(\alpha+u_nx_n))}{(1+(\alpha+u_nx_n))}\Bigg\}^{-1}\\
&= \mathrm{d}t_{n+1}\Bigg[-\frac{(1-(\alpha+u_nx_n))(t_{n+1}-t_n)}{(\alpha+u_nx_n)(u_{n+1}+u_{n-1})}\\
&\quad + \frac{(x_{n-1}u_{n-2}-x_{n-1}u_{n-1}-u_nx_{n+1})(t_{n+1}-t_n)}{u_n(u_{n+1}+u_{n-1})(x_{n+1}+x_{n-1})}\Bigg]^{-1}.
\end{aligned} \tag{30}$$

The following conserved quantities are obtained:

$$\begin{aligned}
I_9 &= -\frac{1}{u_n(x_{n-1}+x_{n+1})}\ln(u_n+u_{n-2})\\
&\quad + \frac{x_{n-1}}{u_n(x_{n-1}+x_{n+1})}\ln(x_{n-1}u_{n-2}-u_nx_{n+1})\\
&\quad + \ln(u_n(x_{n-1}+x_{n+1})+x_{n-1}u_{n-1})\\
&= \frac{1}{u_n(x_{n-1}+x_{n+1})}\ln\Big\{[(x_{n-1}u_{n-2}-u_nx_{n+1})^{x_{n-1}}\\
&\quad \times (u_n(x_{n-1}+x_{n+1})+x_{n-1}u_{n-1})^{u_n(x_{n-1}+x_{n+1})}]\\
&\quad \times (u_n+u_{n-2})^{-1}\Big\},
\end{aligned} \tag{31}$$

$$\begin{aligned}
I_{10} &= \ln(u_n(x_{n-1}+x_{n+1})+x_{n-1}u_{n-1})\\
&\quad - \frac{u_{n+1}}{(u_n+u_{n-2})(x_{n-1}u_{n-2}-u_nx_{n+1})},
\end{aligned} \tag{32}$$

$$\begin{aligned}
I_{11} &= (u_{n+1}+u_{n-1})^{(\alpha+u_nx_n)(x_{n-1}u_{n-2}-x_{n-1}u_{n-1}-u_nx_{n+1})}\\
&\quad \times [-u_n(1-(\alpha+u_nx_n))(t_{n+1}-t_n)(x_{n+1}+x_{n-1})\\
&\quad + (\alpha+u_nx_n)(x_{n-1}u_{n-2}-x_{n-1}u_{n-1}-u_nx_{n+1})\\
&\quad \times (t_{n+1}-t_n)]^{u_n(\alpha+u_nx_n)(x_{n+1}+x_{n-1})}.
\end{aligned} \tag{33}$$

4. Illustrate example

The ultra thin rod elastic equation of motion is a model of a biological macromolecule chain of mechanical behaviors. Ultra thin elastic rod dynamics plays an important role in the DNA of balance and stability studies. The equation of motion of a hyperelastic homogeneous bar, whose cross-sectional area is variable along the bar, is expressed as

$$\rho u_{xx} = T_x + \frac{S'(x)}{S(x)}T, \tag{34}$$

where ρ is the constant mass density, $u = u(x,t)$ is the displacement, x is the coordinate of point P in the present reference frame, t is the time, $T(u_x)$ is the tension, and $S(x)$ is the cross-sectional area that varies according to the following exponential law:

$$S = S_0\,\mathrm{e}^{\mu x}, \quad T = x_{tt}u_x. \tag{35}$$

Then equation (16) becomes

$$u_{tt} = \frac{1}{\rho}x_{tt}u_{xx} + \frac{\mu}{\rho}x_{tt}u_x. \tag{36}$$

With Eq. (7), we can obtain the discrete wave equation

$$\begin{aligned}
&(\rho-1)\{(u_{n+1}+u_{n-1})[x_n(u_{n+1}+u_{n-1})\\
&- u_n(x_{n+1}+x_{n-1})]+x_{n+1}u_{n+1}(u_{n+2}+u_n)\\
&+ u_{n-1}x_{n-1}(u_n+u_{n-2})\}\\
&= \mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}\{[x_n(u_{n+1}+u_{n-1})-u_n(x_{n+1}+x_{n-1})]\\
&\times (x_{n+1}+x_{n-1})-[x_{n+1}u_{n+1}(x_{n+2}+x_n)\\
&+ u_{n-1}x_{n-1}(x_{n-2}+x_n)]\}.
\end{aligned} \tag{37}$$

We introduce the infnitesimal transformation on the following variables:

$$\begin{aligned}
t^*_{n-1} &= t_{n-1}+\varepsilon\lambda_1(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
t^*_n &= t_n+\varepsilon\lambda_2(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
t^*_{n+1} &= t_{n+1}+\varepsilon\lambda_3(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
x^*_{n-2} &= x_{n-2}+\varepsilon\xi_1(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
x^*_{n-1} &= x_{n-1}+\varepsilon\xi_2(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),
\end{aligned}$$

$$
\begin{aligned}
x_n^* &= x_n + \varepsilon\xi_3(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
x_{n+1}^* &= x_{n+1} + \varepsilon\xi_4(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
x_{n+2}^* &= x_{n+2} + \varepsilon\xi_5(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
u_{n-2}^* &= u_{n-2} + \varepsilon\eta_1(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
u_{n-1}^* &= u_{n-1} + \varepsilon\eta_2(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
u_n^* &= u_n + \varepsilon\eta_3(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
u_{n+1}^* &= u_{n+1} + \varepsilon\eta_4(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),\\
u_{n+2}^* &= u_{n+2} + \varepsilon\eta_5(t_{n-1},t_n,t_{n+1},x_{n-2},x_{n-1},x_n,x_{n+1},x_{n+2},u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2}),
\end{aligned}
\tag{38}
$$

where ξ_i and η_j constitute the Lie group of infnitesimal transformations. For obtaining the discrete analogue of the conserved quantities for Eqs. (11) and (37), we introduce the infnitesimal transformation operator

$$
\begin{aligned}
\mathrm{X} = {} & \lambda_1\frac{\partial}{\partial t_{n-1}} + \lambda_2\frac{\partial}{\partial t_n} + \lambda_3\frac{\partial}{\partial t_{n+1}} + \lambda_4\frac{\partial}{\partial t_{n+2}} + \xi_1\frac{\partial}{\partial x_{n-2}}\\
& + \xi_2\frac{\partial}{\partial x_{n-1}} + \xi_3\frac{\partial}{\partial x_n} + \xi_4\frac{\partial}{\partial x_{n+1}}\\
& + \xi_5\frac{\partial}{\partial x_{n+2}} + \eta_1\frac{\partial}{\partial u_{n-2}} + \eta_2\frac{\partial}{\partial u_{n-1}}\\
& + \eta_3\frac{\partial}{\partial u_n} + \eta_4\frac{\partial}{\partial u_{n+1}} + \eta_5\frac{\partial}{\partial u_{n+2}}.
\end{aligned}
\tag{39}
$$

The invariants of Eqs. (11) and (37) under the infinitesimal transformation lead to the determining equations

$$
\begin{aligned}
& \eta_4\Big((\rho-1)x_n(u_{n+1}+u_{n-1}) - u_n(x_{n+1}+x_{n-1})\\
& + x_{n+1}(u_{n+2}+u_n) - \mu\frac{1}{x_{n+1}+x_{n-1}}\{[x_n(u_{n+1}+u_{n-1})\\
& - u_n(x_{n+1}+x_{n-1})](x_{n+1}+x_{n-1})\\
& - [x_{n+1}u_{n+1}(x_{n+2}+x_n) + u_{n-1}x_{n-1}(x_{n-2}+x_n)]\}\\
& + (\rho-1)(u_{n+1}+u_{n-1})[x_n + x_{n+1}(u_{n+2}+u_n)]\\
& - \mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}[x_n(x_{n+1}+x_{n-1}) - x_{n+1}(x_{n+2}+x_n)]\Big)\\
& + \eta_2\Big((\rho-1)[x_n(u_{n+1}+u_{n-1}) - u_n(x_{n+1}+x_{n-1})]\\
& + (\rho-1)(u_{n+1}+u_{n-1})x_n + (\rho-1)x_{n-1}(u_n+u_{n-2})\\
& - \mu\frac{1}{x_{n+1}+x_{n-1}}\{[x_n(u_{n+1}+u_{n-1}) - u_n(x_{n+1}+x_{n-1})]\\
& \times (x_{n+1}+x_{n-1}) - [x_{n+1}u_{n+1}(x_{n+2}+x_n)\\
& + u_{n-1}x_{n-1}(x_{n-2}+x_n)] - \mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}[x_n(x_{n+1}+x_{n-1})\\
& - x_{n-1}(x_{n-2}+x_n)]\Big) + \eta_3\Big[-(\rho-1)(x_{n+1}+x_{n-1})\\
& \times (u_{n+1}+u_{n-1}) + (\rho-1)x_{n+1}u_{n+1} + (\rho-1)x_{n-1}u_{n-1}\\
& + \mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}(x_{n+1}+x_{n-1})^2\Big]\\
& + \eta_5[(\rho-1)x_{n+1}u_{n+1}(u_{n+1}+u_{n-1})]\\
& + \eta_1[(\rho-1)x_{n-1}u_{n-1}(u_{n+1}+u_{n-1})]\\
& + \xi_2\Big\{-(\rho-1)(u_{n+1}+u_{n-1})u_n + (\rho-1)u_{n-1}(u_n+u_{n-2})\\
& - \mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}[x_n(u_{n+1}+u_{n-1}) - u_n(x_{n+1}+x_{n-1})]\Big\}\\
& + \xi_3\Big[(\rho-1)(u_{n+1}+u_{n-1})^2 - \mu(u_{n+1}+u_{n-1})^2\\
& - \mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}(x_{n+1}u_{n+1}+u_{n-1}x_{n-1})\Big]\\
& + \xi_4\Big\{(\rho-1)(u_{n+1}+u_{n-1})u_n + (\rho-1)u_{n+1}(u_{n+2}+u_n)\\
& + \mu\frac{u_{n+1}+u_{n-1}}{(x_{n+1}+x_{n-1})^2}[x_n(u_{n+1}+u_{n-1})\\
& - u_n(x_{n+1}+x_{n-1})](x_{n+1}+x_{n-1}) - x_{n+1}u_{n+1}(x_{n+2}+x_n)\\
& + u_{n-1}x_{n-1}(x_{n-2}+x_n) - \mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}\\
& \times [-u_n(x_{n+1}+x_{n-1})]\Big\} + \xi_5\mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}x_{n+1}u_{n+1}\\
& + \xi_1\mu\frac{u_{n+1}+u_{n-1}}{x_{n+1}+x_{n-1}}u_{n-1}x_{n-1} = 0,
\end{aligned}
\tag{40}
$$

and the restrained equations

$$
\begin{aligned}
& \eta_5[1-(\alpha+u_{n+1}x_{n+1})(t_{n+2}-t_{n+1})]\\
= {} & \eta_4[x_{n+1}(u_{n+2}+u_n)(t_{n+2}-t_{n+1}) + (1-2\alpha(t_{n+2}-t_{n+1}))]\\
& + \eta_3(\alpha+u_{n+1}x_{n+1})(t_{n+2}-t_{n+1})\\
& + \xi_4u_{n+1}(u_{n+2}+u_n)(t_{n+2}-t_{n+1})\\
& + [(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n) - 2\alpha u_{n+1}](\lambda_4-\lambda_3),
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
& \eta_3[1-(\alpha+u_{n-1}x_{n-1})(t_n-t_{n-1})]\\
= {} & \eta_2[(1-2\alpha(t_n-t_{n-1})) + x_{n-1}(u_{n-2}+u_n)(t_n-t_{n-1})]\\
& + \xi_2u_{n-1}(u_{n-2}+u_n)(t_n-t_{n-1})\\
& + \eta_1(\alpha+u_{n-1}x_{n-1})(t_n-t_{n-1})\\
& + (\lambda_2-\lambda_1)[(\alpha+u_{n-1}x_{n-1})(u_{n-2}+u_n) - 2\alpha u_{n-1}].
\end{aligned}
\tag{42}
$$

Let

$$
\begin{aligned}
& \lambda_1 = \lambda_3 = \xi_4 = \xi_3 = \xi_2 = \eta_4 = \eta_2 = \eta_3 = 0,\\
& \eta_1 = \xi_1 = u_{n+1}x_{n+1}, \eta_5 = \xi_5 = -u_{n-1}x_{n-1},\\
& \lambda_2 = -\frac{x_{n+1}u_{n+1}(\alpha+u_{n-1}x_{n-1})(t_n-t_{n-1})}{(\alpha+u_{n-1}x_{n-1})(u_{n-2}+u_n) - 2\alpha u_{n-1}},\\
& \lambda_4 = \frac{x_{n-1}u_{n-1}[1-(\alpha+u_{n+1}x_{n+1})](t_{n+2}-t_{n+1})}{(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n) - 2\alpha u_{n+1}},
\end{aligned}
\tag{43}
$$

then the characteristic equations are

$$\begin{aligned}
&\frac{\mathrm{d}u_{n-2}}{u_{n+1}x_{n+1}} = \frac{\mathrm{d}u_{n+2}}{-u_{n-1}x_{n-1}} \\
&= \frac{\mathrm{d}t_n}{-\dfrac{x_{n+1}u_{n+1}(\alpha+u_{n-1}x_{n-1})(t_n-t_{n-1})}{(\alpha+u_{n-1}x_{n-1})(u_{n-2}+u_n)-2\alpha u_{n-1}}} \\
&= \frac{\mathrm{d}t_{n+2}}{\dfrac{x_{n-1}u_{n-1}[1-(\alpha+u_{n+1}x_{n+1})](t_{n+2}-t_{n+1})}{(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n)-2\alpha u_{n+1}}} \\
&= \frac{\mathrm{d}x_{n-2}}{u_{n+1}x_{n+1}} \\
&= \frac{\mathrm{d}x_{n+2}}{-u_{n-1}x_{n-1}},
\end{aligned} \tag{44}$$

the system will possess the conserved quantities

$$I_1 = u_{n+2}u_{n+1}x_{n+1} - u_{n-2}u_{n-1}x_{n-1}, \tag{45}$$

$$I_2 = u_{n+1}x_{n+1}x_{n+2} + u_{n-1}x_{n-1}x_{n-2}, \tag{46}$$

$$\begin{aligned}
I_3 = {} & \frac{(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n)-2\alpha u_{n+1}}{x_{n-1}u_{n-1}[1-(\alpha+u_{n+1}x_{n+1})]} \\
& \times \ln(t_{n+2}-t_{n+1}) \\
& + \frac{(\alpha+u_{n-1}x_{n-1})(u_{n-2}+u_n)-2\alpha u_{n-1}}{x_{n+1}u_{n+1}(\alpha+u_{n-1}x_{n-1})} \\
& \times \ln(t_{n+1}-t_n),
\end{aligned} \tag{47}$$

$$\begin{aligned}
I_4 = {} & x_{n+1}u_{n+1}\ln[(u_{n-2}+u_n)-2\alpha u_{n-1}] \\
& + \ln(t_n-t_{n-1}),
\end{aligned} \tag{48}$$

$$\begin{aligned}
I_5 = {} & \frac{[1-(\alpha+u_{n+1}x_{n+1})]x_{n+2}}{(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n)-2\alpha u_{n+1}} \\
& + \ln(t_{n+2}-t_{n+1}).
\end{aligned} \tag{49}$$

Let

$$\begin{aligned}
&\lambda_1=\lambda_3=\xi_4=\xi_3=\xi_2=\eta_4=\eta_2=\eta_3=0, \\
&\eta_5=\frac{1}{x_{n+1}u_{n+1}},\quad \eta_1=-\frac{1}{x_{n-1}u_{n-1}}, \\
&\xi_5=\frac{1}{x_{n+1}u_{n+1}},\quad \xi_1=-\frac{1}{x_{n-1}u_{n-1}}, \\
&\lambda_4=\frac{1}{x_{n+1}u_{n+1}}\frac{1-(\alpha+u_{n+1}x_{n+1})(t_{n+2}-t_{n+1})}{(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n)-2\alpha u_{n+1}}, \\
&\lambda_2=\frac{1}{x_{n-1}u_{n-1}}\frac{(\alpha+u_{n-1}x_{n-1})(t_n-t_{n-1})}{(\alpha+u_{n-1}x_{n-1})(u_{n-2}+u_n)-2\alpha u_{n-1}},
\end{aligned} \tag{50}$$

then the characteristic equations are

$$\begin{aligned}
&-\frac{x_{n+1}u_{n+1}[(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n)-2\alpha u_{n+1}]\mathrm{d}t_{n+2}}{1-(\alpha+u_{n+1}x_{n+1})(t_{n+2}-t_{n+1})} \\
&= \frac{x_{n-1}u_{n-1}[(\alpha+u_{n-1}x_{n-1})(u_{n-2}+u_n)-2\alpha u_{n-1}]\mathrm{d}t_n}{(\alpha+u_{n-1}x_{n-1})(t_n-t_{n-1})} \\
&= -x_{n-1}u_{n-1}\mathrm{d}x_{n-2} = x_{n+1}u_{n+1}\mathrm{d}x_{n+2} = -x_{n-1}u_{n-1}\mathrm{d}u_{n-2} \\
&= x_{n+1}u_{n+1}\mathrm{d}u_{n+2},
\end{aligned}$$

the system will possess the conserved quantities

$$I_6 = x_{n+1}u_{n+1}u_{n+2} + x_{n-1}u_{n-1}u_{n-2}, \tag{51}$$

$$I_7 = x_{n+1}x_{n+2}u_{n+1} + x_{n-1}x_{n-2}u_{n-1}, \tag{52}$$

$$\begin{aligned}
I_8 = {} & \frac{x_{n+1}u_{n+1}[(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n)-2\alpha u_{n+1}]}{(\alpha+u_{n+1}x_{n+1})} \\
& \times \ln(1-(\alpha+u_{n+1}x_{n+1})(t_{n+2}-t_{n+1})) \\
& - \frac{x_{n-1}u_{n-1}[(\alpha+u_{n-1}x_{n-1})(u_{n-2}+u_n)-2\alpha u_{n-1}]}{(\alpha+u_{n-1}x_{n-1})} \\
& \times \ln(t_n-t_{n-1}),
\end{aligned} \tag{53}$$

$$\begin{aligned}
I_9 = {} & \frac{[(\alpha+u_{n+1}x_{n+1})(u_{n+2}+u_n)-2\alpha u_{n+1}]\mathrm{d}t_{n+2}}{(\alpha+u_{n+1}x_{n+1})} \\
& \times \ln(1-(\alpha+u_{n+1}x_{n+1})(t_{n+2}-t_{n+1})) + x_{n+2},
\end{aligned} \tag{54}$$

$$\begin{aligned}
I_{10} = {} & \frac{[(\alpha+u_{n-1}x_{n-1})(u_{n-2}+u_n)-2\alpha u_{n-1}]}{(\alpha+u_{n-1}x_{n-1})} \\
& \times \ln(t_n-t_{n-1}) - u_{n-2}.
\end{aligned} \tag{55}$$

Certainly, we can also obtain other conserved quantities.

5. Conclusion

In this paper, we present a method to reduce to a discrete wave equation with the Ablowitz–Ladik–Lattice and obtain the Lie symmetries theories. Our results give a significant step to profoundly study the Lie symmetries in the mechanical and physical fields.

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