The propagation of shape changing soliton in a nonuniform nonlocal media

L. Kavitha[a)b)c)†], C. Lavanya[a)], S. Dhamayanthi[a)], N. Akila[a)], and D. Gopi[d)]

[a)] *Department of Physics, Periyar University, Salem-636 011, India*
[b)] *The Abdus Salam International Center for Theoretical Physics, Trieste, Italy*
[c)] *Center for Nanoscience and Nanotechnology, Periyar University, Salem-636 011, India*
[d)] *Department of Chemistry, Periyar University, Salem-636 011, India*

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Magnetization dynamics in uniformly magnetized ferromagnetic media is studied by using Landau–Lifshitz–Gilbert equation. The nonlinear evolution equation is integrable with site-dependent and biquadratic exchange interaction by means of Landau–Lifshitz (LL) equation which is well understood. In the present work, we construct the exact solitary solutions of the nonlinear evolution equation, particularly, we employ the modified extended tangent hyperbolic function method. We show the shape changing property of solitons for the given integrable system in the presence of damping as well as inhomogeneities.

Keywords: solitons, classical spin models, Maxwell equations, nonlinear dynamics

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1. Introduction

The classical Heisenberg ferromagnetic spin chains with biquadratic interaction, Dzyaloshinsky–Moriya interaction, and octupole–dipole interaction are some of the physically interesting spin models, the nonlinear dynamics of which have been understood and integrable models have been identified in recent years.[1–4] In this context, the studies of the classical isotropic site-dependent Heisenberg spin chains are considered to be pioneering nonlinear spin models for they are modelled by the inhomogeneous nonlinear Schrödinger equations, which are considered to be important when they exhibit integrability in their own respect.[5,6] Also, many effects of further interactions leading to nonintegrability and damping effects, etc., are some of the significant topics of current interest which are being pursued vigorously. It is also of equal interest to study the nature of nonlinear spin excitations in a Heisenberg ferromagnet with varying exchange interactions in the presence of Gilbert damping without holding the conditions of integrability.[7,8] The interactions of solitons with the imperfections in a ferromagnetic medium is an important aspect of nonlinear physics. In recent years, the investigation of the nonlinear dynamics of inhomogeneous systems has assumed much significance because these systems are considered to be realistic. Also, the importance becomes particularly apparent when solitons are used to model the nonlinear wave propagation in dispersive and inhomogeneous media such as radio waves in the ionosphere, waves in the ocean, optical pulses in glass fibers, laser radiations in a plasma, and impurities or defects in magnetic systems.[9–15] Li *et al.*, transformed the modified Landau–Lifshitz (LL) equation into an equation of nonlinear Schrödinger type, and studied the interaction of a periodic and a one-soliton solution for the spin-polarized current in a uniaxial ferromagnetic nanowire and reported that the soliton solution shows the characteristic breather behavior of the soliton as it propagates along the ferromagnetic nanowire.[16] Also, He and Liu investigated the effect of spin-transport on nonlinear excitations in a ferromagnetic nanowire with nonuniform magnetizations and these solitons propagate along the wire's axis.[17] Further, Liu *et al.*, formulated the LL equation for an anisotropic (easy-plane) ferromagnet as a Riemann–Hilbert problem on a Riemann surface of the spectral parameter yielding all possible double wall solutions, including states of colliding walls, breather modes of bound walls, and a set of solutions corresponding to marginally bound walls.[18] Most of the studies on magnetic spin chains have been based on the homogeneous Heisenberg Hamiltonian when the exchange integral associated with the coupling interaction between nearest-neighbor pair of spins is a single constant J. But due to the very presence of various magnetic defects such as imperfect grain boundaries, impurities as well as elastic defects such as misfitting precipitates and dislocation, there arises an inhomogeneity in the magnetic lattice that makes the dynamics more complex. Defects are the local deviations from a given ordered structure and have attracted large interest in many areas like biology,[19] liquid crystals,[20,21] soap bubbles,[22] alloys,[23] and soft matter[24–26] as their formation and evolution are one of the fundamental problems to understand the complexity in these systems. Young *et al.*, described the spatio-temporally chaotic state sustained by the

†Corresponding author. E-mail: louiskavitha@yahoo.co.in

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induced nucleation of penta-hepta defects in a model for rotating hexagonal convection.[27] In experiments performed by Pertsinidis *et al.*, dislocations enhance the diffusion of divacancies in two-dimensional colloidal crystals. The hopping of defects does not follow a pure random walk, but exhibits some memory effects due to the slow relaxation rates of distortions in the over-damped colloidal crystal.[28] Neubecker *et al.*, demonstrated theoretically and experimentally the possibility of an active manipulation of defects in terms of an externally induced motion in a liquid crystal light valve under single optical feedback.[20] Also, physically apart from the presence of defects, in general the inhomogeneity occurs in the magnetic lattices due to the following two factors: (i) if the distance between the neighboring atoms varies along the lattice, as seen in some charge transfer complexes TCNQ and in organometallic insulators, TTF bisdithiolenes, and (ii) the wave function itself varies from site to site although the atoms themselves may equally be spaced, e.g. a one-dimensional magnetic insulator placed in a weak, static, inhomogeneous electric field with the deliberate introduction of imperfections in the vicinity of a bond so as to alter the electronic wave functions without causing appreciable lattice distortion. In inhomogeneous magnetic materials, however, the size of the domain is determined exclusively by the inhomogeneity of the materials independent of the shape of the specimen. The kinds of inhomogeneity which can affect the domain structure are voids, inclusions, fluctuations in alloy compositions internal stress, local directional orders, crystal bounderies, etc. For instance, if a spherical inclusion or void exists in a ferromagnetic substance, the free poles which appear on its surface should make the magnetostatic energy quite large. In the past four decades, a sizable number of soliton possessing nonlinear Schrödinger-type equations have been derived which form the potential platform in studying the various phenomena of diverse physical systems. In this context, the aim of the present study is to construct a series of exact solitary wave solutions of an inhomogeneous higher order nonlinear Schrödinger equation (IHNLS) in the presence of damping and nonlocal potential governing the dynamics of one-dimensional classical isotropic biquadratic Heisenberg spin chain with varying bilinear and biquadratic exchange interactions. We achieve this by employing the tangent hyperbolic function method embedded with symbolic computation. The paper is organized as follows. In Section 2, we formulate the associated dynamical equation for the site-dependent ferromagnetic spin chain and discuss the integrability conditions which hold the existence of solitons. In Section 3, we give the methodology of notion of integrability. In Section 4, we implement the tanh function method to solve the fourth order inhomogeneous nonlinear Schrödinger (NLS) equation with nonlocal potential and damping. In Section 5, we discuss the effect of inhomogeneity in both the presence and absence of damping on the spin solitons. Finally we conclude our results.

2. Dynamics of a deformed ferromagnetic medium modelled by a damped nonlocal higher order nonlinear Schrödinger equation

Magnetic solitons have been identified as one of the interesting class of nonlinear excitations representing the nonlinear spin dynamics of one-dimensional classical continuum heisenberg ferromagnetic spin sytems. For a spin model, the associated Landau–Lifshiftz–Gilbert (LLG) equation can be written as

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{F}_{\text{eff}} + \lambda [\boldsymbol{F}_{\text{eff}} - (\boldsymbol{S} \cdot \boldsymbol{F}_{\text{eff}}) \boldsymbol{S}]. \tag{1}$$

It is also of importance to include the processes leading to damping and diffussion of the spin motions in Eq. (1). Such processes are essentially related to the relativistic interaction due to the conductivity of the medium. The simplest expression for such a process which causes least analytical distress in the system is a term of the form $\hat{\lambda}[\boldsymbol{S} \times [\boldsymbol{S} \times \boldsymbol{F}_{\text{eff}}]]$, where $\hat{\lambda} = \hat{\alpha}\hat{\gamma}$ and $\hat{\alpha}$ is the dimensionless Gilbert damping parameter, which quantifies the decay of the excited system back to equilibrium. The damping term introduced in a phenomelogical manner many decades ago. Clearly the magnetization must be time-dependent for damping to occur, hence the proportionality to $\partial \boldsymbol{S}/\partial t$. The LL equation is a rather simple approximation to very intricate the dynamic processes. The limitations of the approximations entering into the LL equation are likely to be tested by the next generation of magnetodynamic devices. While many generalizations for the LL equation are possible, we focus on investigating the importance of nonlocal contributions to damping. The nonlocality of the magnetization dynamics can be interepted as a nonlocal exchange coupling with explicit time dependence. It is generally assumed in both analyzing experimental results and in performing micromagnetic simulations that damping is a local phenomenon. While no clear experimental evidence exists to contradict this assumption, the possibility that the damping is nonlocal — that it depends, for example, on the local gradient of the magnetization would have particular implications for experiments that quantify spin-current polarization,[29] for storage,[30] logic devices,[31] and mode dynamics[32] in patterned samples, and the behavior of nano contact oscillators. Furthermore, the damping motion of the magnetization vector, particularly the damping of large motion plays a significant role in the performance of magnetic recording devices. The equation of motion in the presence of damping takes the form

$$\begin{aligned} \frac{\partial \boldsymbol{S}}{\partial t}(\boldsymbol{r},t) \;=\; & \hat{\gamma}\big[\boldsymbol{S}(\boldsymbol{r},t) \times \boldsymbol{F}_{\text{eff}}\big] \\ & + \hat{\lambda}\,\big[\boldsymbol{F}_{\text{eff}} - (\boldsymbol{S}(\boldsymbol{r},t) \cdot \boldsymbol{F}_{\text{eff}})\boldsymbol{S}(\boldsymbol{r},t)\big]. \end{aligned} \tag{2}$$

Generally, damping involves loss of energy from the macroscopic motion of the local magnetization field by transfering energy to microscopic thermal motion. The mechanism by which this occurs includes coupling of the magnetization field to spin waves, eddy currents, lattice vibrations, and the effect of polycrystalline structure, strains, and crystal defects such as voids, interstitial atoms, and foreign atoms. When the external magnetic field is not strong enough to eliminate all the domain walls, the domain structure plays a dominant role in the damping, and the local rate of energy loss may vary by large amounts from point to point within a ferromagnet. For low damping, equation (2) describes the precession of the magnetization around the field, $\boldsymbol{F}_{\rm eff}$ while the damping term causes the precession angle to decrease and the spin gets relaxed along the direction parallel to $\boldsymbol{F}_{\rm eff}$ and in turn the system loses energy. The corresponding angular frequency and relaxation time are $\omega = \hat{\gamma}|\boldsymbol{F}_{\rm eff}|$ and $\hat{\tau} = [\hat{\lambda}|\boldsymbol{F}_{\rm eff}|]$ respectively. It may also be noted that equation (2) describes the dynamic domain wall damping caused by the viscous drag on a moving domain wall. The damping constants may be determined from ferromagnetic resonance experiments

$$\begin{aligned}\boldsymbol{F}_{\rm eff} = & \left[A - \frac{b}{2}A_x + \frac{b^2}{2}A_{xx} + \frac{a^2}{2}J_{bg}(S\cdot S_{xx})\right]S_{xx} \\ & + \left[\frac{b}{a}A_x - \frac{b^2}{2a}A_{xx} + \frac{a^2}{3}J_{bg}(S\cdot S_{xxx})\right]S_x \\ & + \frac{a^2}{12}AS_{xxxx} + \frac{ab}{6}A_xS_{xxx}.\end{aligned}$$

Where $A(x) = J_e f(x) + J_b g(x)$, $f(x)$ and $g(x)$ characterize the inhomogeneity in bilinear and biquadratic interactions, respectively. In the above equation, the suffices t and x represent partial derivatives time and space respectively. This equation is the LL equation that describes the dynamics of spins in a one-dimensional classical continuum isotropic inhomogeneous biquadratic Heisenberg spin chain.

Study of Eq. (2) is of interest not only from a physical point of view, but also of great interest in applied mathematics. However equation (2) is a highly nontrivial vector nonlinear partial differential equation and it is very difficult to solve it in its natural form. Therefore it is advantageous to go to a representation by mapping the continuum spin system onto a moving helical space curve in E^3 and obtain invariant equations in terms of the evolution of the curvature and torsion. Like the discovery of solitons, the investigation through the space curve first started in hydrodynamics and was then applied to different branches of science. The time evolution of the space curve is associated with a geometric space. Therefore, the analysis of the evolution equation for the spin density is facilitated through a knowledge of the underlying geometry of the system. The spin density $\boldsymbol{S}(x,t)$ in Eq. (1) is mapped on the unit tangent vector $\boldsymbol{e}_1(x,t)$ of the space curve. Then the unit normal vectors $\boldsymbol{e}_2(x,t)$ and $\boldsymbol{e}_3(x,t)$ can also be defined in the usual way to form a local co-ordinate system of the space curve. The change of orientation of the trihedron $\boldsymbol{e}_i$ $(i = 1,2,3)$ which defines the space curve uniquely within rigid motion is determined by the Serret–Frenet equations:

$$\begin{pmatrix}\boldsymbol{e}_{1x}\\ \boldsymbol{e}_{2x}\\ \boldsymbol{e}_{3x}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix}, \tag{3}$$

where $\boldsymbol{e}_{ix} = \boldsymbol{d}\times\boldsymbol{e}_i$, $\boldsymbol{d} = -\kappa\boldsymbol{e}_3 + \tau\boldsymbol{e}_1$. The vectors $\boldsymbol{e}_i$, $i = 1,2,3$ are the usual unit tangent, normal, and binormal vectors respectively on the space curve. The classical spin vector $\boldsymbol{S}(x,t)$ satisfying Eq. (1) is now identified with the tangent vector $\boldsymbol{e}_1(x,t)$ so that Eq. (1) can be written as

$$\boldsymbol{e}_{1t} = \boldsymbol{e}_1\times\boldsymbol{e}_{1xx} + \lambda[\boldsymbol{e}_{1xx} - (\boldsymbol{e}_1\cdot\boldsymbol{e}_{1xx})]\boldsymbol{e}_1. \tag{4}$$

The evolution of $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ is found similarly. Thus the evolution of the trihedral $\boldsymbol{e}_i$ takes the rigid body form

$$\boldsymbol{e}_{it} = \boldsymbol{\omega}\times\boldsymbol{e}_i, \quad \boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2 + \omega_3\boldsymbol{e}_3.$$

Then the evolution of the trihedral $\boldsymbol{e}_i$ preserving their orthogonality is determined by the equations (3) and (4) and it is written as

$$\begin{pmatrix}\boldsymbol{e}_{1t}\\ \boldsymbol{e}_{2t}\\ \boldsymbol{e}_{3t}\end{pmatrix} = \begin{pmatrix}0 & \omega_3+\lambda x & \omega_2+\lambda k\tau\\ -\omega_3-\lambda k_x & 0 & \omega_1+\dfrac{\lambda}{k}(2k_x\tau + k\tau_x)\\ -\omega_2+\lambda k\tau & \omega_1+\dfrac{\lambda}{k}(2k_x\tau+k\tau_x) & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix}. \tag{5}$$

The condition for the compatability of Serret–Frenet equation (1) with the system of Eq. (5) for the evolution of the trihedron given by

$$(\boldsymbol{e}_{ix})_t = (\boldsymbol{e}_{it})_x, \quad i = 1,2,3,$$

leads to the evolution equations for curvature and torsion of the space curve. After using the following transformation

$$q = \frac{k}{2}\exp\left[\mathrm{i}\int_{-\infty}^{x}\tau(x,t)\,\mathrm{d}x\right],$$

we obtain the generalized fourth-order damped nonlinear Schrödinger equation as

$$\begin{aligned}&\mathrm{i}q_t + (1-\mathrm{i}\lambda)\left[h_{xx}q + 2h_xq_x + hq_{xx}\right]\\ &+2q\left[hqq^* + \int_{-\infty}^{x}h_x'\,\mathrm{d}x'qq^* + \mathrm{i}\lambda h\int_{-\infty}^{x}(qq_x^* - q^*q_x)\,\mathrm{d}x\right]\\ &+\nu\left[q_{xxxx} + 8|q|^2q_{xx} + 2q^2q_{xx}^* + 4|q_x|^2q + 6q^*q_x^2 + 6q|q|^4\right]\\ &= 0,\end{aligned} \tag{6}$$

where $h(x)$ represents the inhomogeneity, q is a function of x and t, and λ is the damping parameter. Equation (6) represents the nonlinear spin dynamics of a one-dimensional Heisenberg

ferromagnet with varying bilinear and biquadratic exchange interaction in the classical continuum limit. The dynamics of the spin system is expected to be integrable only when the inhomogeneity assumes a linear function. Further, apart from the integrable conditions, i.e., when the inhomogeneity arises in the form of a nonlinear function the dynamics has been investigated by employing an appropriate perturbation theory along the lines of Kodama and Ablowitz.[33] As a consequence, it has been analyzed that how the soliton behaves when the nonlinear inhomogeneity is introduced as a small perturbation. Solving recursively this system of equations and taking care of the secularity problem appropriately, one can in principle obtain the required perturbed solution and the relevant physical information can also be revealed from the nature of the solution. However, it is a standard feature of nonlinear systems, that exact analytic solutions are possible only in exceptional cases.

From Eq. (6) we could obtain several integrable models and the integrability will be confirmed by the Painlevè singularity structure analysis and the integrability properties are investigated by constructing the Lax pair and soliton solutions.[34] Finally we get the following fourth order NLS-type equation:

$$\begin{aligned} &\mathrm{i}q_t+(hq)_{xx}+2h|q|^2q+2q\int_{-\infty}^{x}h'_x|q|^2\mathrm{d}x'+\frac{a^2A}{12}\big[q_{xxxx}\\ &+K_1|q|^2q_{xx}+K_2q^2q^*_{xx}+K_3|q_x|^2q+K_4q^*q_x^2+3K_2q|q|^4\big]\\ &=0, \end{aligned} \tag{7}$$

where,

$$\begin{aligned} &h_x=\hat{h}+\frac{b^2}{6}A_{xx}, \qquad K_1=-12\Big(1+\frac{4J_bg}{A}\Big),\\ &K_2=-8(1+\frac{3J_bg}{A}), \quad K_3=-36\Big(1+\frac{8J_bg}{3A}\Big),\\ &K_4=-14\Big(1+\frac{24J_bg}{7A}\Big). \end{aligned}$$

3. The search for integrable models

In the study of integrability, in the sense of series expansion of the solution, some local properties of the system of equations can be extended to answer the global question of integrability. In this attempt to decide whether the system has the Painlevè property we expand a formal Laurent series solution about an arbitrary singular point in complex time. We begin with a leading order analysis for the inhomogeneous fourth-order NLS equation.

3.1. Integrable model at $O(a^0b^0)$

At $0(a^0b^0)$, equation (7) reduces to the following inhomogeneous cubic NLS equation

$$\mathrm{i}q_t+(hq)_{xx}+2h|q|^2q+2q\int_{-\infty}^{x}|q|^2\mathrm{d}x'=0. \tag{8}$$

We denote q and q^* by σ and ι respectively and on defining a new real function A, Eq. (8) and the complex conjugate equation can be written as

$$\mathrm{i}\sigma_t+(h\sigma)_{xx}+2h_x\sigma_x+h_{xx}\sigma+2\rho\sigma=0, \tag{9}$$

$$-\mathrm{i}\iota_t+(h\iota)_{xx}+2h_x\iota_x+h_{xx}\iota+2\rho\iota=0, \tag{10}$$

$$\rho_x-2h_x\sigma\iota_x-h(\sigma\iota)_x=0. \tag{11}$$

We perform P analysis on Eqs. (9)–(11) by expanding the functions σ, ι, and ρ locally in the form of the Laurent series

$$\sigma=\sigma_0(x,t)\phi^p(x,t)+\sum_{j=1}\sigma_j(x,t)\phi^{p+j}, \tag{12}$$

$$\iota=\iota_0(x,t)\phi^q(x,t)+\sum_{j=1}\iota_j(x,t)\phi^{q+j}, \tag{13}$$

$$\rho=\rho_0(x,t)\phi^s(x,t)+\sum_{j=1}\rho_j(x,t)\phi^{s+j}. \tag{14}$$

Now using the leading order terms of the solutions (12)–(14), i.e., $\sigma=\sigma_0(x,t)\phi^p(x,t)$, $\iota=\iota_0(x,t)\phi^q(x,t)$, and $\rho=\rho_0(x,t)\phi^s(x,t)$ in Eqs. (14)–(16) containing leading order terms alone

$$h\sigma_{xx}+2h_x\sigma_x+h_{xx}\sigma+2\rho\sigma\sim0, \tag{15}$$

$$h\iota_{xx}+2h_x\iota_x+h_{xx}\iota+2\rho\iota\sim0, \tag{16}$$

$$2h_x\sigma\iota+h(\sigma\iota)_x\sim0, \tag{17}$$

and on balancing the dominant terms, we obtain the following leading order results

$$p=q=-1, \quad s=-2, \tag{18}$$

$$\sigma_0\iota_0=-\phi_x^2, \quad \rho_0=-h\phi_x^2. \tag{19}$$

In order to find the resonances, i.e., the powers at which the free coefficients enter into the generalized expansion, we expand

$$\sigma=\sigma_0\phi^p(x,t)\cdots+\alpha_1\phi^{p+r}, \tag{20}$$

$$\iota=\iota_0\phi^q(x,t)\cdots+\alpha_2\phi^{q+r}, \tag{21}$$

$$\rho=\rho_0\phi^s(x,t)\cdots+\alpha_3\phi^{s+r}, \tag{22}$$

on substituting Eqs. (20)–(22) in Eqs. (15)–(17), we obtain the following resonance equations

$$\begin{pmatrix} h(r-1)(r-2)+2\rho_0 & 0 & 2\sigma_0\\ 0 & h(r-1)(r-2)+2\rho_0 & 2\iota_0\\ h\iota_0r & h\sigma_0r & (r-2) \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}=0. \tag{23}$$

In order that α_1, α_2, and α_3 are arbitrary and nontrivial, we evaluate the determinant of the (3×3) matrix on the left-hand side of Eq. (23) and obtain the following polynomial equation

$$\begin{aligned} &vh^2r^5-8h^2r^4+\big[25h^2+4h\rho_0-4h\sigma_0\iota_0\big]r^3\\ &+\big[-38h^2-20h\rho_0+12h\eta_0\iota_0\big]r^2+\big[28h^2+4\rho_0^2+32h\rho_0\\ &-8h\sigma_0\iota_0-8h\sigma_0\iota_0\rho_0\big]r-8h^2-8\rho_0^2-16h\rho_0\\ &=0. \end{aligned} \tag{24}$$

Upon substituting σ_0, ι_0, and ρ_0, from Eq. (19) into Eq. (24), we obtain the following resonances

$$r=-1,0,2,3,4. \tag{25}$$

The resonances, -1 and 0 correspond to the arbitrariness of the singular manifold of the coefficients σ_0 or ι_0 respectively. To check whether sufficient number of arbitrary functions exists without introducing movable critical singularity manifolds, we substitute the Laurent expansion given in Eqs. (12)–(14) in the full Eqs. (9)–(11) and collect coefficients of different powers of ϕ and solve them. The results are as follows:

For $(\phi^{-2},\phi^{-2},\phi^{-2})$:

$$\begin{pmatrix} 2\rho_0 & 0 & 2\sigma_0 \\ 0 & 2\rho_0 & 2\iota_0 \\ h\iota_0 & h\sigma_0 & -1 \end{pmatrix}\begin{pmatrix}\sigma_1\\ \iota_1\\ \rho_1\end{pmatrix} = \begin{pmatrix} \mathrm{i}\sigma_0\phi_t+2h\sigma_{0x}\phi_x+h\sigma_{0x}\phi_{xx}+2h_x\sigma_0\phi_x \\ -\mathrm{i}\iota_0\phi_t+2h\iota_{0x}\phi_x+h\iota_{0x}\phi_{xx}+2h_x\iota_0\phi_x \\ -h_x\phi_x \end{pmatrix}. \tag{26}$$

Solving the above equation, we obtain

$$\sigma_1=\frac{1}{6h\phi_x^2}\left[-3\mathrm{i}\sigma_0\phi_t-6h\sigma_{0x}\phi_x+3h\sigma_0\phi_{xx}\right], \tag{27}$$

$$\iota_1=\frac{1}{6h\phi_x^2}\left[3\mathrm{i}\iota_0\phi_t-6h\iota_{0x}\phi_x+3h\iota_0\phi_{xx}\right], \tag{28}$$

$$\rho_1=h_x\phi_x+h\phi_{xx}. \tag{29}$$

For $(\phi^{-1},\phi^{-1},\phi^{-1})$:

$$\begin{pmatrix} 2\rho_0 & 0 & 2\sigma_0 \\ 0 & 2\rho_0 & 2\iota_0 \\ h\iota_0 & h\sigma_0 & -1 \end{pmatrix}\begin{pmatrix}\sigma_1\\ \iota_1\\ \rho_1\end{pmatrix} = \begin{pmatrix} -\mathrm{i}\sigma_{0t}-h\sigma_{0xx}-2h_x\sigma_{0x}-2\rho_1\sigma_1-\sigma_0h_{xx} \\ \mathrm{i}\iota_0-h\iota_{0xx}-2h_x\iota_{0x}-2\rho_1\iota_1-\iota_0h_{xx} \\ 0 \end{pmatrix}. \tag{30}$$

From the above equation we infer that either σ_2 or ι_2 is arbitrary and obtain the resonance condition

$$\rho_{1x}=4h_x(\iota_0\sigma_1+\sigma_0\iota_1)-2h(\iota_0\sigma_1+\sigma_0\iota_1)_x. \tag{31}$$

Further the value of ρ_2 is evaluated from Eqs. (30) and (31). Substituting the value of ρ_1 in the resonance condition (31), we get

$$h_{xx}=0,\quad h_x=\text{constant},$$

which implies $h(x)=Cx+D$, where C and D are constants.

For (ϕ^0,ϕ^0,ϕ^0):

$$\begin{pmatrix} 0 & 0 & 2\sigma_0 \\ 0 & 0 & 2\iota_0 \\ -2h\sigma_0\phi_x & -2h\iota_0\phi_x & -\phi_x \end{pmatrix}\begin{pmatrix}\sigma_3\\ \iota_3\\ \rho_3\end{pmatrix} = \begin{pmatrix} \sigma_{1t}+\sigma_2\phi_t-h_{xx}\sigma_1-2h_x(\psi_{1x}+\iota_2\phi_x)-h(\iota_{1xx}+2\iota_{2x}\phi_x+\iota_2\phi_{xx})-2(\iota_1\rho_2+\rho_2\iota_1) \\ -\iota_{1t}+\iota_2\phi_t-h_{xx}\iota_1-2h_x(\sigma_{1x}+\sigma_2\phi_x)-h(\sigma_{1xx}+2\sigma_{2x}\phi_x+\sigma_2\phi_{xx})-2(\sigma_1\rho_2+\rho_2\sigma_1) \\ -\rho_{2x}+2h_x(2\sigma_0\sigma_2+2\iota_0\iota_2+\sigma_1^2+\iota_1^2)+2h(\sigma_0\sigma_{2x}+\sigma_1\sigma_{1x}+\sigma_1\sigma_2\phi_x+\sigma_2\sigma_{0x} \\ \qquad +\iota_0\iota_{2x}+\iota_1\iota_{1x}+\iota_1\iota_2\phi_x+\iota_2\iota_{0x}) \end{pmatrix}. \tag{32}$$

From the above derivation we find that either σ_3 or ι_3 is arbitrary and ρ_3 is found to be

$$\begin{aligned}\rho_3 = \frac{1}{\sigma_0\iota_0}[&-\iota_0\iota_{1t}+\sigma_0\sigma_{1t}+(\iota_0\iota_2-\sigma_0\sigma_2)\phi_t \\ &-2h_x(\sigma_0\iota_{1x}+G_0\sigma_{1x})-2h_x\phi_x(\sigma_0\iota_2+\iota_0\sigma_2) \\ &-h(\sigma_0\iota_{1xx}+\iota_0\sigma_{1xx}+2\sigma_0\iota_{2x}+2\sigma_0\iota_{2x}+2\iota_0\sigma_{2x} \\ &+(\sigma_0\iota_2+\iota_0\sigma_2)\phi_{xx}]-2\rho_2(\sigma_0\iota_1+\iota_0\sigma_1) \\ &-2\rho_1(\sigma_0\iota_2+\iota_0\sigma_2)].\end{aligned} \tag{33}$$

Proceeding further and by collecting terms at (ϕ^1,ϕ^1,ϕ^1) we find that σ_4 or ι_4 is arbitrary and ρ_4 is expressed by a definite value. As the equations at this order are unwieldy in nature, we are not presenting the results here. Thus, equations (9)–(11) are found to be free from movable critical manifolds for the specific form of $h(x)$ (inhomogeneity)

$$h(x)=Cx+D,$$

thereby satisfying the P-property. The integrable form of Eq. (8) can therefore be written as

$$\mathrm{i}q_t+2Cq_x+(Cx+D)\left[q_{xx}+2|q|^2q\right]+2Cq\int_{-\infty}^{x}|q|^2\mathrm{d}x'=0. \tag{34}$$

Equation (34) is a completely integrable inhomogeneous NLS equation which admits the soliton solution. The corresponding integrable LL equation is written as

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t}=\boldsymbol{S}\wedge[(Cx+D)\boldsymbol{S}_{xx}+C\boldsymbol{S}_x].$$

Also at the same order, i.e., at $O(a^0b^0)$ when $C=0$ which ultimately implies that $A(x)=J_ef(x)+J_bg(x)$, is a constant, Eq. (8) reduces to the completely integrable cubic NLS equation

$$\mathrm{i}q_t+q_{xx}+2|q|^2q=0.$$

The cubic NLS equation has also been obtained as equivalent to the one-dimensional classical isotropic Heisenberg ferromagnetic spin chain under space curve mapping. The corresponding LL equation is written as

$$\boldsymbol{S}_t=\boldsymbol{S}\wedge\boldsymbol{S}_{xx}.$$

3.2. Integrable model at $O(a^2)$

Moving to the next order, i.e. at $O(a^2)$, we assume that the terms at $O(ab)$ is of higher order than terms at $O(a^2)$, which means that the variation of exchange interaction is slow compared to the regular spin lattice. Under this approximation, equation (6) gives the following inhomogeneous fourth-order NLS-type equation

$$\begin{aligned}&\mathrm{i}q_t+(hq)_{xx}+2hq|q|^2+2q\int_{-\infty}^{x}h_{x'}|q|^2\mathrm{d}x'+\frac{a^2A}{12}\left[q_{xxxx}\right. \\ &\left.+K_1|q|^2q_{xx}+K_2q^2q_{xx}^*+K_3q|q_x|^2+K_4q^*q_x^2+3K_2q|q|^4\right] \\ &=0,\end{aligned} \tag{35}$$

where,

$$h_x=\hat{h}+\frac{b^2}{6}A_{xx},\qquad K_1=-12\left(1+\frac{4J_bg}{A}\right),$$

$$K_2 = -8\Big(1+\frac{3J_bg}{A}\Big), \quad K_3 = -36\Big(1+\frac{8J_bg}{3A}\Big),$$
$$K_4 = -14\Big(1+\frac{24J_bg}{7A}\Big).$$

Now, we seek whether equation (35) is also expected to be integrable when A is a linear function of x or otherwise through P-analysis. For this, we rewrite Eq. (35) and its complex conjugate equation by denoting q and q^* by E and G respectively and by defining a new real function R as before

$$\mathrm{i}\sigma_t + h\sigma_{xx} + 2h_x\sigma_x + h_{xx}\sigma + 2R\sigma + \frac{a^2}{12}A\big[\sigma_{xxxx} + K_1\sigma\iota\sigma_{xx} + K_2\sigma^2\iota_{xx} + K_3\sigma\sigma_x\iota_x + K_4G\sigma_x^2 + 3K_2\sigma^3\iota^2\big] = 0, \tag{36}$$

$$-\mathrm{i}\iota_t + h\iota_{xx} + 2h_x\iota_x + h_{xx}\iota + 2\rho\iota + \frac{a^2}{12}A\big[\iota_{xxxx} + K_1\sigma\iota\iota_{xx} + K_2\iota^2\sigma_{xx} + K_3\iota\sigma_x\iota_x + K_4\sigma\iota_x^2 + 3K_2\sigma^2\iota^3\big] = 0, \tag{37}$$

$$\rho_x - 2h_x\sigma\iota - h(\sigma\iota)_x = 0. \tag{38}$$

We perform P-analysis on Eqs. (36)–(38) by expanding the functions σ, ι, and ρ locally in the form of Laurent series as done in the previous case and using the leading order terms of the solutions (12)–(14) in Eqs. (36)–(38) and after balancing the terms in the leading order equation,

$$\big[\sigma_{xxxx} + K_1\sigma\iota\sigma_{xx} + K_2\sigma^2\iota_{xx} + K_3\sigma\sigma_x\iota_x + K_4\iota\sigma_x^2 + 3K_2\sigma^3\iota^2\big] \sim 0, \tag{39}$$

$$\big[\iota_{xxxx} + K_1\sigma\iota\iota_{xx} + K_2\iota^2\sigma_{xx} + K_3\iota\sigma_x\iota_x + K_4\sigma\iota_x^2 + 3K_2\sigma^2\iota^3\big] \sim 0, \tag{40}$$

$$2h_x\sigma\iota + h(\sigma\iota)_x \sim 0. \tag{41}$$

We obtain the following two branches of leading order results as listed in Table 1.

Table 1. Leading order results of the singularity structure analysis.

Branches	Leading order behavior	Resonances
(i)	$p=q=-1,\ s=-2$	$r=-1,0,1,2,$
	$\sigma_0\iota_0 = \dfrac{-\phi_x^2}{4(1+3J_bg/A)}$	$3,4,5,6$
	$\rho_0 = \dfrac{-\phi_x^2}{4(1+3J_bg/A)}$	
(ii)	$p=q=-1,\ \omega=-2$	$r=-3,-2,-1,0,$
	$\sigma_0\iota_0 = -4\phi_x^2$	$5,6,7,8$
	$\rho_0 = -4h\phi_x^2$	

Now, we find the resonances, i.e., the powers at which the free coefficients enter into the generalized expansion, by substituting Eqs. (18) and (19) into Eqs. (39)–(41), we obtain the resonance matrix as

$$\begin{pmatrix} M_1 & \sigma_0^2M_2 & 0 \\ \iota_0^2M_2 & M_1 & 0 \\ h\iota_0(2-r) & h\sigma_0(2-r) & (r-2) \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = 0, \tag{42}$$

where

$$M_1 = (r-1)(r-2)(r-3)(r-4) + \big[(r-1)(K_1(r-2) - K_3 - 2K_4) + 2K_1 + 4K_2 + K_3 + 9K_2\sigma_0\iota_0\big]\sigma_0\iota_0, \tag{43}$$

$$M_2 = \big[K_2(r-1)(r-2) - K_3(r-1) + K_4 + 2K_1\big] + 6K_2\sigma_0\iota_0. \tag{44}$$

On evaluating the determinant on the left-hand side of Eq. (42), we obtain

$$\begin{aligned}
&r^8 - 20r^7 + (80+2K_1)r^6 + (-800 - 26K_1 - 2K_3 - 4K_4)r^5 \\
&+(1773 - 32K_1 + 10K_2 + 24K_4 + 12K_3 + K_1^2 - K_2^2\sigma_0^2\iota_0^2 \\
&-30K_2\sigma_0\iota_0 - 10K_3\sigma_0\iota_0)r^4 + (-310K_1 - 6K_1^2 - 70K_3 \\
&-2K_1K_3 - 4K_1K_4 - 140K_4 + 310K_2\sigma_0\iota_0 + 6K_1K_2\sigma_0\iota_0 \\
&+70K_3\sigma_0\iota_0 + 2K_1K_3\sigma_0\iota_0)r^3 + (4180 + 628K_1 + 240K_3 \\
&+340K_4 + 280K_2 - 140K_2\sigma_0\iota_0 - 70K_3\sigma_0\iota_0 - 35K_4\sigma_0\iota_0 \\
&-140K_1\sigma_0\iota_0 + 16K_1K_4 + 10K_1K_3 + 8K_1K_2 + 4K_3K_4 + 4K_4^2 \\
&+8K_1^2 - 4K_1K_2\sigma_0\iota_0 - 2K_1K_3\sigma_0\iota_0 - 2K_1K_4\sigma_0\iota_0 - 4K_1^2\sigma_0\iota_0 \\
&-12K_1K_2\sigma_0^2\iota_0^2 - K_3^2\sigma_0^2\iota_0^2 - 9K_2^2\sigma_0^2\iota_0^2 - 6K_2K_3\sigma_0^2\iota_0^2)r^2 \\
&+(-2400 - 544K_1 - 246K_3 - 400K_2 - 296K_4 - 24K_1^2 \\
&-8K_3^2 - 8K_4^2 - 24K_1K_2 - 28K_1K_3 - 28K_1K_4 - 16K_2K_3 \\
&-16K_2K_4 - 8K_3K_4 - 7K_1K_4\sigma_0\iota_0 - 4K_2K_3\sigma_0\iota_0 - 50K_4\sigma_0\iota_0 \\
&-12K_1K_2\sigma_0^2\iota_0^2 + 2K_1K_3\sigma_0^2\iota_0^2 - 5K_2K_3\sigma_0^2\iota_0^2 - 300K_2\sigma_0^2\iota_0^2 \\
&-12K_2K_4\sigma_0^2\iota_0^2 + 12K_2^2\sigma_0^2\iota_0^2 + K_3^2\sigma_0^2\iota_0^2 + 18K_2^2\sigma_0^3\iota_0^3 \\
&+6K_2K_3\sigma_0^3\iota_0^3)r + 576 + 192K_1 + 96K_3 + 192K_2 + 96K_4 \\
&+24K_3\sigma_0G_0 + 24K_4\sigma_0\iota_0 + 16K_1^2 + 16K_2^2 + 4K_3^2 + 16K_1K_3 \\
&+32K_1K_2 + 16K_1K_4 + 12K_3K_4 + 16K_2K_4 + 4K_1K_3\sigma_0\iota_0 \\
&-4K_2^2\sigma_0\iota_0 + 2K_3^2\sigma_0\iota_0 + 2K_2K_3\sigma_0\iota_0 - 24K_2\sigma_0\iota_0 \\
&-4K_1K_2\sigma_0\iota_0 - 2K_2K_4\sigma_0\iota_0 - 24K_4\sigma_0\iota_0 - 6K_2^2\sigma_0^2\iota_0^2 \\
&-3K_2K_3\sigma_0^2\iota_0^2 - 5K_2K_4\sigma_0^2\iota_0^2 - K_3K_4\sigma_0^2\iota_0^2 - K_4^2\sigma_0^2\iota_0^2 = 0.
\end{aligned} \tag{45}$$

In both the branches the resonances -1 and 0 correspond to the arbitrariness of the manifold σ_0 or ι_0 respectively. To check whether sufficient numbers of arbitrary functions exist without introducing movable critical singularity manifolds, we substitute the Laurent expansions (12)–(14) into the full equations (36)–(38) and collect the coefficients of different powers of ϕ. At $(\phi^{-4},\phi^{-4},\phi^{-2})$, we obtain $\rho_1 = -2h_x + h\phi_{xx}$ and σ_1 or ι_1 is arbitrary corresponding to $r=1$ with the resonance condition

$$\sigma_0\iota_1 + \iota_0\sigma_1 = \phi_{xx}. \tag{46}$$

From the equations at $(\phi^{-3},\phi^{-3},\phi^{-1})$, we find that ρ_2 is arbitrary and σ_2 or ι_2 is arbitrary corresponding to $r=2,2$ with the following resonance conditions

$$\begin{aligned}
\sigma_0\iota_2 - \iota_0\sigma_2 = \frac{1}{6\phi_x^2}\big[&-3(\sigma_0\iota_{0xx} - \iota_0\sigma_{0xx}) - 2(\phi_x\phi_{xx})_x - 4\phi_{xx}^2 \\
&+4(2\iota_0\sigma_{1x} - \sigma_0\iota_{1x})\phi_x + 4(\iota_{0x}\phi_x - \iota_0\phi_{xx})\sigma_1 \\
&+8\sigma_{0x}\phi_x\phi_{xx}\big],
\end{aligned} \tag{47}$$

$$h_{xx} = 0. \tag{48}$$

From the above results, we have σ_1 or ι_1 and σ_2 or ι_2, and ρ_2 being arbitrary. Further the condition $h_{xx}=0$ implies that h is a linear function of x, say $h = Cx + D$ where C and D are con-

stants. Proceeding further, we find that sufficient number of arbitrary functions enter without introduction of movable critical manifolds at $r = 3$, 4, 5, and 6. However, as the compatibility conditions corresponding to these resonances are unwieldy in nature we are not presenting them here. For the branch (ii) we verify that arbitrary functions enter at $r = 5$, 6, 7, and 8. Thus Eq. (35) possesses the Painlevè property and hence it is a candidate to be integrable when $J_b g/A = -5/12$ and $h = Cx + D$. The integrable form of Eq. (35) (the fourth-order inhomogeneous NLS equation) can therefore be written as

$$\begin{aligned} &\mathrm{i}q_t + 2Cq_x + (Cx+D)\left[q_{xx} + 2|q|^2 q\right] + 2Cq\int_{-\infty}^{x} |q|^2 \mathrm{d}x' \\ &+ \frac{a^2}{12}\left[q_{xxxx} + 8|q|^2 q_{xx} + 2q^2 q_{xx}^* + 4|q_x|^2 q + 6q^* q_x^2 + 6q|q|^4\right] \\ &= 0. \end{aligned} \tag{49}$$

The associated LL equation is written as

$$\begin{aligned} \boldsymbol{S}_t &= \boldsymbol{S} \wedge \left[\left[(Cx+D)\boldsymbol{S}_{xx} + C\boldsymbol{S}_x\right] + \frac{a^2}{12}\left[\boldsymbol{S}_{xxxx} - \frac{5}{2}(\boldsymbol{S}\cdot\boldsymbol{S}_{xx})\boldsymbol{S}_{xx}\right.\right. \\ &\left.\left. - \frac{5}{3}(\boldsymbol{S}\cdot\boldsymbol{S}_{xxx})\boldsymbol{S}_x\right]\right]. \end{aligned} \tag{50}$$

At the same order i.e., at $O(a^2)$, when $C = 0$, that is when the exchange interaction is not varying (constant) and for the choice $J_b g/A = -5/12$, Eq. (26) reduces to the following completely integrable fourth-order NLS equation

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + 2q|q|^2 + \frac{a^2}{12}\left[q_{xxxx} + 8|q|^2 q_{xx} + 2q^2 q_{xx}^* + 4|q_x|^2 q\right. \\ &\left. + 6q^* q_x^2 + 6|q|^4 q\right] = 0. \end{aligned} \tag{51}$$

By substituting the values of $\sigma_0 t_0$ into Eq. (45), we obtain all the resonances as real values only for the following set of parametric choices i) $J_b g/A = -17/48$ and ii) $J_b g/A = -5/12$. For the choices of parameters given in case i) the two branches merge into a single branch with the leading order results, $\sigma_0 t_0 = -4\phi_x^2$, $\rho_0 = -4h\phi_x^2$, and the resonances $r = -1$, -1, 0, 0, 2, 3, 4, 7, 8. Analyzing further, we find that the system admits a logarithmic singularity manifold at $r = 0$, thereby destroying the Painlevè nature of solution and therefore we stop treating case i) further. However, for the case ii) we obtain the following two branches as listed in Table 2.

Table 2. Leading order results of the singularity structure analysis.

Branches	Leading order behavior	Resonances
i)	$p = q = -1$, $s = -2$	$r = -1, 0, 1, 2, 2$
	$\sigma_0 t_0 = -\phi_x^2$, $\rho_0 = h - \phi_x^2$	3, 4, 5, 6
ii)	$p = q = -1$, $\omega = -2$	$r = -3, -2, -1, 0,$
	$\sigma_0 t_0 = -4\phi_x^2$	2, 5, 6, 7, 8
	$\rho_0 = -4h\phi_x^2$	

The associated LL equation can be written as

$$\begin{aligned} \boldsymbol{S}_t &= \boldsymbol{S} \wedge \left[\boldsymbol{S}_{xx} + \frac{a^2}{12}\left[\boldsymbol{S}_{xxxx} - \frac{5}{2}(\boldsymbol{S}\cdot\boldsymbol{S}_{xx})\boldsymbol{S}_{xx}\right]\right. \\ &\left. - \frac{5}{3}(\boldsymbol{S}\cdot\boldsymbol{S}_{xxx})\boldsymbol{S}_x\right]. \end{aligned} \tag{52}$$

When linear inhomogeneity is present in the system the Lax pair and soliton solution can be written as follows.[34] The Lax pair U, V associated with the AKNS-type linear eigenvalue problem $\Psi_x = U\Psi$, $\Psi_t = V\Psi$, where $\Psi = (\Psi_1, \Psi_2)^{\mathrm{T}}$ corresponding to Eq. (50) can be written as[35]

$$U = \begin{pmatrix} -\mathrm{i}\hat{\lambda} & q \\ -q^* & \mathrm{i}\hat{\lambda} \end{pmatrix}, \tag{53}$$

$$\begin{aligned} V &= \left[\mathrm{i}|q|^2 + \mathrm{i}\hat{\nu}[q^* q_{xx} + qq_{xx}^* - |q_x|^2 + 3|q|^4]\right. \\ &+ 2\hat{\lambda}\hat{\nu}(qq_x^* - q_x q^*) - 2\mathrm{i}\hat{\lambda}^2[1 + Cx + D + 2\hat{\nu}|q|^2] \\ &\left. + 8\hat{\nu}\hat{\lambda}^4\right]\sigma_z + \mathrm{i}\sigma_z M_x + i\hat{\nu}[6M^2 M_x\sigma_z + \sigma_z M_{xxx}] \\ &- 4\mathrm{i}\hat{\nu}\lambda^2\sigma_z M_x + 2(Cx+D)M + 2\hat{\lambda}[M + \hat{\nu}(M_{xx} - 2M^3)] \\ &- 8\hat{\nu}\hat{\lambda}^3 M, \end{aligned} \tag{54}$$

where $\hat{\lambda}$ is the spectral parameter and $\hat{\nu} = a^2 A/12$,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{55}$$

$$M = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{56}$$

Knowing the Lax pair, the multisoliton solutions can be constructed by making use of the inverse scattering transform (IST) method[36] or using the Bäcklund transformation technique.[37,38] For example, the one-soliton solution of Eq. (50) is found to be

$$q(1) = -2\rho\,\mathrm{sech}\,\zeta\exp(-2\mathrm{i}\epsilon), \tag{57}$$

where,

$$\begin{aligned} \zeta &= 2[\rho(\hat{\delta}_1 + x) - 32\hat{\nu}\mu^2 + 4(8\rho^2+1)\mu\rho t] \\ &+ 8D\int(\mu^2 - \rho^2)\mathrm{d}t + \hat{\delta}_1, \end{aligned} \tag{58}$$

$$\begin{aligned} \epsilon &= \mu[(\hat{\delta}_2 + x) + 2[(\mu^2 - \rho^2)t - 4\hat{\nu}(\mu^4 + \rho^4)t + 24\mu^2\rho^2 t] \\ &+ 4D\int \mu\rho\,\mathrm{d}t + \hat{\delta}_2, \end{aligned} \tag{59}$$

where ρ and μ correspond to the scattering parameters and $\hat{\delta}_1$ and $\hat{\delta}_2$ are phase constants. The corresponding spin components can be written as[39]

$$S^x = \frac{-2\rho}{(\mu^2+\rho^2)}\left[\mu\cos\left(\frac{\mu}{\rho}\right)\zeta - \rho\sin\left(\frac{\mu}{\rho}\right)\zeta\tanh\zeta\right]\mathrm{sech}\,\zeta, \tag{60}$$

$$S^y = \frac{-2\rho}{(\mu^2+\rho^2)}\left[\mu\sin\left(\frac{\mu}{\rho}\right)\zeta + \rho\cos\left(\frac{\mu}{\rho}\right)\zeta\tanh\zeta\right]\mathrm{sech}\,\zeta, \tag{61}$$

$$S^z = 1 - \left[\frac{2\rho^2}{(\mu^2+\rho^2)}\right]\mathrm{sech}^2\,\zeta. \tag{62}$$

It may be noted that when terms at $O(ab)$ are assumed to be lesser order than terms at $O(a^2)$ and defining $a \to \mathrm{i}a$ we have the following inhomogeneous Hirota-type equation

$$\begin{aligned} &\mathrm{i}q_t + (hq)_{xx} + 2hq|q|^2 + 2q\int_{-\infty}^{x} h_{x'}|q|^2\mathrm{d}x' \\ &+ \frac{\mathrm{i}ab}{12}h_x[q_{xxx} + 6|q|^2 q_x] = 0. \end{aligned} \tag{63}$$

Equation (57) when $h(x)$ is in the form of a linear function of x has been proved to be completely integrable while carrying out P-analysis on a generalized x-dependent Hirota equation.

4. Exact analytic solution for the fourth order INLS equation

The rapid development of science and technology and the need to simulate complex physical phenomena demand an increasing need to solve quite a number of complicated higher dimensional and nonlinear partial differential equations (PDEs) with variable coefficients. Searching for explicit solutions for these equations by using various methods is a goal for many researchers. The investigation of these explicit exact solutions of soliton equations plays an important role in the study of nonlinear physical phenomena. The wave phenomena observed in fluid dynamics, plasma physics, and chemical physics are often modelled by the bell-shaped and kink-shaped soliton solutions. During the last decades, effective methodologies such as inverse scattering method,[40,41] the tanh–sech method,[42] extended tanh method,[43–51] sine–cosine method,[52–56] Jacobi elliptic function method,[57–59] and Hirota bilinear method[60,61] have been proposed for the determination of solitons. Among these proposed methods the tanh function method provides an effective and direct algebraic method for solving nonlinear equations. Based on the localized nature of soliton solutions, the tanh function method overcomes the complex integration process to obtain explicit solutions to various types of nonlinear equations. The soliton solutions of the nonlinear equations are usually expressed as polynomials of tanh functions. Based on the extension of the tanh function method, Fan[62,63] developed a new algebraic method with symbolic computation for obtaining a series of travelling wave solutions in the unified way. Recently, Wakil *et al.*[64,65] proposed a modified extended tanh function method to obtain new exact solutions for some nonlinear evolution equations. Fan and Hon[66] proposed a generalized tanh method to obtain the more general explicit solutions to nonlinear soliton equations. The tanh method is developed by Malfliet[67–70] where the tanh is used as a new variable, since all derivaties of a tanh are represented by a tanh itself. Consider the general nonlinear wave partial differential equation say in two variables

$$H(u,u_t,u_x,u_{xx},u_{xt},u_{tt},\ldots)=0. \tag{64}$$

Firstly, we consider its travelling solutions $u(x,t)=u(\xi)$, $\xi=x-ct$ or $\xi=x+ct$, where c is the velocity of wave and the equation becomes an ordinary differential equation. In order to seek the travelling wave solutions, we introduce a finite series of tanh-function in the form of

$$u(\xi)=a_0+\sum_{i=1}^{N}a_i\phi^i+b_i\phi^{-i},\ \phi'=b+\phi^2, \tag{65}$$

where b is a parameter to be determined, $\phi=\phi(\xi)$ and $\phi'=\mathrm{d}\phi/\mathrm{d}\xi$. The parameter N can be found by balancing the highest order derivative term with the nonlinear term. Substituting these equations in the ordinary differential equation will yield a system of algebraic equations with respect to a_i, b_i, b, and c since all the coefficients of ϕ^i have to vanish. Then we can determine a_0, a_i, b_i, b, and c. The Riccati equation has the following general solutions:

(I) If $b<0$

$$\phi=-\sqrt{-b}\tanh(\sqrt{-b}\xi),\quad \phi=-\sqrt{-b}\coth(\sqrt{-b}\xi),$$

it depends on the initial conditions.

(II) If $b=0$

$$\phi=-1/\xi.$$

(III) If $b>0$

$$\phi=\sqrt{b}\tan(\sqrt{b}\xi),\quad \phi=-\sqrt{b}\cot(\sqrt{b}\xi).$$

The modified extended tanh function is a mechanization method, in which the processes of generating an algebraic system from Eq. (6) and solving the same are two key procedures and laborious to do by hand. But they can be implemented with the help of symbolic computation packages. The outputs of solving the algebraic system from computer comprise a list of the form a_0, b_0, and b. In general if b or any of the parameter is left or unspecified, then it is to be regarded as arbitrary for the solution of Eq. (6). Especially we are concerned with the sign of b which shows the number of travelling wave solutions for a given partial differential equation. The proposed method not only gives a unified formulation to construct various travelling wave solutions but also provides a rule to classify the types of solutions according to the given parameter.

5. Zero damped dynamics

In order to solve Eq. (6), we put $q(x,t)=E$ and $q^*(x,t)=G$,

$$\begin{aligned}&\mathrm{i}E_t+h_{xx}E+2h_xE_x+hE_{xx}+2E\left[hEG+\int_{-\infty}^{x}h'_xEG\mathrm{d}x'\right]\\&+\nu\left[E_{xxxx}+8EGE_{xx}+2E^2G_{xx}+4E_x\iota_xE+6GE_x^2+6E^3G^2\right]\\&=0.\end{aligned} \tag{66}$$

Therefore equation (6) becomes,

$$\begin{aligned}&\mathrm{i}E_t+h_{xx}E+2h_xE_x+hE_{xx}+2ER+\nu\left[E_{xxxx}+8EGE_{xx}\right.\\&\left.+2E^2G_{xx}+4E_xG_xE+6GE_x^2+6E^3G^2\right]=0.\end{aligned} \tag{67}$$

Taking the complex conjugate of Eq. (6) and simpifying the equation we get,

$$\begin{aligned}&-\mathrm{i}G_t+h_{xx}G+2h_xG_x+hG_{xx}+2G\left[hGE+\int_{-\infty}^{x}h'_xEG\mathrm{d}x'\right]\\&+\nu\left[G_{xxxx}+8EGG_{xx}+2G^2E_{xx}+4G_xE_xG+6EG_x^2+6G^3E^2\right]\\&=0,\end{aligned} \tag{68}$$

hence equation (6) becomes

$$\begin{aligned}&-\mathrm{i}G_t+h_{xx}G+2h_xG_x+hG_{xx}+2GR+\nu\left[G_{xxxx}+8EGG_{xx}\right.\\&\left.+2G^2c_{xx}+4G_xE_xG+6EG_x^2+6G^3E^2\right]=0,\end{aligned} \tag{69}$$

$$R_x-2h_xEG-hE_xG-hEG_x=0, \tag{70}$$

where

$$R = hEG + \int_{-\infty}^{x} h_{x'} EG \mathrm{d}x', \tag{71}$$

equation (6) admits the travelling wave solutions as

$$E = u(\xi)\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \quad \iota = u(\xi)\mathrm{e}^{-\mathrm{i}(kx-\omega t)}, \quad R = v(\xi), \tag{72}$$

where,

$$\xi = t - \eta x + \xi_0. \tag{73}$$

Substituting the above transformation we get,

$$\begin{aligned}
&\pm \mathrm{i}u'(\xi) + \omega u(\xi) + h_{xx}u(\xi) + h\eta^2 u''(\xi) \mp \mathrm{i}h2k\eta u'(\xi) \\
&- hk^2 u(\xi) - 2h_x \eta u'(\xi) \pm 2\mathrm{i}h_x k u(\xi) + 2u(\xi)v(\xi) \\
&+ \nu\eta^4 u''''(\xi) \mp 4\mathrm{i}\nu k\eta^3 u'''(\xi) - 6\nu\eta^2 k^2 u''(\xi) \pm 4\mathrm{i}\nu\eta k^3 u'(\xi) \\
&+ \nu k^4 u(\xi) + 10\nu\eta^2 u^2(\xi)u''(\xi) \mp 24\mathrm{i}\nu k\eta u^2(\xi)u'(\xi) \\
&- 12\nu k^2 u^3(\xi) + 10\nu\eta^2 u(\xi)u'^2(\xi) + 6\nu u^5(\xi) = 0, \qquad (74) \\
&-\eta v'(\xi) - 2h_x u^2(\xi) + 2h\eta u(\xi)u'(\xi) = 0, \qquad (75)
\end{aligned}$$

upon separating the real from imaginary parts we gain,

$$\begin{aligned}
&\omega u(\xi) + h_{xx}u(\xi) + h\eta^2 u''(\xi) - hk^2 u(\xi) - 2h_x\eta u'(\xi) \\
&+ 2u(\xi)v(\xi) + \nu\eta^4 u''''(\xi) - 6\nu\eta^2 k^2 u''(\xi) + \nu k^4 u(\xi) \\
&+ 10\nu\eta^2 u^2(\xi)u''(\xi) - 12\nu k^2 u^3(\xi) + 10\nu\eta^2 u(\xi)u'^2(\xi) \\
&+ 6\nu u^5(\xi) = 0, \qquad (76) \\
&u'(\xi) - 2hk\eta u'(\xi) + 2h_x k u(\xi) - 4\nu k\eta^3 u'''(\xi) \\
&+ 4\nu\eta k^3 u'(\xi) - 24\nu k\eta u^2(\xi)u'(\xi) = 0, \qquad (77) \\
&-u'(\xi) + 2hk\eta u'(\xi) - 2h_x k u(\xi) + 4\nu k\eta^3 u'''(\xi) \\
&- 4\nu\eta k^3 u'(\xi) + 24\nu k\lambda u^2(\xi)u'(\xi) = 0, \qquad (78) \\
&-\eta v'(\xi) - 2h_x u^2(\xi) + 2h\eta u(\xi)u'(\xi) = 0. \qquad (79)
\end{aligned}$$

Now let us assume the solutions of $u(\xi)$ and $v(\xi)$ as

$$u(\xi) = a_0 + \sum_{i}^{N} (a_i \phi^i + b_i \phi^{-i}), \tag{80}$$

$$v(\xi) = s_0 + \sum_{j}^{N} (c_j \phi^j + d_j \phi^{-j}), \tag{81}$$

substituting $\phi'(\xi) = b + \phi^2$ and equating the powers of higher order linear term with nonlinear term, we obtain the values of i and j as 1 and 2 respectively. With the aid of Matlab we substitute the value of i and j in Eqs. (80) and (81) and obtain a system of equations as follows:

$$\begin{aligned}
&16\nu\eta^4 a_1\phi^6 b^2 + 24\nu\eta^4 b_1 b^4 + 6\nu b_1^5 + 2\phi^6 a_0 c_1 \\
&+ 60\nu a_0^2\phi^8 a_1^3 + 30\nu\eta^2 b_1^3 b^2 + 6\nu a_0^5\phi^5 + 40\nu\eta^2 a_1^2\phi^6 b_1 b \\
&+ 30\nu\eta^2 a_1\phi^2 b_1^2 b^2 + 20\nu\eta^2 a_0\phi^7 b_1 a_1 + 60\nu\eta^2 a_0\phi^3 b_1^2 b \\
&- 12\nu k^2\eta^2\phi^8 a_1 + 20\nu\eta^2 a_0\phi^3 a_1 b^2 b_1 + 40\nu\eta^2 a_0\phi^5 a_1 b_1 b \\
&+ 40\nu\eta^2 a_1^3\phi^8 b + 50\nu\eta^2 a_0\phi^9 a_1^2 - 12\nu\eta^2 k^2\phi^4 b_1 b \\
&+ 20\nu\eta^2 a_0^2\phi^8 a_1 + 20\nu\eta^2 a_0^2\phi^4 b_1 b + 2h_x\eta\phi^5 b_1 \\
&+ 20\nu\eta^2 a_0^2\phi^2 b^2 b_1 + 10\nu\eta^2 a_1^3\phi^6 b^2 - 36k^2\nu\phi^4 a_0^2 b_1 \\
&- 36k^2\nu\phi^7 a_0 a_1^2 - 36k^2\nu\phi^3 a_0 b_1^2 + 30\nu a_0\phi^9 a_1^4 \\
&+ 10\nu\eta^2 a_0\phi^5 b^2 a_1^2 - 72k^2\nu\phi^5 a_0 a_1 b_1 + 30\nu a_1^4\phi^8 b_1 \\
&+ 10\nu\eta^2 a_0\phi^5 b_1^2 - 36k^2\nu\phi^6 a_1^2 b_1 - 36k^2\nu\phi^4 a_1 b_1^2 \\
&+ 60\nu\eta^2 a_0\phi^7 a_1^2 b + 50\nu\eta^2 a_0\phi b_1^2 b^2 + 20\nu\eta^2 a_0^2\phi^6 b a_1 \\
&+ 2\phi^6 a_1 s_0 + 2\phi^3 a_0 d_2 + 2\phi^7 a_0 c_2 + 2\phi^4 a_0 d_1 + 60\nu a_1^3\phi^6 b_1^2 \\
&+ 60\nu a_1^2\phi^4 b_1^3 + 30\nu a_0\phi b_1^4 + 30\nu a_0^4\phi^6 a_1 + 30\nu a_1\phi^2 b_1^4 \\
&+ \omega\phi^4 b_1 + \omega\phi^6 a_1 + 60\nu a_0^3\phi^7 a_1^2 + 60\nu a_0^3\phi^3 b_1^2 + 2\phi^5 a_1 d_1 \\
&+ 2\phi^7 a_1 c_1 + 40\nu\eta^2 a_1\phi^4 b_1^2 b + 10\nu\eta^2 a_1^2\phi^4 b^2 b_1 \\
&+ 16\nu\eta^4\phi^4 b_1 b^2 - 12\nu k^2\eta^2\phi^6 b a_1 + \omega\phi^5 a_0 + 2h\eta^2\phi^2 b_1 b^2 \\
&+ 30\nu\eta^2 a_1^2\phi^8 b_1 + 10\nu\eta^2 b_1^2\phi^6 a_1 + 2h\eta^2\phi^4 b_1 b \\
&- 36k^2\nu\phi^6 a_0^2 a_1 + 40\nu\eta^2 b_1^3\phi^2 b - 2h_x\eta\phi^5 b a_1 \\
&+ 2h_x\eta\phi^3 b_1 b + 2h\eta^2\phi^6 b a_1 - 12\nu k^2\eta^2\phi^2 b_1 b^2 \\
&+ 40\nu\eta^4 b a_1\phi^8 + 40\nu\eta^4 b_1 b^3\phi^2 + 2\phi^3 b_1 d_1 + 2\phi^6 b_1 c_2 \\
&+ 2\phi^2 b_1 d_2 + 2\phi^5 a_0 s_0 + 30\nu a_0^4\phi^4 b_1 + 60\nu a_0^2\phi^2 b_1^3 + 2\phi^8 a_1 c_2 \\
&+ 2\phi^4 a_1 d_2 + 2\phi^4 b_1 s_0 + 2\phi^5 b_1 c_1 + h_{xx}\phi^4 b_1 + h_{xx}\phi^6 a_1 \\
&+ h_{xx}\phi^5 a_0 + 6\nu a_1^5\phi^{10} + \nu k^4\phi^6 a_1 + \nu k^4\phi^5 a_0 + 24\nu\eta^4\phi^{10} a_1 \\
&+ \nu k^4\phi^4 b_1 + 180\nu a_0^2\phi^6 a_1^2 b_1 + 120\nu a_0\phi^3 a_1 b_1^3 \\
&+ 180\nu a_0\phi^5 a_1^2 b_1^2 - 2h_x\eta\phi^7 a_1 - hk^2\phi^4 b_1 + 30\nu\eta^2 a_1^3\phi^{10} \\
&- 12k^2\nu\phi^2 b_1^3 - 12k^2\nu\phi^8 a_1^3 - 12k^2\nu\phi^5 a_0^3 + 120\nu a_0\phi^7 a_1^3 b_1 \\
&+ 10\nu\eta^2 b_1^3\phi^4 + 180\nu a_0^2\phi^4 a_1 b_1^2 + 120\nu a_0^3\phi^5 a_1 b_1 - hk^2\phi^5 a_0 \\
&+ 2h\eta^2\phi^8 a_1 - hk^2\phi^6 a_1 = 0, \qquad (82) \\
&24\nu k\eta^3 b_1 b^3 + 24\nu k\eta b_1^3 b + 4\nu\eta k^3\phi^6 a_1 - 24\nu k\eta a_0^2\phi^4 b a_1 \\
&- 4\nu\eta k^3\phi^4 b_1 + 2hk\eta\phi^2 b b_1 + 2hk\eta\phi^4 b_1 - 2hk\eta\phi^6 a_1 \\
&- 32\nu k\eta^3 b a_1\phi^6 - 2hk\eta\phi^4 b a_1 - 8\nu k\eta^3 a_1\phi^4 b^2 \\
&+ 32\nu k\eta^3 b_1 b^2\phi^2 + 8\nu k\eta^3\phi^4 b_1 b - 24\nu k\eta^3\phi^8 a_1 \\
&+ 4\nu\eta k^3\phi^4 b a_1 - 4\nu\eta k^3\phi^2 b b_1 - 24\nu k\eta a_1^3\phi^6 b \\
&- 24\nu k\eta a_1^2\phi^6 b_1 + 24\nu k\eta a_1\phi^4 b_1^2 + 24\nu k\eta a_1\phi^2 b_1^2 b \\
&+ 24\nu k\eta\phi^2 b_1^3 - 24\nu k\eta a_1^3\phi^8 + 48\nu k\eta a_0\phi b_1^2 b \\
&+ 48\nu k\eta a_0\phi^3 b_1^2 - 24\nu k\eta a_1^2\phi^4 b b_1 - 48\nu k\eta a_0\phi^5 a_1^2 b \\
&- 48\nu k\eta a_0\phi^7 a_1^2 + 24\nu k\eta a_0^2\phi^4 b_1 + 2h_x k\phi^5 a_1 \\
&+ 24\nu k\eta a_0^2\phi^2 b b_1 - 24\nu k\eta a_0^2\phi^6 a_1 - b_1 b\phi^2 + a_1\phi^4 b \\
&+ a_1\phi^6 - b_1\phi^4 + 2h_x k\phi^4 a_0 + 2h_x k\phi^3 b_1 = 0, \qquad (83) \\
&\eta b c_1\phi^3 + \eta b d_1\phi - 2\eta b c_2\phi^4 + 2\eta b d_2 - \eta\phi^5 c_1 + \eta\phi^3 d_1 \\
&- 2\eta\phi^6 c_2 + 2\eta\phi^2 d_2 - 2h_x\phi^3 a_0^2 - 4h_x\phi^4 a_0 a_1 - 4h_x\phi^2 a_0 b_1 \\
&- 2h_x\phi^5 a_1^2 - 4h_x\phi^3 a_1 b_1 - 2h_x\phi b_1^2 + 2h\eta a_0\phi^3 a_1 b \\
&- 2h\eta a_0\phi b_1 b + 2h\eta\phi^5 a_0 a_1 - 2h\eta a_0\phi^3 b_1 + 2h\eta a_1^2\phi^4 b \\
&+ 2h\eta a_1^2\phi^6 - 2h\eta b_1^2 b - 2h\eta b_1^2\phi^2 = 0. \qquad (84)
\end{aligned}$$

Collecting the coefficients of each power of ϕ,

$$\begin{aligned}
&\phi^{10} : 24\nu\eta^4 a_1 + 6\nu a_1^5 + 30\nu\eta^2 a_1^3 = 0, \\
&\phi^9 : 50\nu\eta^2 a_0 a_1^2 + 30\nu a_0 a_1^4 = 0, \\
&\phi^8 : 30\nu\eta^2 a_1^2 b_1 + 2h\eta^2 a_1 + 2a_1 c_2 + 60\nu a_0^2 a_1^3 + 20\nu\eta^2 a_0^2 a_1 \\
&\quad + 40\nu\eta^2 a_1^3 b + 30\nu a_1^4 b_1 - 12\nu k^2\eta^2 a_1 + 40\nu\eta^4 b a_1 \\
&\quad - 12k^2\nu a_1^3 = 0, \\
&\phi^7 : 2a_1 c_1 + 20\nu\eta^2 a_0 b_1 a_1 + 2a_0 c_2 + 60\nu\eta^2 a_0 a_1^2 b
\end{aligned}$$

$$
\begin{aligned}
&-36k^2\nu a_0a_1^2-2h_x\eta a_1+60\nu a_0^3a_1^2+120\nu a_0a_1^3b_1=0,\\
\phi^6:\ &20\nu\eta^2a_0^2ba_1+2a_0c_1-12\nu k^2\eta^2ba_1+\nu k^4a_1\\
&+10\nu\eta^2b_1^2a_1+40\nu\eta^2a_1^2b_1b+2a_1s_0+2h\eta^2ba_1\\
&-hk^2a_1+h_{xx}a_1+30\nu a_0^4a_1+180\nu a_0^2a_1^2b_1+10\nu\eta^2a_1^3b^2\\
&-36k^2\nu a_1^2b_1+16\nu\eta^4a_1b^2+60\nu a_1^3b_1^2+2b_1c_2\\
&+\omega a_1-36k^2\nu a_0^2a_1=0,\\
\phi^5:\ &h_{xx}a_0+\nu k^4a_0+2b_1c_1+120\nu a_0^3a_1b_1+10\nu\eta^2a_0b^2a_1^2\\
&+10\nu\eta^2a_0b_1^2+2h_x\eta b_1+2a_0s_0+\omega a_0-hk^2a_0\\
&+180\nu a_0a_1^2b_1^2+6\nu a_0^5-72k^2\nu a_0a_1b_1+2a_1d_1\\
&-12k^2\nu a_0^3-2h_x\eta ba_1+40\nu\eta^2a_0a_1b_1b=0,\\
\phi^4:\ &h_{xx}b_1-12\nu k^2\eta^2b_1b+10\nu\eta^2a_1^2b^2b_1+2a_0d_1\\
&+60\nu a_1^2b_1^3+10\nu\eta^2b_1^3+2a_1d_2+\omega b_1\\
&+20\nu\eta^2a_0^2b_1b+2h\eta^2b_1b+40\nu\eta^2a_1b_1^2b-36\nu k^2a_1b_1^2\\
&+30\nu a_0^4b_1-hk^2b_1+180\nu a_0^2a_1b_1^2+16\nu\eta^4b_1b^2\\
&-36k^2\nu a_0^2b_1+2b_1s_0+\nu k^4b_1=0,\\
\phi^3:\ &2a_0d_2+120\nu a_0a_1b_1^3+2h_x\eta b_1b+60\nu\eta^2a_0b_1^2b\\
&-36\nu k^2a_0b_1^2+20\nu\eta^2a_0a_1b^2b_1+2b_1d_1+60\nu a_0^3b_1^2\\
&=0,\\
\phi^2:\ &60\nu a_0^2b_1^3+20\nu\eta^2a_0^2b^2b_1+30\nu a_1b_1^4+2h\eta^2b^2b_1\\
&-12\nu k^2\eta^2b_1b^2+40\nu\eta^4b_1b^3+30\nu\eta^2a_1b_1^2b^2\\
&-12k^2\nu b_1^3+40\nu\eta^2b_1^3b+2b_1d_2=0,\\
\phi^1:\ &50\nu\eta^2a_0b_1^2b^2+30\nu a_0b_1^4=0,\\
\phi^0:\ &6\nu b_1^5+24\nu k\eta^4b_1b^4+30\nu\eta^2b_1^3b^2=0;
\end{aligned}
\tag{85}
$$

$$
\begin{aligned}
\phi^8:\ &-24\nu k\eta a_1^3-24\nu k\eta^3a_1=0,\\
\phi^7:\ &-48\nu k\eta a_0a_1^2=0,\\
\phi^6:\ &-24\nu k\eta a_1^2b_1-2hk\eta a_1+4\nu\eta k^3a_1-24\nu k\eta a_1^3b\\
&-32\nu k\eta^3ba_1+a_1-24\nu k\eta a_0^2a_1=0,\\
\phi^5:\ &-48\nu k\eta a_0a_1^2b+2h_xka_1=0,\\
\phi^4:\ &-24\nu k\eta a_0^2ba_1-4\nu\eta k^3b_1+2hk\eta b_1-24\nu k\eta a_1^2bb_1\\
&+24\nu k\eta a_1b_1^2+a_1b-b_1+24\nu k\eta a_0^2b_1+4\nu\eta k^3ba_1\\
&+8\nu k\eta^3b_1b-2hk\eta ba_1+2h_xka_0-8\nu k\eta^3a_1b^2=0,\\
\phi^3:\ &2h_xkb_1+48\nu k\eta a_0b_1^2=0,\\
\phi^2:\ &24\nu k\eta a_1b_1^2b-4\nu\eta k^3bb_1+2hk\eta bb_1-b_1b\\
&+32\nu k\eta^3b_1b^2+24\nu k\eta a_0^2bb_1+24\nu k\eta b_1^3=0,\\
\phi^1:\ &48\nu k\eta a_0b_1^2b=0,\\
\phi^0:\ &24\nu k\eta b_1^3b+24\nu k\eta^3b_1b^3=0;
\end{aligned}
\tag{86}
$$

and

$$
\begin{aligned}
\phi^6:\ &-2\eta c_2+2h\eta a_1^2=0,\\
\phi^5:\ &-\eta c_1-2h_xa_1^2+2h\eta a_0a_1=0,\\
\phi^4:\ &2h\eta ba_1^2-2\eta bc_2-4h_xa_0a_1=0,\\
\phi^3:\ &\eta d_1+2h\eta a_0ba_1-2h_xa_0^2-4h_xa_1b_1-2h\lambda a_0b_1\\
&-\eta bc_1=0,\\
\phi^2:\ &-2h\eta b_1^2-4h_xa_0b_1+2\eta d_2=0,\\
\phi^1:\ &-2h\eta a_0bb_1-2h_xb_1^2+\eta bd_1=0,\\
\phi^0:\ &-2h\eta bb_1^2+2\eta bd_2=0.
\end{aligned}
\tag{87}
$$

Further using Matlab we solve the resulting system of algebraic equations and obtain the following set of distinct solutions of parameters.

Case 1

$$
\begin{aligned}
&a_1=\frac{24\nu\eta a_0^3}{h_x},\ b_1=0,\ c_1=\frac{-48\nu a_0^4\eta(24\nu a_0^2-h)}{h_x},\\
&c_2=\frac{-576\eta^2a_0^6\nu^2(-h+48\nu a_0^2)}{h_x^2},\\
&d_1=0,\ d_2=0,\ b=\frac{h_x^2}{576a_0^4\nu^2\eta^2},\\
&E=\left\{a_0+\frac{24\nu\eta a_0^3}{h_x}\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}\right.\right.\\
&\quad\left.\left.\times\tan\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]\right\}\exp(\mathrm{i}(kx-\omega t)),\\
&G=\left\{a_0+\frac{24\nu\eta a_0^3}{h_x}\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}\right.\right.\\
&\quad\left.\left.\times\tan\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]\right\}\exp(-\mathrm{i}(kx-\omega t)),\\
&R=\left\{s_0+\frac{-48\nu a_0^4\eta(24\nu a_0^2-h)}{h_x}\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}\right.\right.\\
&\quad\left.\times\tan\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]\\
&\quad+\frac{-576\eta^2a_0^6\nu^2(-h+48\nu a_0^2)}{h_x^2}\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}\right.\\
&\quad\left.\left.\times\tan\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^2\right\}.
\end{aligned}
\tag{88}
$$

Case 2

$$
\begin{aligned}
&a_1=0,\ b_1=\frac{-h_x}{24\nu\eta a_0},\ c_1=0,\ c_2=0,\\
&d_1=\frac{h_x(24\nu a_0^2-h)}{12\nu\eta},\ d_2=\frac{-h_x^2(-h+48\nu a_0^2)}{576\nu^2\eta^2a_0^2},\\
&b=\frac{h_x^2}{576a_0^4\nu^2\eta^2},\\
&E=\left\{a_0+\frac{-h_x}{24\nu\eta a_0}\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}\right.\right.\\
&\quad\left.\left.\times\tan\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}\right\}\\
&\quad\times\exp(\mathrm{i}(kx-\omega t)),\\
&G=\left\{a_0+\frac{-h_x}{24\nu\eta a_0}\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}\right.\right.\\
&\quad\left.\left.\times\tan\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}\right\}
\end{aligned}
$$

$$\times \exp(-\mathrm{i}(kx-\omega t)),$$

$$R=\left\{s_0+\frac{h_x(24\nu a_0^2-h)}{12\nu\eta}\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}+\frac{-h_x^2(-h+48\nu a_0^2)}{576\nu^2\eta^2a_0^2}\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{576a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-2}\right\}. \quad (89)$$

Case 3

$$a_1=\frac{6\nu\eta a_0^3}{h_x},\ b_1=\frac{-h_x}{24\nu\eta a_0},\ c_1=\frac{-12\nu a_0^4\eta(6\nu a_0^2-h)}{h_x},$$

$$c_2=\frac{-36\eta^2a_0^6\nu^2(-h+48\nu a_0^2)}{h_x^2},\ d_1=\frac{h_x(6\nu a_0^2-h)}{12\nu\eta},$$

$$d_2=\frac{-h_x^2(-h+48\nu a_0^2)}{576\nu^2\eta^2a_0^2},\ b=\frac{h_x^2}{144a_0^4\nu^2\eta^2},$$

$$E=\left\{a_0+\frac{6\nu\eta a_0^3}{h_x}\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]+\frac{-h_x}{24\nu\eta a_0}\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}\right\}\times\exp(\mathrm{i}(kx-\omega t)),$$

$$G=\left\{a_0+\frac{6\nu\eta a_0^3}{h_x}\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]+\frac{-h_x}{24\nu\eta a_0}\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}\right\}\exp(-\mathrm{i}(kx-\omega t)),$$

$$R=\left\{s_0+\frac{-12\nu a_0^4\eta(6\nu a_0^2-h)}{h_x}\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}\times\tanh\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]+\frac{-36\eta^2a_0^6\nu^2(-h+48\nu a_0^2)}{h_x^2}\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{2}+\frac{h_x(6\nu a_0^2-h)}{12\nu\eta}\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}+\frac{-h_x^2(-h+48\nu a_0^2)}{576\nu^2\eta^2a_0^2}\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}\times\tan\left[\sqrt{\frac{h_x^2}{144a_0^4\nu^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-2}. \quad (90)$$

The propagation of solitons in the presence of inhomogeneities concerns a wide variety of physical systems. A soliton in an integrable system moves with constant velocity and shape. In realistic situations, however, because of the inhomogeneity of the media, solitons may exhibit more complex motion with changing velocity and shape which may be used as a desirable effect for fast transport, fast communication, or even for a possible soliton gun.[71] If the system is inhomogeneous, the coefficients of equations may be expected to become dependent on the function describing the inhomogeneity and on its derivatives. From the above solution, the $h(x)$ arises from the site dependence or inhomogeneity of the coupling between the spins. When $h=$ constant the system yields the homogeneous Heisenberg spin chain. Now we elucidate a series of exact propagating soliton solutions for the following types of inhomogeneities such as (i) quadratic inhomogeneity, (ii) inverse inhomogeneity, (iii) localized inhomogeneity in the form of tangent hyperbolic function, and (iv) exponential function which introduces a novel type of solitons.

a) Linear inhomogeneity

We have plotted for this case $h(x)=Cx+D$, which corresponds to the solution E, as presented in the first case, as shown in Figs. 1(a)–1(c), respectively. It is observed from the plots that, the shape of the soliton is stable and robust when $C=0.81$ (a), $C=0.01$ (b), and $C=0.833$ (c). A further increase in the value of C, makes the soliton distorted with number of unstable modes developed subsequently as depicted in Fig. 1.

b) Quadratic inhomogeneity

For the case of quadratic inhomogeneity $h(x)=Bx^2+Cx+D$, it leads an intriguing change in the soliton shape as clearly seen in Fig. 2, we have plotted the solutions E as represented in Eq. (88), for quadratic inhomogeneity, and the solution assumes the hyperbolic solitary excitations as portrayed in Fig. 2, for various values of B, C, and D.

c) Inverse inhomogeneity

We have plotted Eq. (88) for the case of inverse inhomogeneity $h(x)=C/x$ in Fig. 3, for the parameters (a) $C=0.001$, (b) $C=0.001$, and (c) $C=1.41$. Surprisingly, the solution assumes the form of parabolic solitons as shown in Fig. 3.

d) Localized inhomogeneity

If the inhomogeneity is assumed to be localized in the form of a hyperbolic function, $h(x) = A\tanh x$, the solution propagates in the form of typical parabolic solitons as shown in Fig. 4. It is known that in real magnetic materials the appearance of local variation of magnetic parameters can be related to various types of structural and chemical inhomogeneities, and also to various local actions (mechanical, thermal or light). The presence of such inhomogeneities (or defects) can lead to the formation of different kinds of magnetic inhomogeneities, which influence the process of the magnetization reversal of a sample.[72,73] Recently Ekomasov *et al.*, have considered a magnetic inhomogeneity of soliton or kink type which is highly localized in nature. The ranges of the parameters which determine the possibility of existence of each of the obtained magnetic inhomogeneity have also been determined, which has potential application in constructing multilayered ferromagnetic structures with a periodic and spatially inhomogeneous anisotropy for data storage devices. Figure 4 depicts the soliton excitations and the evolution occurs only in the middle region, keeping the soliton tail intact.

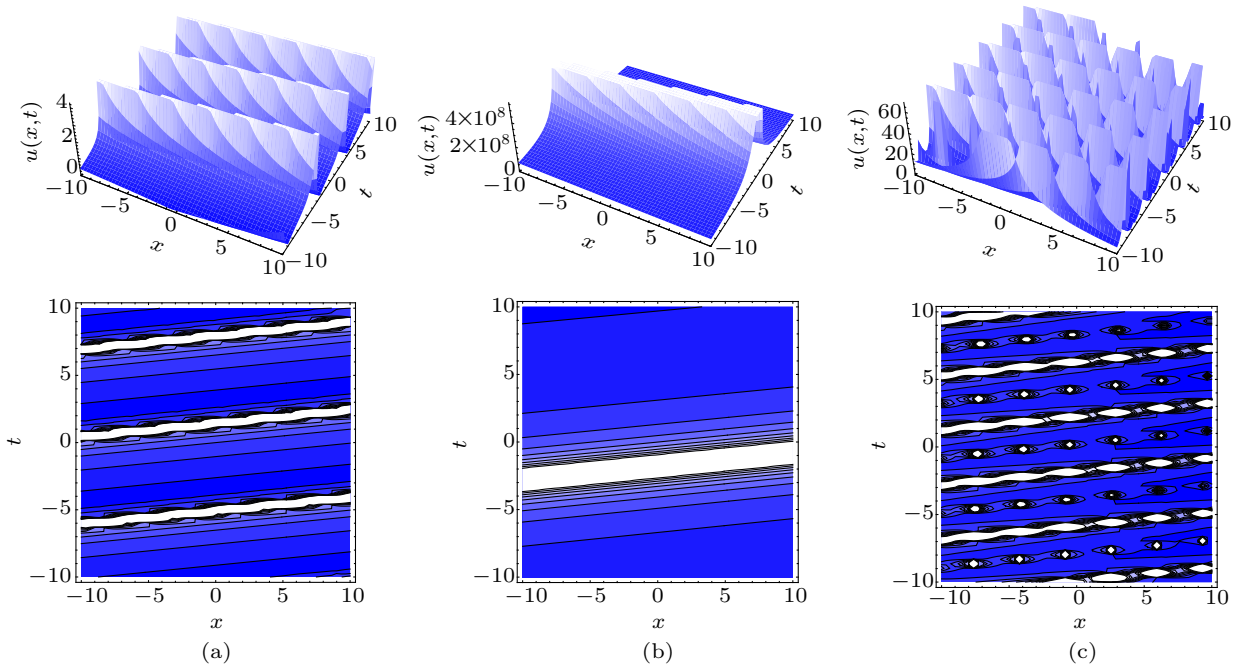


Fig. 1. (color online) Snapshots of a soliton for linear inhomogeneity (a) $C = 0.81$, $D = 0$, (b) $C = 0.01$, $D = 0.7$, and (c) $C = 0.833$, $D = 0.19$.

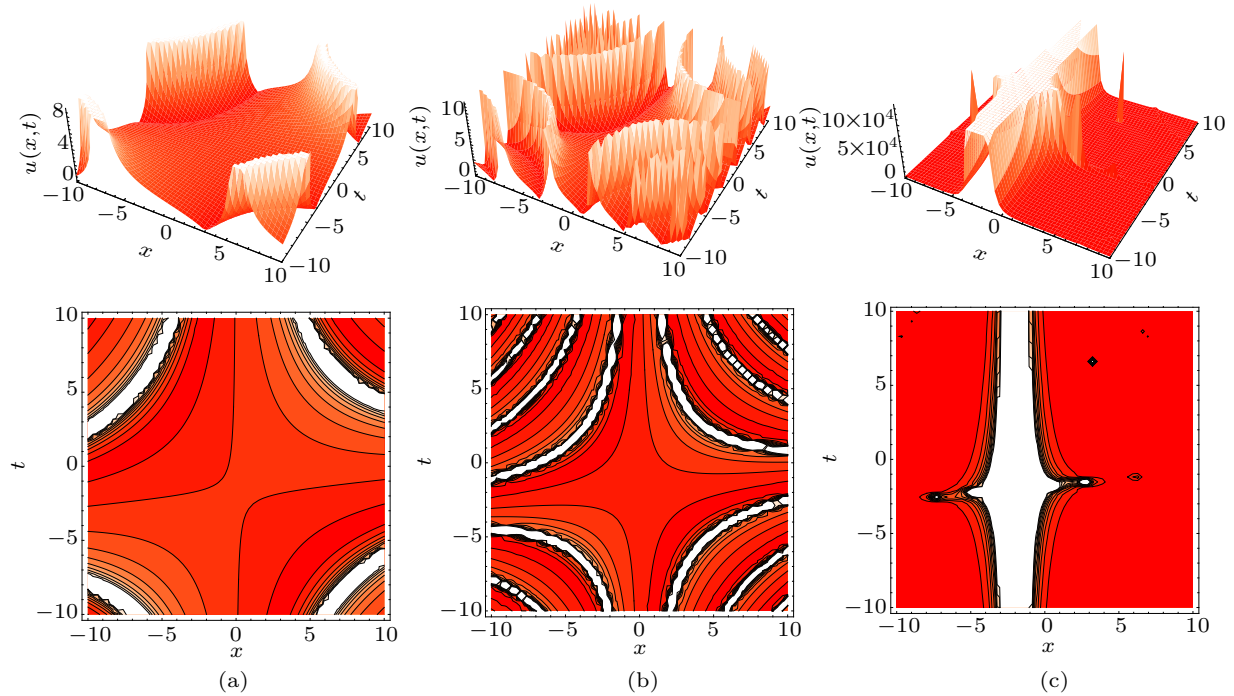


Fig. 2. (color online) Snapshots of a soliton for nonlinear inhomogeneity (a) $B = 0.23$, $C = 0$, $D = 0$, (b) $B = 0.7$, $C = 0$, $D = 0$, and (c) $B = 0.02$, $C = 0.08$, $D = 0.06$.

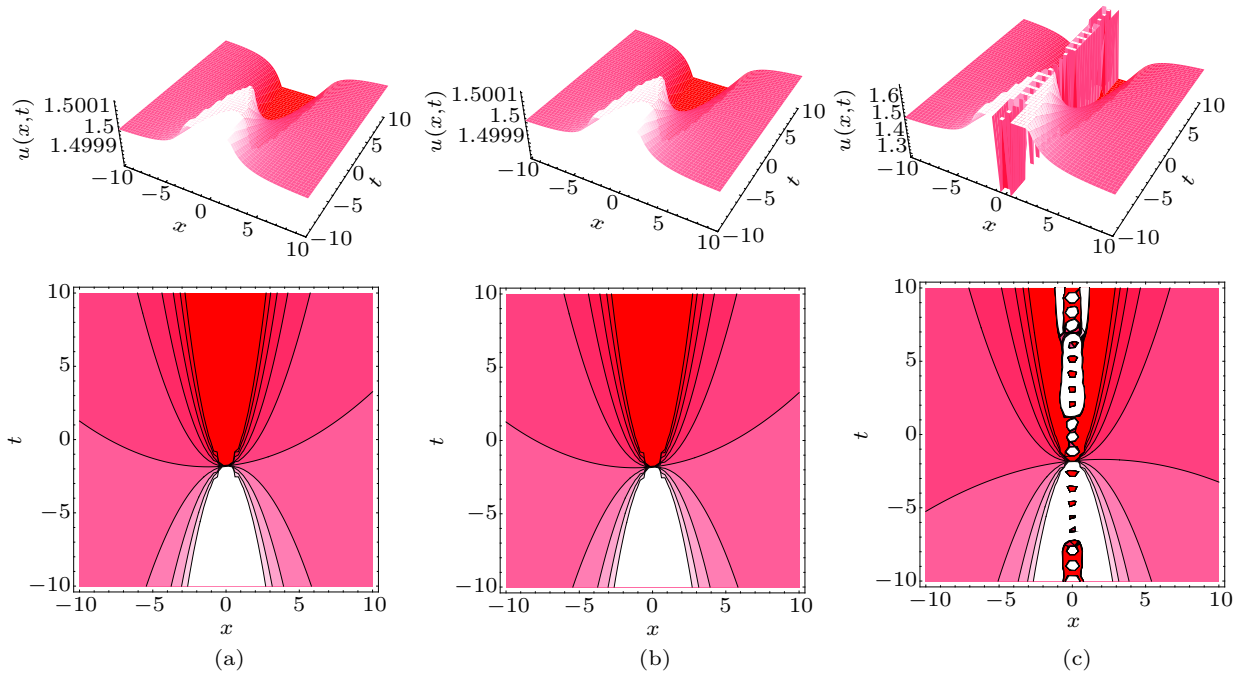


Fig. 3. (color online) Snapshots of a soliton for inverse inhomogeneity (a) $C = 0.001$, (b) $C = 0.001$, and (c) $C = 1.41$.

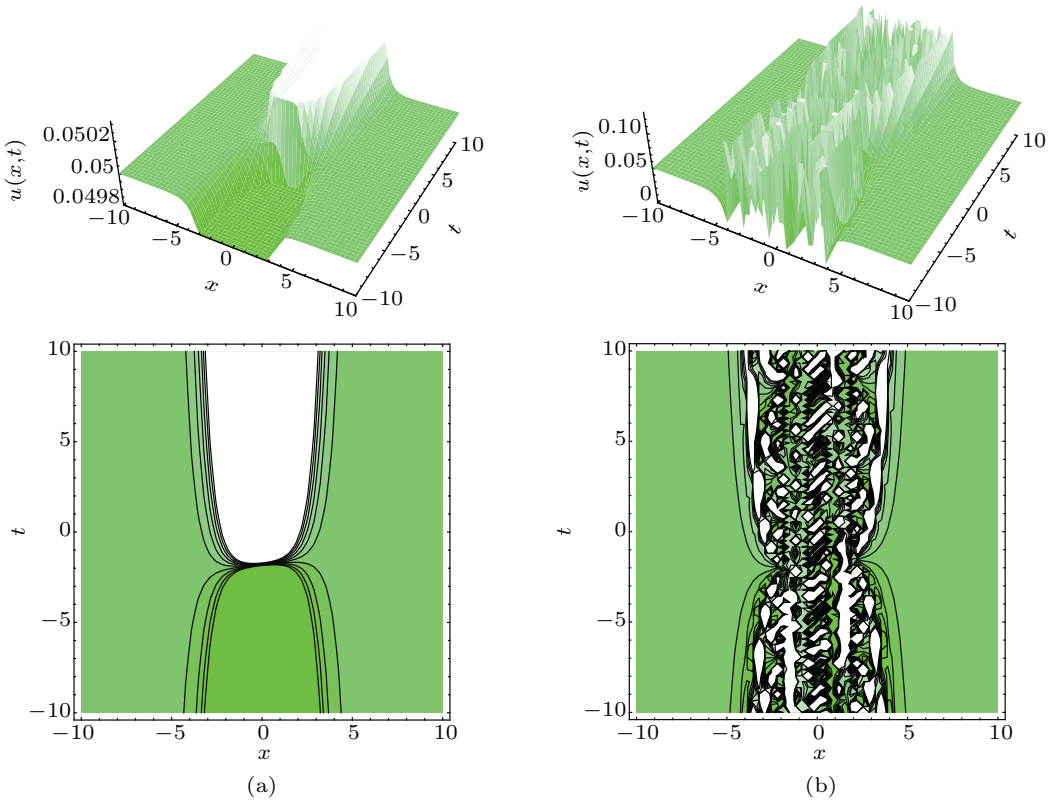


Fig. 4. (color online) Snapshots of a soliton for localized inhomogeneity (a) $C = 0.001$ and (b) $C = 0.89$.

e) Exponential inhomogeneity

To illustrate this type of exponential inhomogeneity, Villarroel *et al.*[74] have studied the propagation of optical beams under the interplay of dispersion and Kerr nonlinearity modelled by a perturbed optical NLS fibers with impurities distributed randomly on the fiber. More, especially with the assumption that the distance between the impurities or inhomogeneities that are exponentially distributed, which eventually resulted in the disappearance of the solitons into radiation and may be relevant for long-distance fiber-optic communication systems. Figure 5 represents the periodic soliton for exponential inhomogeneity in the zero damped case. An increase of inhomogeneity will lead to the periodic soliton being annihilated. Furthermore the annihilated soliton is created again and the figure shows the recreation of the soliton nature and correspondingly, the contour plots are also given.

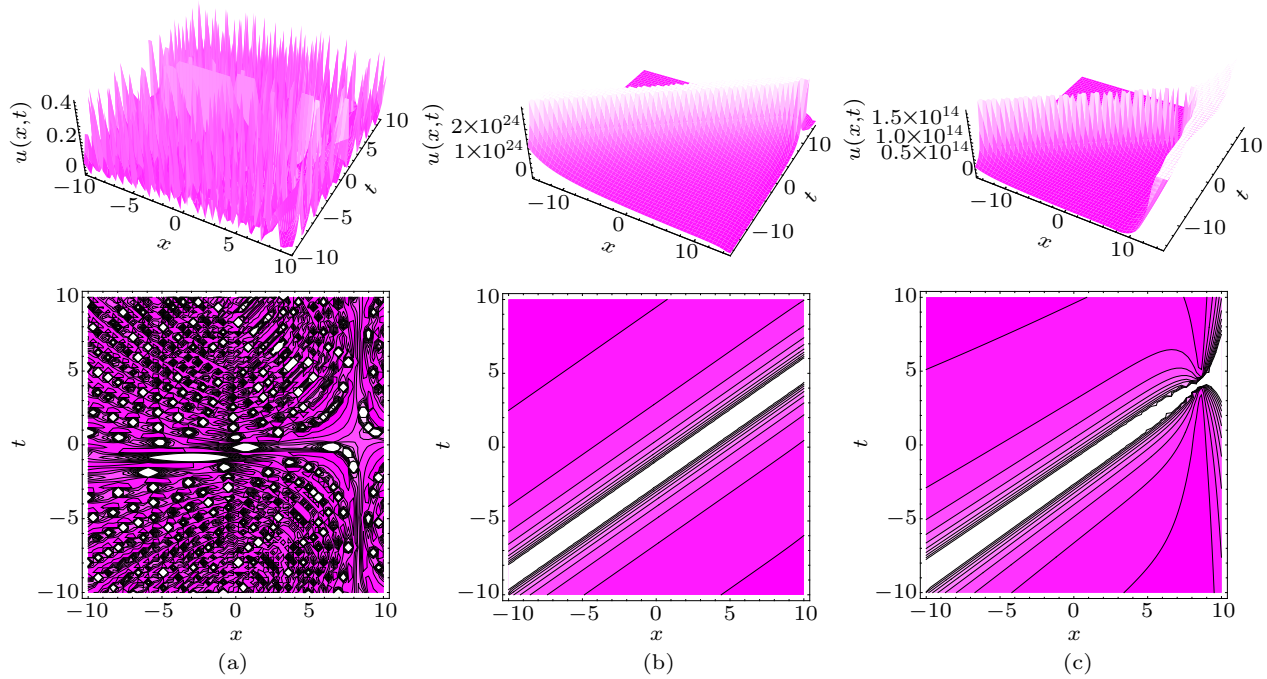


Fig. 5. (color online) Snapshots of a soliton for exponential inhomogeneity (a) $C = 0.08$, (b) $C = 0.00001$ and (c) $C = 0.005$.

6. Damped inhomogeneous dynamics

In this section, we implement the tangent hyperbolic function method in the presence of inhomogeneity as well as damping. To see the effect of nonlinear inhomogeneities and damping on soliton dynamics, we consider the following inhomogeneous damped NLS equation

$$\begin{aligned}&\mathrm{i}q_t+(1-i\lambda)[h_{xx}q+2h_xq_x+hq_{xx}]+2q\Big[hqq^*+\int_{-\infty}^{x}h'_x\mathrm{d}x'qq^*\\&+\mathrm{i}\lambda h\int_{-\infty}^{x}(qq^*_x-q^*q_x)\mathrm{d}x\Big]+\nu\left[q_{xxxx}+8|q|^2q_{xx}+2q^2q^*_{xx}\right.\\&\left.+4|q_x|^2q+6q^*q_x^2+6q|q|^4\right]=0.\end{aligned}\tag{91}$$

In a similar way, we take the complex conjugate and separate the real from imaginary parts, then we get

$$\begin{aligned}&\omega u(\xi)+h_{xx}u(\xi)-2h_x\eta u'(\xi)+h\eta^2u''(\xi)-hk^2u(\xi)+\\&2\lambda kh_xu(\xi)-2\lambda hk\eta u'(\xi)+2u(\xi)v(\xi)+\nu\eta^4u''''(\xi)\\&-6\nu\eta^2k^2u''(\xi)+\nu k^4u(\xi)+10\nu u(\xi)^2\eta^2u''(\xi)\\&-12\nu u(\xi)^3k^2+10\nu\eta^2u(\xi)u'(\xi)^2+6\nu u(\xi)^5=0,\end{aligned}\tag{92}$$

$$\begin{aligned}&u'(\xi)+2h_xku(\xi)-2hk\eta u'(\xi)-\lambda u(\xi)h_{xx}+2\lambda\eta h_xu'(\xi)\\&-\lambda h\eta^2u''(\xi)+k^2\lambda hu(\xi)-4\nu k\eta^3u'''(\xi)+4\nu k^3\eta u'(\xi)\\&-24\nu k\eta u(\xi)^2u'(\xi)=0,\end{aligned}\tag{93}$$

$$\begin{aligned}&-u'(\xi)-2h_xku(\xi)+2hk\eta u'(\xi)+\lambda u(\xi)h_{xx}-2\lambda\eta h_xu'(\xi)\\&+\lambda h\eta^2u''(\xi)-k^2\lambda hu(\xi)+4\nu k\eta^3u'''(\xi)-4\nu k^3\eta u'(\xi)\\&+24\nu k\eta u(\xi)^2u'(\xi)=0,\end{aligned}\tag{94}$$

and

$$-\lambda v'(\xi)-2h_xu^2(\xi)+2h\eta u(\xi)u'(\xi)-2hk\lambda u^2(\xi)=0.\tag{95}$$

The tanh method admits the use of a finite expansion of tanh function. Assume the solutions for $u(\xi)$ and $v(\xi)$ as

$$u(\xi)=a_0+\sum_i^N(a_i\phi^i+b_i\phi^{-i}),\tag{96}$$

$$v(\xi)=s_0+\sum_j^N(c_j\phi^j+d_j\phi^{-j}),\tag{97}$$

where $\phi=\phi(\xi)$, $\phi'=\mathrm{d}\phi/\mathrm{d}\xi$, and $\phi'=b+\phi^2$. On balancing the higher order linear term with nonlinear term we obtain the values of i and j as 1 and 2 respectively. Again with the aid of Matlab by substituting the values of i and j into Eqs. (96) and (97), we obtain the following system of equations,

$$\begin{aligned}&2\phi^5b_1c_1+60\nu a_1^3\phi^6b_1^2+30\nu a_1^4\phi^8b_1+10\nu\eta^2b_1^3\phi^4\\&-12\nu k^2\phi^2b_1^3-12\nu k^2\phi^8a_1^3-12\nu k^2\phi^5a_0^3+30\nu\eta^2a_1^3\phi^{10}\\&+\nu k^4\phi^4b_1+\nu k^4\phi^6a_1+\nu k^4\phi^5a_0+24\nu\eta^4\phi^{10}a_1+2h_x\eta\phi^5b_1\\&-2h_x\eta\phi^7a_1-hk^2\phi^4b_1+60\nu a_0^2\phi^2b_1^3+60\nu a_0^2\phi^8a_1^3\\&+60\nu a_0^3\phi^3b_1^2+60\nu a_0^3\phi^7a_1^2+30\nu a_0^4\phi^4b_1+h_{xx}\phi^6a_1\\&+h_{xx}\phi^4b_1+2\phi^3b_1d_1+30\nu a_0^4\phi^6a_1+30\nu a_0\phi b_1^4+30\nu a_0\phi^9a_1^4\\&+60\nu a_1^2\phi^4b_1^3+2\phi^6a_1s_0+2\phi^3a_0d_2+2\phi^7a_0c_2+2\phi^4a_0d_1\\&+2\phi^6a_0c_1+2\phi^5a_0s_0+\omega\phi^4b_1+6\nu a_0^5\phi^5+2\phi^4b_1s_0\\&+2\phi^4a_1d_2+2\phi^8a_1c_2+2\phi^5a_1d_1+2\phi^7a_1c_1\\&+2\phi^6b_1c_2+2\phi^2b_1d_2+6\nu a_1^5\phi^{10}+\omega\phi^5a_0+\omega\phi^6a_1\\&+h_{xx}\phi^5a_0-hk^2\phi^6a_1+30\nu a_1\phi^2b_1^4+2h\eta^2\phi^8a_1-hk^2\phi^5a_0\\&+2\lambda kh_x\phi^5a_0+180\nu a_0^2\phi^4a_1b_1^2-12\nu\eta^2k^2\phi^2b_1b^2\\&-12\nu\eta^2k^2\phi^4b_1b+16\nu\eta^4\phi^4b_1b^2-12\nu\eta^2k^2\phi^8a_1\\&+40\nu\eta^4b_1b^3\phi^2-12\nu\eta^2k^2\phi^6ba_1+16\nu\eta^4a_1\phi^6b^2\\&+40\nu\eta^4ba_1\phi^8+2\lambda hk\eta\phi^5b_1+2\lambda hk\eta\phi^3b_1b-2\lambda hk\eta\phi^7a_1\\&+2\lambda kh_x\phi^4b_1-2\lambda hk\eta\phi^5ba_1+2h_x\eta\phi^3b_1b+2\lambda kh_x\phi^6a_1\\&+2h\eta^2\phi^2b_!b^2-2h_x\eta\phi^5ba_1+2h\eta^2\phi^6ba_1+120\nu a_0\phi^3a_1b_1^3\\&+20\nu\eta^2a_0\phi^7b_1a_1+50\nu\eta^2a_0\phi b_1^2b^2+40\nu\eta^2a_0\phi^5a_1b_1b\\&+50\nu\eta^2a_0\phi^9a_1^2+60\nu\eta^2a_0\phi^7a_1^2b+20\nu\eta^2a_0\phi^3a_1b_1b^2\\&+20\nu\eta^2a_0^2\phi^8a_1+20\nu\eta^2a_0^2\phi^4b_1b+20\nu\eta^2a_0^2\phi^2b^2b_1\\&+20\nu\eta^2a_0^2\phi^6ba_1+180\nu a_0^2\phi^6a_1^2b_1+120\nu a_0^3\phi^5a_1b_1\\&+120\nu a_0^3\phi^5a_1b_1+120\nu a_0\phi^7a_1^3b_1+180\nu a_0\phi^5a_1^2b_1^2\end{aligned}$$

$$+10\nu\eta^2a_0\phi^5b^2a_1^2+10\nu\eta^2a_0\phi^5b_1^2+10\nu\eta^2a_1^3\phi^6b^2$$
$$-36\nu k^2\phi^4a_1b_1^2-36\nu k^2\phi^6a_1^2b_1-36\nu k^2\phi^3a_0b_1^2$$
$$-72\nu k^2\phi^5a_0a_1b_1-36\nu k^2\phi^7a_1^2a_0-36\nu k^2\phi^4a_0^2b_1$$
$$-36\nu k^2\phi^6a_0^2a_1+40\nu\eta^2\phi^2b_1^3b+40\nu\eta^2\phi^4a_1b_1^2b$$
$$+10\nu\eta^2b_1^2\phi^6a_1+30\nu\eta^2a_1\phi^2b_1^2b^2+30\nu\eta^2a_1^2\phi^8b_1$$
$$+10\nu\eta^2a_1^2\phi^4b^2b_1+40\nu\eta^2a_1^2\phi^6b_1b+60\nu\eta^2a_0\phi^3b_1^2b$$
$$+40\nu\eta^2a_1^3\phi^8b+2h\eta^2\phi^4b_1b+24\nu\eta^4b_1b^4+30\nu\eta^2b_1^3b^2$$
$$+6\nu b_1^5=0, \tag{98}$$
$$-2hk\eta\phi^6a_1+2h_xk\phi^3b_1+2h_xk\phi^5a_1+2h_xk\phi^4a_0$$
$$-\lambda h_{xx}\phi^3b_1+a_1\phi^4b-b_1b\phi^2+24\nu k\eta b_1^3b+24\nu k\eta^3b_1b^3$$
$$-b_1\phi^4-\lambda h_{xx}\phi^4a_0-\lambda h_{xx}\phi^5a_1+2hk\eta\phi^2b_1b$$
$$-24\nu k\eta a_0^2\phi^6a_1+a_1\phi^6-48\nu k\eta a_0\phi^7a_1^2+48\nu k\eta a_0\phi^3b_1^2$$
$$-48\nu k\eta a_0\phi^5a_1^2b-24\nu k\eta^3\phi^8a_1-24\nu k\eta\phi^4a_0^2ba_1$$
$$+4\nu\eta k^3\phi^6a_1-4\nu\eta k^3\phi^4b_1-4\nu\eta k^3\phi^2b_1b+4\nu\eta k^3\phi^4ba_1$$
$$+8\nu\eta^3k\phi^4b_1b-32\nu k\eta^3ba_1\phi^6+32\nu k\eta^3b_1b^2\phi^2$$
$$+24\nu k\eta a_1\phi^4b_1^2-8\nu k\eta^3a_1\phi^4b^2+24\nu k\eta a_1\phi^2b_1^2b$$
$$+24\nu k\eta\phi^2b_1^3-24\nu k\eta a_1^2\phi^4b_1b-24\nu k\eta a_1^2\phi^6b_1$$
$$-24\nu k\eta a_1^3\phi^6b-24\nu k\eta a_1^3\phi^8+48\nu k\eta a_0\phi b_1^2b+2\eta\lambda h_x\phi^6a_1$$
$$-2\eta\lambda h_x\phi^2b_1b+2hk\eta\phi^4b_1+2\lambda\eta h_x\phi^4ba_1-2hk\eta\phi^4ba_1$$
$$+24\nu k\eta\phi^2a_0^2b_1b-2\lambda\eta^2h\phi b_1b^2-2\lambda h\eta^2\phi^3b_1b$$
$$-2\lambda h\eta^2\phi^5a_1b-2\lambda\eta h_x\phi^4b_1+k^2\lambda h\phi^3b_1+k^2\lambda h\phi^4a_0$$
$$+k^2\lambda h\phi^5a_1+24\nu k\eta a_0^2\phi^4b_1-2\lambda h\eta^2\phi^7a_1=0, \tag{99}$$
$$-\eta bc_1\phi^3+\eta bd_1\phi-2\eta bc_2\phi^4+2\eta bd_2-\eta\phi^5c_1+\eta\phi^3d_1$$
$$-2\eta\phi^6c_2+2\eta\phi^2d_{+2}-2h_x\phi^3a_0^2-4h_x\phi^4a_0a_1-4h_x\phi^2a_0b_1$$
$$-2h_x\phi^5a_1^2-4h_x\phi^3a_1b_1-2h_x\phi b_1^2+2h\eta a_0\phi^3a_1b$$
$$-2h\eta a_0\phi b_1b+2h\eta a_0\phi^5a_1-2h\eta a_0\phi^3b_1+2h\eta a_1^2\phi^4b$$
$$+2h\eta a_1^2\phi^6-2h\eta b_1^2b-2h\eta b_1^2\phi^2-2hk\phi^5\lambda a_1^2$$
$$-4hk\lambda\phi^4a_0a_1-2hk\lambda\phi b_1^2-4hk\lambda\phi^3a_1b_1-4hk\lambda\phi^2a_0b_1$$
$$-2hk\lambda a_0^2\phi^3=0. \tag{100}$$

Upon collecting the coefficients of each power of ϕ

$$\phi^{10}:30\nu\eta^2a_1^3+24\nu\eta^4a_1+6\nu a_1^5=0,$$
$$\phi^9:30\nu a_0a_1^4+50\nu\eta^2a_0a_1^2=0,$$
$$\phi^8:2h\eta^2a_1-12\nu\eta^2k^2a_1+30\nu a_1^4b_1+20\nu\eta^2a_0^2a_1$$
$$+60\nu a_0^2a_1^3+30\nu\eta^2a_1^2b_1+40\nu\eta^2a_1^3b-12k^2\nu a_1^3$$
$$+2a_1c_2+40\nu\eta^4ba_1=0,$$
$$\phi^7:20\nu\eta^2a_0b_1a_1+60\nu a_0^3a_1^2+2a_0c_2+60\nu\eta^2a_0a_1^2b$$
$$-2h\lambda\eta ka_1+120\nu a_0a_1^3b_1-36k^2\nu a_0a_1^2+2a_1c_1$$
$$-2h_x\eta a_1=0,$$
$$\phi^6:h_{xx}a_1-hk^2a_1+2a_1s_0+10\nu\eta^2b_1^2a_1+\nu k^4a_1$$
$$-36k^2\nu a_0^2a_1+16\nu\eta^4a_1b^2+2a_0c_1+2h_xk\lambda a_1$$
$$+2h\lambda^2ba_1+180a_0^2a_1^2b_1+60\nu a_1^3b_1^2+40\nu\eta^2a_1^2b_1b$$
$$-36k^2\nu a_1^2b_1+2b_1c_2+30\nu a_0^4a_1+10\nu\eta^2a_1^3b^2+\omega a_1$$
$$-12\nu k^2\eta^2ba_1+20\nu\eta^2a_0^2ba_1=0,$$
$$\phi^5:2h\lambda\eta kb_1+h_{xx}a_0+120\nu a_0^3a_1b_1+180\nu a_0a_1^2b_1^2+2b_1c_1$$
$$-2h\lambda\eta kba_1-2h_x\eta ba_1-72k^2\nu a_0a_1b_1+2a_0s_0$$
$$+2h_x\eta b_1-12k^2\nu a_0^3+6\nu a_0^5-hk^2a_0+40\nu\eta^2a_0a_1b_1b$$
$$+10\nu\eta^2a_0b^2a_1^2+\nu k^4a_0+2h_xk\lambda a_0+2a_1d_1$$
$$+10\nu\eta^2a_0b_1^2+\omega a_0=0,$$
$$\phi^4:h_{xx}b_1+2a_0d_1+2a_1d_2+60\nu a_1^2b_1^3+20\nu\eta^2a_0^2b_1b$$
$$+2h\eta^2bb_1+10\nu\eta^2a_1^2b_1b^2+180\nu a_0^2a_1b_1^2+2h_xk\lambda b_1$$
$$-36k^2\nu a_1b_1^2+\omega b_1-hk^2b_1+10\nu\eta^2b_1^3+30\nu a_0^4b_1$$
$$+40\nu\eta^2a_1b_1^2b-12\nu k^2\eta^2b_1b+16\nu\eta^4b_1b^2+\nu k^4b_1$$
$$-36k^2\nu a_0^2b_1+2b_1s_0=0,$$
$$\phi^3:2b_1d_1+2a_0d_2+120\nu a_0a_1b_1^3+2h_x\eta b_1b-36k^2\nu a_0b_1^2$$
$$+2h\lambda\eta kbb_1+60\nu a_0^3b_1^2+20\nu\eta^2a_0a_1b^2b_1$$
$$+60\nu\eta^2a_0b_1^2b=0,$$
$$\phi^2:-12k^2\nu b_1^3+40\nu\eta^2b_1^3b+2h\eta^2b^2b_1+60\nu a_0^2b_1^3$$
$$+40\nu\eta^4b_1b^3+2b_1d_2+30\nu\eta^2a_1b_1^2b^2-12\nu k^2\eta^2b_1b^2$$
$$+30\nu a_1b_1^4+20\nu\eta^2a_0^2b^2b_1=0,$$
$$\phi^1:30\nu a_0b_1^4+50\nu\eta^2a_0b_1^2b^2=0,$$
$$\phi^0:30\nu\eta^2b_1^3b^2+6\nu b_1^5+24\nu\eta^4b_1b^4=0, \tag{101}$$

and

$$\phi^8:-24\nu k\eta^3a_1-24\nu k\eta a_1^3=0,$$
$$\phi^7:-2h\lambda\eta^2a_1-48\nu k\eta a_0a_1^2=0,$$
$$\phi^6:4\nu\eta k^3a_1-24\nu k\eta a_1^3b-2hk\eta a_1-24\nu k\eta a_0^2a_1$$
$$-32\nu k\eta^3a_1b+2h_x\lambda\eta a_1-24\nu k\eta a_1^2b_1+a_1=0,$$
$$\phi^5:h\lambda k^2a_1-\lambda h_{xx}a_1-2h\lambda\eta^2ba_1-48\nu k\eta a_0a_1^2b$$
$$+2h_xka_1=0,$$
$$\phi^4:-2h_x\lambda\eta b_1-24\nu k\eta a_0^2ba_1-24\nu k\eta a_0^2b_1+h\lambda k^2a_0$$
$$+2h_x\lambda\eta ba_1-24\nu k\eta a_1^2bb_1-8\nu k\eta^3a_1b^2+2h_xka_0$$
$$+8\nu k\eta^3b_1b+2hk\eta b_1+24\nu k\eta a_1b_1^2+a_1b+4\nu\eta k^3ba_1$$
$$-b_1-4\nu\eta k^3b_1-2hk\eta ba_1-\lambda h_{xx}a_0=0,$$
$$\phi^3:h\lambda k^2b_1-2h\eta^2\lambda b_1b+2h_xkb_1+48\nu k\eta a_0b_1^2$$
$$-\lambda h_{xx}b_1=0,$$
$$\phi^2:-b_1b-2h_x\lambda\eta b_1b+2hk\eta bb_1+24\nu k\eta b_1^3$$
$$+24\nu k\eta a_1b_1^2b+24\nu k\eta a_0^2bb_1+32\nu k\eta^3b_1b^2$$
$$-4\nu\eta k^3b_1b=0,$$
$$\phi^1:-2h\lambda\eta^2b^2b_1+48\nu k\eta a_0b_1^2b=0,$$
$$\phi^0:24\nu k\eta b_1^3b+24\nu k\eta^3b_1b^3=0. \tag{102}$$

By using the symbolic computation and solving the resulting system of algebraic equations, we obtain the following sets of traveling wave solutions as follows.

Case a

$$a_1=0,\ b_1=\frac{24a_0^3k\nu}{h\lambda\eta},\ c_1=0,\ c_2=0,$$
$$d_1=\frac{2a_0^2(h_x\lambda+h\lambda^2k+24k\nu a_0^2)}{\lambda\eta},$$
$$d_2=\frac{48a_0^4k\nu(12k\nu a_0^2+h\lambda^2k+h_x\lambda)}{h\lambda^2\eta^2},$$
$$\text{and}\ b=\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}.$$

The travelling wave solutions are given by

$$E=\left\{a_0+\frac{24a_0^3k\nu}{h\lambda\eta}\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\right.\right.$$
$$\left.\left.\times\tan\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}\right\}$$

$$\times \exp(\mathrm{i}(kx-\omega t)), \tag{103}$$

$$G=\left\{a_0+\frac{24a_0^3k\nu}{h\lambda\eta}\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}\right\}\times\exp(-\mathrm{i}(kx-\omega t)), \tag{104}$$

$$R=\left\{s_0+\frac{2a_0^2(h_x\lambda+h\lambda^2k+24k\nu a_0^2)}{\lambda\eta}\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-1}+\frac{48a_0^4k\nu(12k\nu a_0^2+h\lambda^2k+h_x\lambda)}{h\lambda^2\eta^2}\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^{-2}\right\}. \tag{105}$$

Case b

$$a_1=\frac{-1}{24}\frac{h\lambda\eta}{\nu ka_0},\ b_1=0,$$

$$c_1=\frac{-1}{288}\frac{\lambda\eta h^2(h_x\lambda+h\lambda^2k+24\nu ka_0^2)}{\nu^2k^2a_0^2},$$

$$c_2=\frac{1}{6912}\frac{h^3\lambda^2\eta^2(12k\nu a_0^2+h\lambda^2k+h_x\lambda)}{\nu^3k^3a_0^4},$$

$$d_1=0,\ d_2=0,\ b=\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}.$$

We have

$$E=\left\{a_0+\frac{-1}{24}\frac{h\lambda\eta}{\nu ka_0}\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right\}\exp(\mathrm{i}(kx-\omega t)), \tag{106}$$

$$G=\left\{a_0+\frac{-1}{24}\frac{h\lambda\eta}{\nu ka_0}\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right\}\exp(-\mathrm{i}(kx-\omega t)), \tag{107}$$

$$R=\left\{s_0+\frac{-1}{288}\frac{\lambda\eta h^2(h_x\lambda+h\lambda^2k+24\nu ka_0^2)}{\nu^2k^2a_0^2}\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]\right]+\frac{1}{6912}\frac{h^3\lambda^2\eta^2(12k\nu a_0^2+h\lambda^2k+h_x\lambda)}{\nu^3k^3a_0^4}\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{576\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]\right]^2\right\}. \tag{108}$$

Case c

$$a_1=\frac{-1}{24}\frac{h\lambda\eta}{\nu ka_0},\ b_1=\frac{6a_0^3k\nu}{h\lambda\eta},$$

$$c_1=\frac{-1}{288}\frac{\lambda\eta h^2(h_x\lambda+h\lambda^2k+24\nu ka_0^2)}{\nu^2k^2a_0^2},$$

$$c_2=\frac{1}{1728}\frac{h^3\lambda^2\eta^2(3k\nu a_0^2+h\lambda^2k+h_x\lambda)}{a_0^4\nu^3k^3},$$

$$d_1=\frac{a_0^2(h_x\lambda+h\lambda^2k+24k\nu a_0^2)}{2\lambda\eta},$$

$$d_2=\frac{12a_0^4k\nu(3k\nu a_0^2+h\lambda^2k+h_x\lambda)}{h\lambda^2\eta^2},\ b=\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}.$$

We gain

$$E=\left\{a_0+\frac{-1}{24}\frac{h\lambda\eta}{\nu ka_0}\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]+\frac{6a_0^3k\nu}{h\lambda\eta}\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]^{-1}\right\}\times\exp(\mathrm{i}(kx-\omega t)), \tag{109}$$

$$G=\left\{a_0+\frac{-1}{24}\frac{h\lambda\eta}{\nu ka_0}\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]+\frac{6a_0^3k\nu}{h\lambda\eta}\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]^{-1}\right\}\times\exp(-\mathrm{i}(kx-\omega t)), \tag{110}$$

$$R=\left\{s_0+\frac{-1}{288}\frac{\lambda\eta h^2(h_x\lambda+h\lambda^2k+24\nu ka_0^2)}{\nu^2k^2a_0^2}\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^4}}(t-\eta x+\xi_0)\right]+\frac{1}{1728}\frac{h^3\lambda^2\eta^2(3k\nu a_0^2+h\lambda^2k+h_x\lambda)}{a_0^4\nu^3k^3}\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\left[\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^4}}(t-\eta x+\xi_0)\right]^2+\frac{a_0^2(h_x\lambda+h\lambda^2k+24k\nu a_0^2)}{2\lambda\eta}\left[\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]^{-1}+\frac{12a_0^4k\nu(3k\nu a_0^2+h\lambda^2k+h_x\lambda)}{h\lambda^2\eta^2}\left[\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}\times\tan\sqrt{\frac{144\nu^2k^2a_0^4}{h^2\lambda^2\eta^2}}(t-\eta x+\xi_0)\right]^{-2}\right\}. \tag{111}$$

In this section, we try to demonstrate the shape changing soliton under the influence of both the effects of inhomogeneity and damping. In the presence of inhomogeneity and the damping Eqs, (103)–(105) lead to the evolution of soliton in the periodic nature which depends upon the parameters A, B, and C. When the inhomogeneity takes the form of a linear function in the presence of damping, it leads to the following conclusions. We assume the function that takes the form of a linear inhomogeneity function i.e., $h(x) = Ax + B$. Figure 6 displays the evolution of stable soliton pattern under the influence of linear inhomogeneity and damping for the first case of solution. Upon increasing the value of parameter B, the system is found to support the creation and annihilation of a soliton depending on the strength of the inhomogeneity involved. Next, we try to analyze the effect of nonlinear inhomogeneity that takes the form of $h(x) = Bx^2 + Cx + D$, again the solution exhibits a peculiar property when the value of inhomogeneity is increased as seen in Fig. 7. When the value of C is increased a stable periodic solitary pattern found to develop from an annihilated structure. For the case of inverse inhomogeneity, when the strength of the inhomogeneity is increased, surprisingly, a hyperbolic solitary pattern is found to evolve from an annihilated pattern as depicted in Fig. 8. More interestingly, when the exponential inhomogeneity is introduced and upon the increase of the strength of the exponential inhomogeneity, the stable solitary pattern transforms into a stable kink soliton, as depicted in Fig. 9.

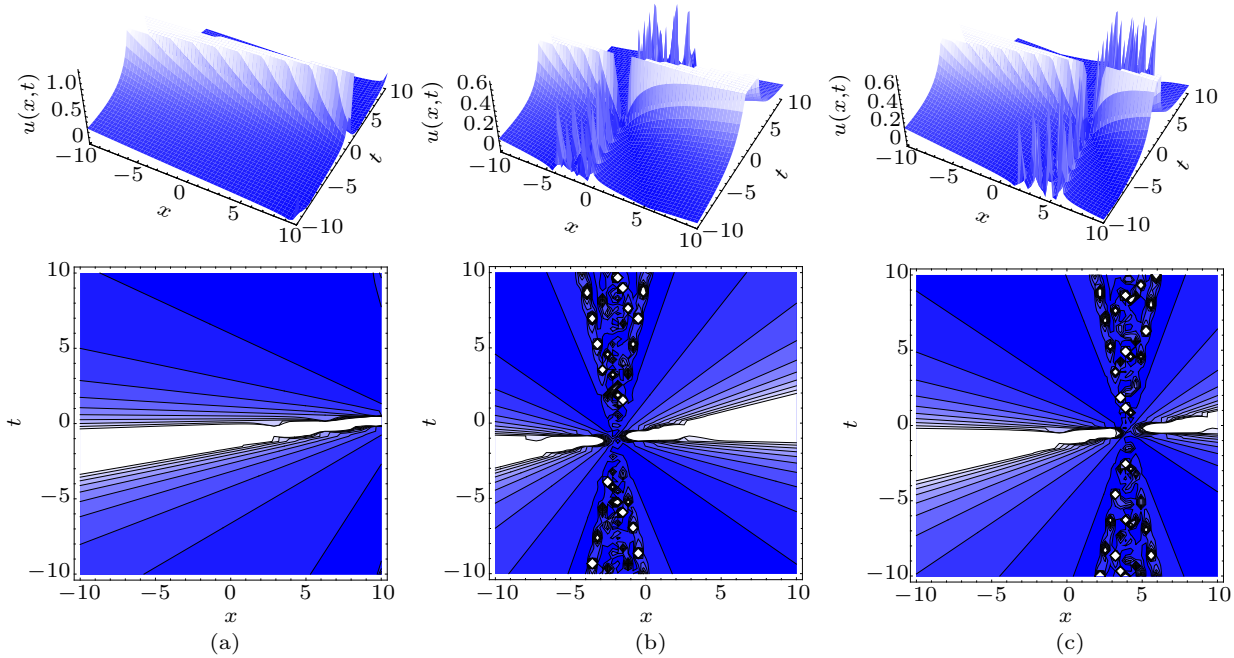


Fig. 6. (color online) Soliton evolution for the case of linear inhomogeneity with damping, λ=1, (a) $A = 0.1$, $B = -1.2$, (b) $A = 0.1$, $B = -0.2$, and (c) $A = 0.1$, $B = 0.2$.

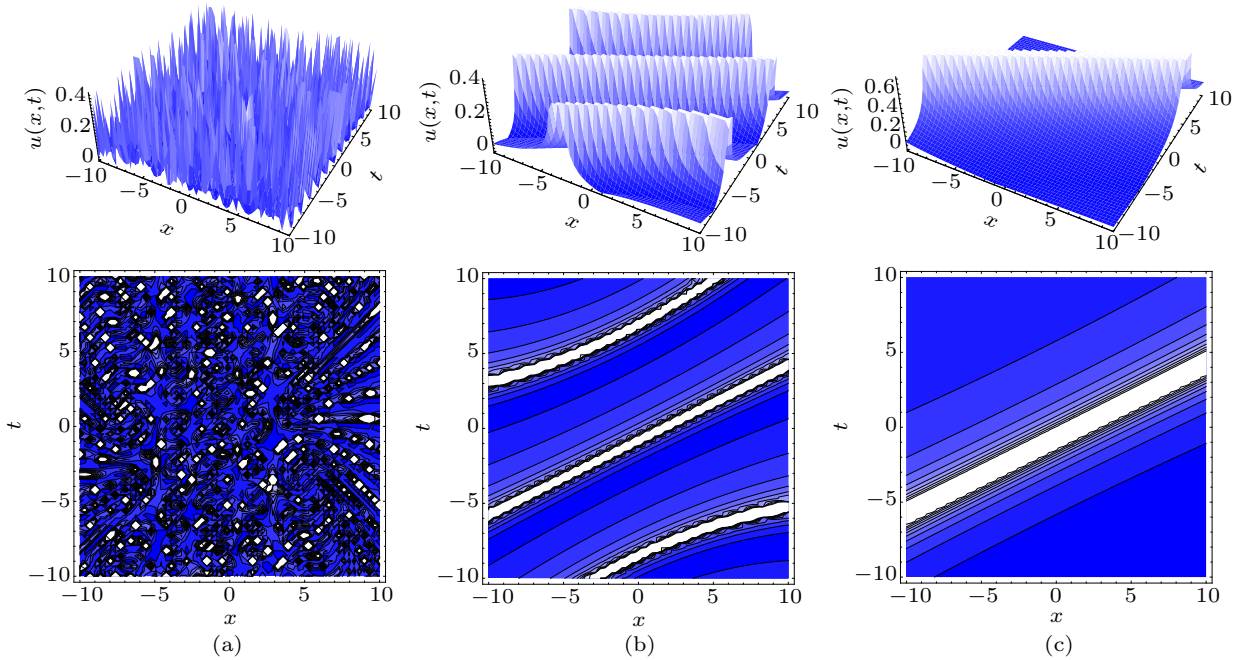


Fig. 7. (color online) Soliton evolution for the case of nonlinear inhomogeneity with damping $\lambda = 0.5$, (a) $A = 0.001$, $B = 0.002$, $C = 0.01$, (b) $A = 0.001$, $B = 0.002$, $C = 0.53$, and (c) $A = 0.001$, $B = 0.002$, $C = 2.4$.

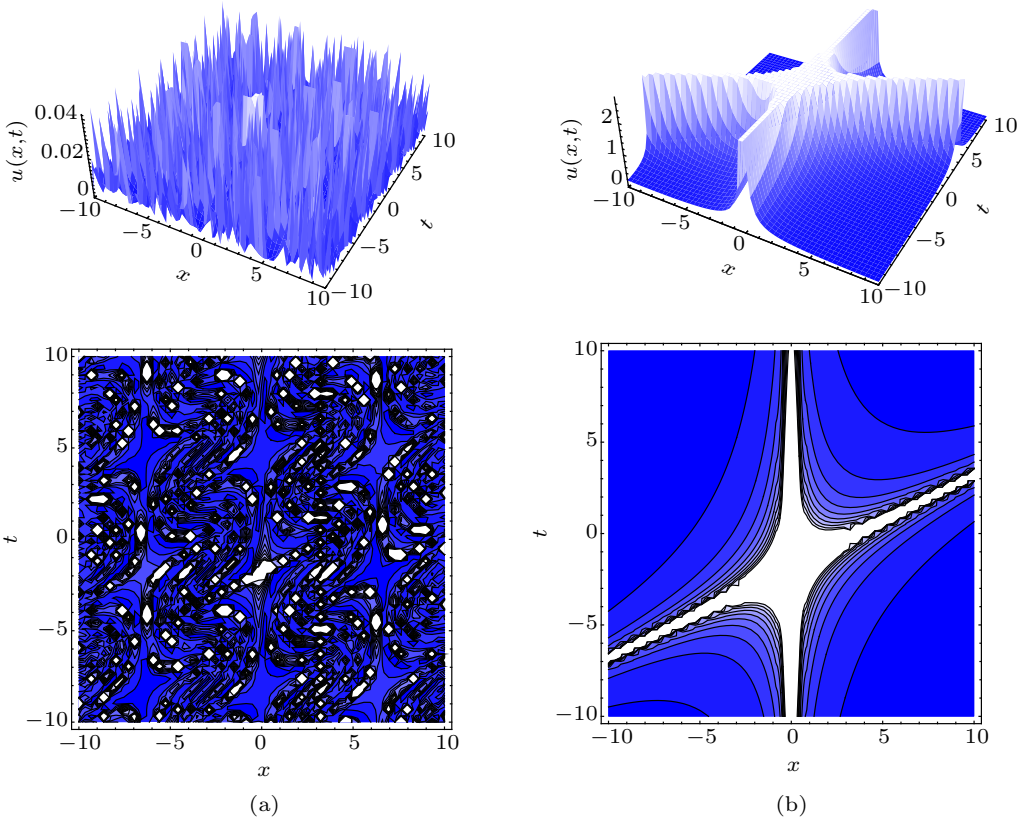


Fig. 8. (color online) Soliton evolution for the case of inverse inhomogeneity with damping $\lambda = 1$, (a) $A = 0.001$, and (b) $A = 1.45$.

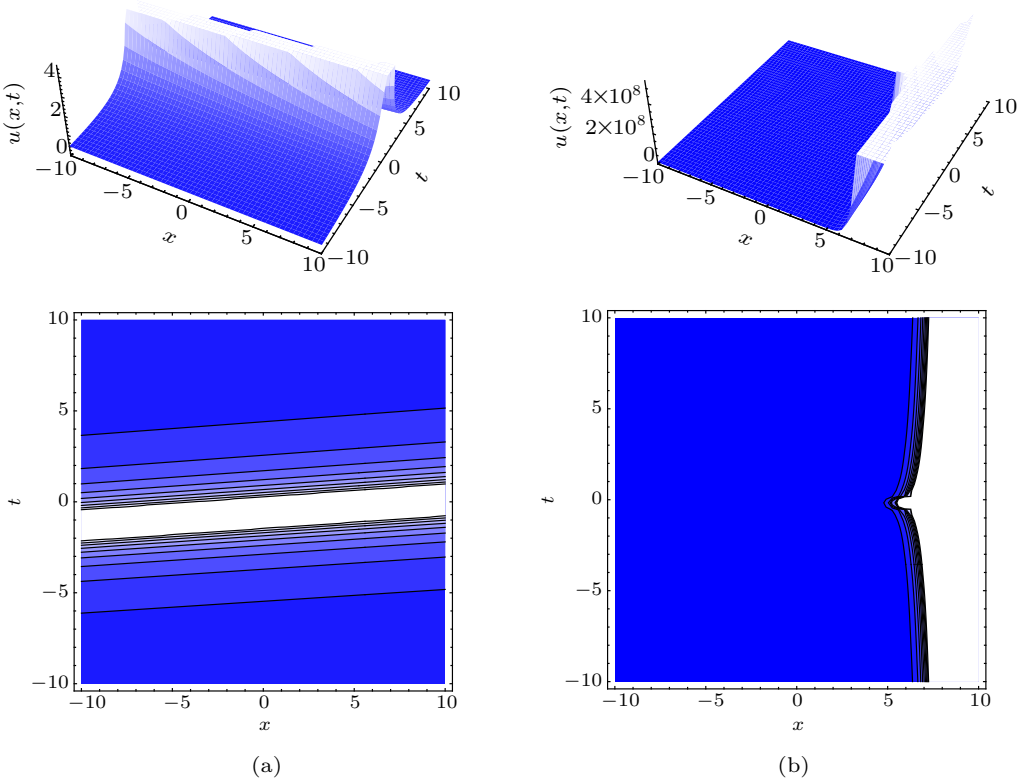


Fig. 9. (color online) Soliton evolution for the case of exponential inhomogeneity with damping $\lambda = 0.5$, (a) $A = 0.001$, and (b) $A = 2.9$.

7. Conclusion

In summary, we have investigated the nonlinear excitations in an inhomogeneous ferromagnetic media along with the higher order biquadratic exchange interaction and damping. The associated dynamics of the Heisenberg spin chain is governed by the LL equation which is highly nontriv-

ial to solve and it is not amenable to easy analysis. It is shown using space curve formalism that the relevant nonlinear evolution equation is described by the generalized nonlinear Schrödinger equation. In order to elucidate the integrable property of the equation, we employed *P*-analysis and found that the equation possesses integrability. Further, we have constructed the exact solitary wave solutions for the INLS equation by invoking modified extended tanh function method aided with symbolic computation. Moreover, attention has been paid to understand the effect of various inhomogeneities and damping on the nature of soliton solutions and we elaborately discussed the classes of coherent structures that evolved under different inhomogeneities.

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