# Periodic, quasiperiodic, and chaotic breathers in two-dimensional discrete $\beta$-Fermi–Pasta–Ulam lattice*

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(Received 20 December 2012; revised manuscript received 29 January 2013)

Using numerical method, we investigate whether periodic, quasiperiodic, and chaotic breathers are supported by the two-dimensional discrete Fermi–Pasta–Ulam (FPU) lattice with linear dispersion term. The spatial profile and time evolution of the two-dimensional discrete $\beta$-FPU lattice are segregated by the method of separation of variables, and the numerical simulations suggest that the discrete breathers (DBs) are supported by the system. By introducing a periodic interaction into the linear interaction between the atoms, we achieve the coupling of two incommensurate frequencies for a single DB, and the numerical simulations suggest that the quasiperiodic and chaotic breathers are supported by the system, too.

**Keywords:** breather, quasiperiodic breather, chaotic breather

**PACS:** 63.20.Ry, 63.20.Pw, 05.50.+q **DOI:** 10.1088/1674-1056/22/8/086302

## 1. Introduction

Localized excitations in nonlinear lattice systems have been the topic of increasing focus in view of their significant role in energy localization and transport during the last decade since the discovery of the discrete breathers that are time-periodic and spatially localized excitations by Sievers and Takeno.[1] Later, rigorous proofs for existence in particular Fermi–Pasta–Ulam (FPU) models were provided, such as breather or gap breather in diatomic FPU chain,[2–6] the problem of $\beta$-FPU,[7,8] asymptotic dynamics of breather in FPU chain,[9] interaction of a kink soliton with a breather in an FPU chain,[10] bright and dark breathers in FPU lattice,[11] etc.

Soon after the first development, the question of the existence of quasiperiodic and even chaotic breathers arose in the study of Hamiltonian lattices. The term "quasiperiodic breathers" was introduced by Johansson and Aubry.[12] Typical features of quasiperiodic breathers can be described as follows: at the anticontinuous limit, quasiperiodic solutions are trivially obtained as solutions where different sites oscillate with frequencies $\omega_n$ which are incommensurate, of which the simplest one consists of two sites oscillating with incommensurate frequencies $\omega_1$ and $\omega_2$ while all other sites are at rest.[12] The term "chaotic breathers" was introduced by Cretegny *et al.*[13] Typical features of chaotic breathers can be described as follows: the destabilization of the initial highest-frequency ($\pi$-mode) disturbances leads to a number of localized breather-like structures which gather the energy of the system, which then further merge into a few localized structures or a single one with higher energy. They are stable over a relatively long time. Each localized structure is qualitatively similar to discrete breathers (DBs) but moves irregularly. In most situations, chaotic breathers finally break and damp out into a thermalization (equilibrium) state. The most obvious difference between this kind of localized excitation and a DB is that the former one moves in an erratic (chaotic) way and mostly has a finite lifetime. Thus, it was named a chaotic breather.[13]

Several studies have been most recently dedicated to the exploration of quasiperiodic breathers and chaotic breathers.[14–17] But these studies are restricted to simpler one-dimensional FPU modes or absence of linear dispersion. Although Butt and Wattis,[18] and Xu and Tian[19] also investigated the two-dimensional discrete FPU lattice, they only discussed DBs in the system and obtained approximate solutions by the methods of multiple-scale, and Xu and Tian[20] did not considered the linear dispersion term of the two-dimensional discrete FPU lattice. The present paper is devoted to the exact solutions of periodic, quasiperiodic, and chaotic breathers in two-dimensional discrete FPU lattice with linear dispersion by the methods of separation of variables and numerical simulation.

## 2. Two-dimensional discrete FPU lattice with the linear dispersion

The two-dimensional FPU system is a lattice of $N$ equal mass coupled by nonlinear interaction with the equations of

*Project supported by the National Natural Science Foundation of China (Grant No. 11247255) and the Natural Science Foundation of Heilongjiang Province, China (Grant No. A200506).

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motion containing quadratic interaction term (the $\alpha$ model)

$$\begin{aligned}\frac{\mathrm{d}^2u_{l,m}}{\mathrm{d}t^2} &= K\left(u_{l+1,m}+u_{l-1,m}+u_{l,m+1}+u_{l,m-1}-4u_{l,m}\right)\\ &\quad+\alpha\left[\left(u_{l+1,m}-u_{l,m}\right)^2-\left(u_{l-1,m}-u_{l,m}\right)^2\right.\\ &\quad\left.+\left(u_{l,m+1}-u_{l,m}\right)^2-\left(u_{l,m-1}-u_{l,m}\right)^2\right] \qquad (1)\end{aligned}$$

or cubic interaction term (the $\beta$ model)

$$\begin{aligned}\frac{\mathrm{d}^2u_{l,m}}{\mathrm{d}t^2} &= K\left(u_{l+1,m}+u_{l-1,m}+u_{l,m+1}+u_{l,m-1}-4u_{l,m}\right)\\ &\quad+\beta\left[\left(u_{l+1,m}-u_{l,m}\right)^3-\left(u_{l,m}-u_{l-1,m}\right)^3\right.\\ &\quad\left.+\left(u_{l,m+1}-u_{l,m}\right)^3-\left(u_{l,m}-u_{l,m-1}\right)^3\right], \qquad (2)\end{aligned}$$

where $u_{l,m}$ is the displacement of the particle from its original position. $K$ is the linear coupling constant, $\alpha$ and $\beta$ are the parameters controlling the strength of the quadratic and cubic nonlinear coupling, respectively.

## 3. Periodic, quasiperiodic, and chaotic breathers

We also suppose that the solutions of this system can be written as a product of a purely spatial part and a function of time only as follows:

$$u_{l,m}(t)=\Phi^0_{l,m}G\left(\delta^0_{l,m}t\right), \qquad (3)$$

where $\Phi^0_{l,m}$ corresponds to purely spatial profile of the solution and is independent of time, $G\left(\delta^0_{l,m}t\right)$ describes its time evolution,

$$\delta^0_{l,m}=\sqrt{\left|\left(\Phi^0_{l+1,m}+\Phi^0_{l-1,m}+\Phi^0_{l,m+1}+\Phi^0_{l,m-1}-4\Phi^0_{l,m}\right)/\Phi^0_{l,m}\right|}$$

is also purely spatial term and independent of time. Thus equations (1) and (2) can be rewritten as follows:

$$\frac{\ddot{G}(\delta^0_{l,m}t)-KG(\delta^0_{l,m}t)}{\alpha G^2(\delta^0_{l,m}t)}=\frac{(\Phi^0_{l+1,m}-\Phi^0_{l,m})^2-(\Phi^0_{lm}-\Phi^0_{l-1,m})^2+(\Phi^0_{l,m+1}-\Phi^0_{l,m})^2-(\Phi^0_{lm}-\Phi^0_{l,m-1})^2}{\Phi^0_{l+1,m}+\Phi^0_{l-1,m}+\Phi^0_{l,m+1}+\Phi^0_{l,m-1}-4\Phi^0_{l,m}} \qquad (4)$$

and

$$\frac{\ddot{G}(\delta^0_{l,m}t)-KG(\delta^0_{l,m}t)}{\beta G^3(\delta^0_{l,m}t)}=\frac{(\Phi^0_{l+1,m}-\Phi^0_{l,m})^3-(\Phi^0_{lm}-\Phi^0_{l-1,m})^3+(\Phi^0_{l,m+1}-\Phi^0_{l,m})^3-(\Phi^0_{lm}-\Phi^0_{l,m-1})^3}{\Phi^0_{l+1,m}+\Phi^0_{l-1,m}+\Phi^0_{l,m+1}+\Phi^0_{l,m-1}-4\Phi^0_{l,m}}. \qquad (5)$$

The terms on the right-hand side of Eqs. (4) and (5) are purely spatial terms and independent of time, and the terms on the left-hand sides are dependent on time and spatial term $\delta^0_{l,m}$. When $\delta^0_{l,m}$ is constant the $\alpha$ model has not solution, so we only study the $\beta$ model. At this point, we have

$$\frac{\ddot{G}(\delta^0_{l,m}t)-KG(\delta^0_{l,m}t)}{\beta G^3(\delta^0_{l,m}t)}=\frac{(\Phi^0_{l+1,m}-\Phi^0_{l,m})^3-(\Phi^0_{lm}-\Phi^0_{l-1,m})^3+(\Phi^0_{l,m+1}-\Phi^0_{l,m})^3-(\Phi^0_{lm}-\Phi^0_{l,m-1})^3}{\Phi^0_{l+1,m}+\Phi^0_{l-1,m}+\Phi^0_{l,m+1}+\Phi^0_{l,m-1}-4\Phi^0_{l,m}}=C^0_{(l,m)\beta}, \qquad (6)$$

where $C^0_{(l,m)\beta}$ is constant. Equation (6) can be rewritten as follows:

$$\begin{aligned}&C^0_{(l,m)\beta}(\Phi^0_{l+1,m}+\Phi^0_{l-1,m}+\Phi^0_{l,m+1}+\Phi^0_{l,m-1}-4\Phi^0_{l,m})\\ &-(\Phi^0_{l+1,m}-\Phi^0_{l,m})^3+(\Phi^0_{lm}-\Phi^0_{l-1,m})^3\\ &-(\Phi^0_{l,m+1}-\Phi^0_{l,m})^3+(\Phi^0_{lm}-\Phi^0_{l,m-1})^3=0, \qquad (7a)\\ &\ddot{G}(\delta^0_{l,m}t)-KG(\delta^0_{l,m}t)-C^0_{(l,m)\beta}\beta G^3(\delta^0_{l,m}t)=0. \qquad (7b)\end{aligned}$$

The numerical solutions of Eq. (7a) is shown in Fig. 1.

Figure 1 shows us that equation (7a) has spatially localized solution. The numerical solutions of Eq. (7b) can be shown in Fig. 2.

Figures 2(a) and 2(b) show us that the stable DB can exist in two-dimensional $\beta$-FPU model with linear dispersion. Figures 2(c) and 2(d) show us that the quasiperiodic breather can exist in two-dimensional $\beta$-FPU model with linear dispersion by adding to the coefficient of the linear dispersion term a periodic modulation. Such a type of parametric driving can be experimentally implemented by introducing a periodic interaction into the linear interaction of all oscillators. Figures 2(e) and 2(f) show us that the chaotic breather can exist in two-dimensional $\beta$-FPU model with linear dispersion by adding to the coefficient of the linear dispersion term a periodic modulation.

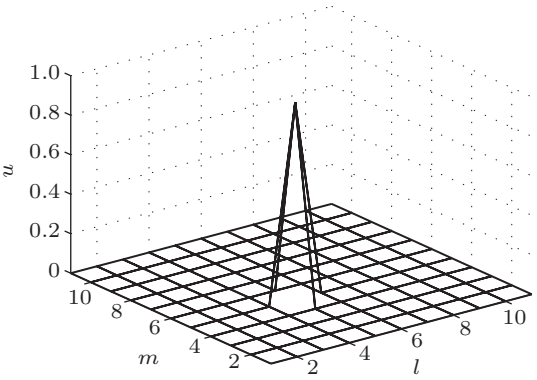



**Fig. 1.** Numerical solution of Eq. (7a), where $\delta^0_{l,m}=2$ and $C^0_{(l,m)\beta}=1$.

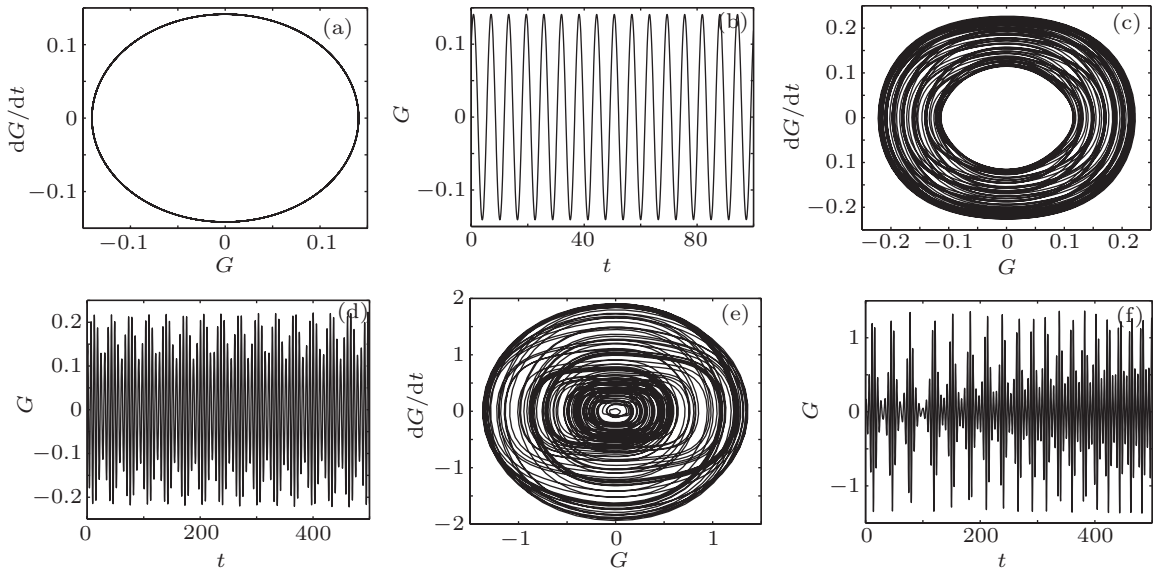



**Fig. 2.** (color online) Numerical solutions of Eq. (7b). Panels (a) and (b) show phase portrait and vibration of the periodic solutions of Eq. (7b) for $K = 1$, $\beta = \pm 1$, and $C^0_{(l,m)\beta} = 1$. Panels (c) and (d) exhibit phase portrait and vibration of the quasiperiodic solutions of Eq. (7b) for $K = 1 + 0.3\cos(2.268t)$, $\beta = \pm 1$, and $C^0_{(l,m)\beta} = 1$. Panels (e) and (f) are phase portrait and vibration of the quasiperiodic solutions of Eq. (7b) for $K = 1 + 0.9\cos(2.268t)$, $\beta = \pm 1$, and $C^0_{(l,m)\beta} = 1$.

## 4. Conclusions

In this paper, we segregated spatial profile and time evolution of two-dimensional $\beta$-FPU model with linear dispersion and obtain exact DBs. We also achieved the coupling of two incommensurate frequencies for single breather by introducing a periodic interaction into the linear interaction of all oscillators and then obtained the quasiperiodic and chaotic breathers in the two-dimensional $\beta$-FPU model with linear dispersion. This paper will provide theoretic guidance for observing quasiperiodic and chaotic breathers experimentally.

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