Barut–Girardello and Gilmore–Perelomov coherent states for pseudoharmonic oscillators and their nonclassical properties: Factorization method

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In this paper we try to introduce the ladder operators associated with the pseudoharmonic oscillator, after solving the corresponding Schrödinger equation by using the factorization method. The obtained generalized raising and lowering operators naturally lead us to the Dirac representation space of the system which is much easier to work with, in comparison to the functional Hilbert space. The $SU(1,1)$ dynamical symmetry group associated with the considered system is exactly established through investigating the fact that the deduced operators satisfy appropriate commutation relations. This result enables us to construct two important and distinct classes of Barut–Girardello and Gilmore–Perelomov coherent states associated with the system. Finally, their identities as the most important task are exactly resolved and some of their nonclassical properties are illustrated, numerically.

Keywords: pseudoharmonic oscillator, factorization method, Barut–Girardello coherent states, Gilmore–Perelomov coherent states, nonclassical properties

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1. Introduction

Molecular vibrational and rotational spectroscopies are usually known as important parts of molecular physics. The harmonic oscillator could be considered as a useful basis on which to explain molecular vibrations, but this model is very ideal and simple and therefore has some particular restrictions. Strictly speaking, molecular vibrations are really anharmonic, so if their Schrödinger equation can be exactly solved, they may provide more suitable models for the description of vibrating molecules. As some examples of anharmonic oscillator molecular potentials we may refer to the Morse,[1] Pöschl–Teller,[2,3] and Mie-type[4] potentials which are three typical model potentials, and the molecular vibrations can be described satisfactorily by means of them. In this work, we want to study another anharmonic potential, the so-called pseudoharmonic oscillator, which is the sum of the harmonic oscillator and the inversely quadratic potential,[5] i.e.,

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{\hbar^2}{2m}\frac{\alpha^2}{x^2}, \tag{1}$$

where m, ω, and α respectively represent the mass of the particle, the frequency, and the strength of the external field. Sometimes this system has been called isotonic potential.[6–9] In fact, this potential was proposed by Goldman and Krivchenkov.[10] Some methods that have been used to solve the wave equation for the pseudoharmonic potential can be summarized as follows: Nikiforov–Uvarov method,[11,12] polynomial solution,[13] exact quantization rule,[14] and the Laplace transform approach.[15] Moreover, to obtain the exact solutions of the Schrödinger equation for the pseudoharmonic oscillator using a new method (and in general enlarging the circle of solvable quantum systems) is one of the main problems in quantum physics.[16]

In the present study we firstly want to solve the latter equation by the factorization method which simply leads us to the exact solution of the eigenvalues and eigenfunctions of the considered system. Very recently this method was used for the construction of a new set of generalized coherent states and then the geometric phase of the states was investigated[17] (see also Refs. [18]–[20] and references therein). Our motivation lies in the fact that by using this method we will be able to find out the appropriate generalized creation and annihilation operators for the chosen system, by which we are simply led to the Dirac space with its instruments (ket–bra space) that provides us with a much simpler tool than the usual functional space. Consequently, we can construct two new, important, and distinct classes of generalized coherent states associated with such a physical system, namely Barut–Girardello and Gilmore–Perelomov coherent states. After this exploration, due to the importance of the nonclassicality of quantum states in various theoretical and experimental fields of physics in recent decades,[22,23] we are naturally led to the investigation of the nonclassical signs of our introduced states for completing the work. Moreover, prior to these examinations we will ex-

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actly establish the resolution of the identities of both classes of states, as the most important property of any generalized coherent state.

Notice that our present work can be considered in the line of the dynamical generalization of coherent states.[24–26] Also, it is worth mentioning that the obtained Barut–Girardello and Gilmore–Perelomov coherent states can also be really categorized in the "algebraic"[27–29] and "group theoretical"[30–32] generalization of coherent states, respectively. From the physical point of view, we would like to mention that the strong relation of the laser light to the well-known standard bosonic coherent states corresponding to harmonic oscillators was well established six decades ago by the pioneering works of Glauber, Sudarshan, and Klauder. Indeed, it was only after this discovery that great efforts were made to find their generalizations as well as physical realizations. This is why their various manners of generalizations are still of importance.[33–37] Henceforth, apart from presenting a powerful technique which can be used for other known potentials, due to the direct relation of our introduced coherent states to a particular physical potential (which as we mentioned in the above lines, may be more suitable for the description of real physical phenomena) we hope that our work leads to a physical realization too.

As mentioned before, this potential has received a lot of attention in recent years. In addition, the approach used in this paper is conceptually close to the well-known supersymetry (SUSY) quantum mechanics technique, which is an interesting approach for many authors.[38–44] Altogether, it may clearly be seen that our method to the solution of the eigenvalue problem and the final obtained results are essentially new. In this way, to the best of our knowledge, the approach to the treatment of this particular problem, the corresponding ladder operators, the associated coherent states of the types which are obtained, in addition to establishing the resolution of the identities for both classes of states and the related nonclassicality signs have not appeared in the literature up to now.

The organization of the paper is as follows. In the next section, we are going to review the factorization method briefly. This method is used to solve the Schrödinger equation for the pseudoharmonic oscillator in one-dimensional space in Section 3. According to our obtained results, we construct the creation and annihilation operators, by which we study the dynamical group corresponding to this system in Section 4. Then, after producing the Barut–Girardello and Gilmore–Perelomov coherent states associated with the pseudoharmonic potential, the resolution of the identities corresponding to both classes of introduced states is established in Section 5. We demonstrate some of the nonclassical properties of the states, such as squeezing and sub-Poissonian statistics, numerically in Section 6. Finally, the summary and conclusion are presented in Section 7.

2. The factorization method: A brief review

The factorization method is a useful technique that reduces the dynamical equation of a given system into a simple one which is easier to handle. This method was first introduced by Schrödinger[45–47] and then followed by Dirac[48] to avoid the usage of cumbersome mathematical tools and it has also been successfully applied to resolve some problems for which there are exact solutions.

This method was further developed by Infeld and Hull[49] and Green.[50] The characteristics and advantages of the factorization method can be summarized as follows. Firstly, this method applies only to the discrete energy spectra. Secondly, we can achieve immediately the desired eigenvalues and the normalized eigenfunctions from the given Hamiltonian without any need to use the traditional methods in quantum mechanics. Thirdly, it is possible to avoid deriving the normalization constant, which is sometimes difficult to obtain. The fourth property comes from the fact that we may discover the hidden symmetry of the quantum system through the establishment of an appropriate Lie algebra, which usually can be realized by means of the ladder operators. This method is a very powerful tool for solving second-order differential equations and has attracted the attention of many authors. In this method, which is closely related to SUSY quantum mechanics, the second-order differential operator H_0 (Hamiltonian) is written as the product of two first-order differential operators a_0 and $a_0^\dagger$ plus a constant E_0,

$$H_0 = a_0^\dagger a_0 + E_0. \tag{2}$$

Then, by rewriting the SUSY Hamiltonians H_- and H_+ as H_0 and H_1 respectively and changing suitably the subscripts, we have $H_1 = a_0 a_0^\dagger + E_0$. Also, by factorizing H_1 one has $H_1 = a_1^\dagger a_1 + E_1$, so firstly we have

$$H_1 = a_0 a_0^\dagger + E_0 = a_1^\dagger a_1 + E_1, \tag{3}$$

with $H_2 = a_1 a_1^\dagger + E_1$ as the SUSY partner of H_1. Continuing the above procedure leads to the generation of a hierarchy of Hamiltonians[51,52] given by

$$H_{n+1} = a_{n+1}^\dagger a_{n+1} + E_{n+1} = a_n a_n^\dagger + E_n, \ \ n = 0,1,2,3,\ldots. \tag{4}$$

The complete set of eigenfunctions can be generated by the successive action of these operators on the key eigenfunctions, which are the exact solutions of the first-order differential equation. Then, the recurrence relation may be used to achieve all eigenvalues.

In fact, with the following theorem we can find eigenfunctions and eigenvalues.[53]

Theorem 1 Assume that equations (2)–(4) with real constants E_n are true. Also suppose that there is a null eigenfunction $|\xi_n\rangle$ with a zero eigenvalue for each a_n, i.e.,

$$a_n|\xi_n\rangle = 0. \tag{5}$$

Then,

a) E_n is the n-th eigenvalue of the Hamiltonian.

b) The corresponding eigenfunction of the Hamiltonian H is,

$$|\psi_n\rangle = c_n a_0^\dagger a_1^\dagger \cdots a_{n-1}^\dagger |\xi_n\rangle, \tag{6}$$

where c_n is a normalization constant that may be determined as

$$|c_n| = [(E_n - E_{n-1})(E_n - E_{n-2}) \ldots (E_n - E_0)]^{-1/2}. \tag{7}$$

Three important worthwhile points lie in the theorem.

(i) This method applies only to the discrete energy spectra.

(ii) In the selection of a_n, one should choose it in such a way that E_n gets the maximum value.

(iii) The theorem does not give us any information about the form of a_n. Moreover, evidently the form of a_n depends explicitly on the particular potential $V(x)$. Anyway, since $a_0^\dagger a_0$ is equal to H except for a constant, a_0 must contain a linear momentum term to be consistent with the kinetic energy part of the Hamiltonian. Accordingly, a_n is assumed to have the form

$$a_n = \frac{1}{\sqrt{2m}}(p + \mathrm{i} f_n(x)), \tag{8}$$

where p is the momentum operator and $f_n(x)$ is a real (and differentiable) unknown function of x. This function should guarantee that Eq. (4) is satisfied.

3. Solving the Schrödinger equation of the pseudoharmonic oscillator: Factorization method

Now, we are going to search for the eigenenergies and eigenfunctions of a particular system, i.e., the pseudoharmonic potential in one-dimensional space, by the factorization method. Let us start with the Schrödinger equation for this potential, i.e.,

$$\left(\frac{p^2}{2m} + V(x)\right)\psi(x) = E\psi(x), \tag{9}$$

where the operator p is defined as $(\hbar/\mathrm{i})(\mathrm{d}/\mathrm{d}x)$. According to the form of the pseudoharmonic oscillator potential introduced in Eq. (1) and the assumption in Eq. (8), we can guess that $f_n(x)$ can take the following form

$$f_n(x) = b_n x + \frac{c_n}{x}, \tag{10}$$

where b_n and c_n are constant values which should be chosen such that E_n gets the maximum value. Thus, a_n can be written as:

$$a_n = \frac{1}{\sqrt{2m}}\left[p + \mathrm{i}\left(b_n x + \frac{c_n}{x}\right)\right]. \tag{11}$$

With the help of a_n and its conjugate $a_n^\dagger$, one can easily check that the following relations hold:

$$a_n^\dagger a_n = \frac{1}{2m}\left[p^2 + b_n^2 x^2 + b_n(2c_n + \hbar) + \frac{c_n}{x^2}(c_n - \hbar)\right], \tag{12}$$

$$a_n a_n^\dagger = \frac{1}{2m}\left[p^2 + b_n^2 x^2 + b_n(2c_n - \hbar) + \frac{c_n}{x^2}(c_n + \hbar)\right]. \tag{13}$$

By substituting $n = 0$ into Eq. (12) and using Eq. (4), it is required that

$$\frac{1}{2m}\left[p^2 + b_0^2 x^2 + b_0(2c_0 + \hbar) + \frac{c_0}{x^2}(c_0 - \hbar)\right] + E_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{\hbar^2}{2m}\frac{\alpha}{x^2}. \tag{14}$$

The above equation will be satisfied, if

$$b_0^2 = m^2\omega^2 \longrightarrow b_0 = -m\omega, \tag{15}$$

$$c_0(c_0 - \hbar) = s(s+1)\hbar^2 \longrightarrow c_0 = (s+1)\hbar, \tag{16}$$

$$b_0(2c_0 + \hbar) + 2mE_0 = 0, \tag{17}$$

where the allowed values of b_0 in Eq. (15) can satisfy the condition (ii) of the stated theorem in which E_0 gets the maximum value. Also, note that the change of variable $s = -1/2 + \sqrt{\alpha + 1/4}$ in Eq. (16) has been used. By substituting Eqs. (15) and (16) into Eq. (17), the maximum of the ground state energy will be achieved as:

$$E_0 = \hbar\omega\left(s + \frac{3}{2}\right). \tag{18}$$

For attaining the other eigenvalues we can use the recurrence relation (4), from which one has:

$$\frac{1}{2m}\left[p^2 + b_{n+1}^2 x^2 + b_{n+1}(2c_{n+1} + \hbar) + \frac{c_{n+1}}{x^2}(c_{n+1} - \hbar)\right] + E_{n+1} = \frac{1}{2m}\left[p^2 + b_n^2 x^2 + b_n(2c_n - \hbar) + \frac{c_n}{x^2}(c_n + \hbar)\right] + E_n. \tag{19}$$

The relation (19) will be satisfied, if

$$b_{n+1}^2 = b_n^2 \longrightarrow b_n = b_0 = -m\omega, \tag{20}$$

$$c_{n+1}(c_{n+1} - \hbar) = c_n(c_n + \hbar) \longrightarrow c_n = (s + n + 1)\hbar, \tag{21}$$

$$b_{n+1}(2c_{n+1} + \hbar) + 2mE_{n+1} = b_n(2c_n - \hbar) + 2mE_n. \tag{22}$$

Finally, from Eqs. (20), (21), and (22) we can derive the required eigenvalues as

$$E_n = \hbar\omega\left(2n + s + \frac{3}{2}\right), \quad n = 0, 1, 2, \ldots. \tag{23}$$

By substituting b_n and c_n respectively from Eq. (20) and Eq. (21) into Eq. (11), $a_n(x)$ will be explicitly obtained as

$$a_n(x) = \frac{1}{\sqrt{2m}}\left[p + \mathrm{i}\left(-m\omega x + \frac{(s + n + 1)\hbar}{x}\right)\right]. \tag{24}$$

Equation (23) gives all eigenvalues, if there is an appropriate solution for any arbitrary n in Eq. (5). Thus, we need to solve the following equation

$$a_n(x)\xi_n(x) = 0 \tag{25}$$

or equivalently

$$\frac{\mathrm{d}\xi_n(x)}{\mathrm{d}x}=\Big(\frac{s+n+1}{x}-\frac{m\omega}{\hbar}x\Big)\xi_n(x). \tag{26}$$

The solution of the above differential equation can be simply obtained as

$$\xi_n(x)=x^{s+n+1}\exp\Big(-\frac{m\omega}{2\hbar}x^2\Big). \tag{27}$$

By using Eq. (6) and doing some tedious calculations, the normalized eigenfunctions of the pseudoharmonic oscillator can be obtained as ($\hbar=m=\omega=1$)

$$\psi_n(x)=N_n x^{s+1}\,\mathrm{e}^{-x^2/2}\mathrm{L}_n^{s+1/2}(x^2), \tag{28}$$

where $\mathrm{L}_n^{s+1/2}(x^2)$ is the associated Laguerre polynomial and the normalization constant in Eq. (28) is determined as

$$N_n=\sqrt{\frac{2\Gamma(n+1)}{\Gamma(n+s+\frac{3}{2})}}. \tag{29}$$

4. Introducing ladder operators associated with the pseudoharmonic oscillator

It is remarkable that the operator a_n in Eq. (24) and its conjugate do not play the roles of raising and lowering operators for the eigenstates of the pseudoharmonic potential. But, keeping in mind of these operators, we want to construct a rather new pair of operators which yield our mentioned requirement. For this purpose, we introduce a new operator as follows:

$$A_n\equiv-\frac{\mathrm{d}}{\mathrm{d}x}-x+\frac{s+n+1}{x}, \tag{30}$$

where its conjugate reads as

$$A_n^\dagger\equiv\frac{\mathrm{d}}{\mathrm{d}x}-x+\frac{s+n+1}{x}. \tag{31}$$

By applying the operator A_n on $\psi_n(x)$ and using the following recurrence relations for the associated Laguerre polynomials[54]

$$x\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{L}_n^\alpha(x)=\begin{cases}n\mathrm{L}_n^\alpha(x)-(n+\alpha)\mathrm{L}_{n-1}^\alpha(x),\\(n+1)\mathrm{L}_{n+1}^\alpha(x)-(n+\alpha+1-x)\mathrm{L}_n^\alpha(x),\end{cases} \tag{32}$$

we have

$$A_n\psi_n(x)=-\frac{n}{x}\psi_n(x)+\frac{2}{x}\sqrt{n\Big(n+s+\frac{1}{2}\Big)}\,\psi_{n-1}(x). \tag{33}$$

Similarly, by acting $A_n^\dagger$ on $\psi_n(x)$ one has:

$$A_n^\dagger\psi_n(x)=-\frac{n+1}{x}\psi_n(x)+\frac{2}{x}\sqrt{(n+1)\Big(n+s+\frac{3}{2}\Big)}\,\psi_{n+1}(x). \tag{34}$$

Clearly, the above operators A_n and $A_n^\dagger$ do not yet play the roles of ladder operators. Altogether, it is possible to find out the ladder operators which can regularly raise and lower any eigenfunction of our considered system, i.e.,

$$M_\pm=\pm\frac{x}{2}\frac{\mathrm{d}}{\mathrm{d}x}+N+\frac{s}{2}+\frac{1}{3/2\mp1/2}-\frac{x^2}{2}, \tag{35}$$

where we have introduced the number operator N which satisfies the eigenvalue equation

$$N|n\rangle=n|n\rangle,\quad \psi_n(x)=\langle x|n\rangle \tag{36}$$

with $\{|n\rangle\}_{n=0}^\infty$ the eigenkets of the pseudoharmonic oscillator system. Notice that $N\neq M_+M_-$. It is easy to check that the ladder operators $M_\pm$ have the following properties

$$M_\pm|n\rangle=m_\pm|n\pm1\rangle,\quad n=0,1,2,\ldots, \tag{37}$$

where

$$m_\pm=\sqrt{\Big(n+\frac{1}{2}\pm\frac{1}{2}\Big)\Big(n+s+\frac{2\pm1}{2}\Big)}. \tag{38}$$

We now study the dynamical group associated with the operators M_+ and M_-. Based on the obtained results in Eqs. (37) and (38), we can calculate the following commutator

$$[M_-,M_+]|n\rangle=2m_0|n\rangle, \tag{39}$$

where we have defined

$$m_0=n+\frac{s}{2}+\frac{3}{4}. \tag{40}$$

Thus, we can define the operator

$$M_0=N+\frac{s}{2}+\frac{3}{4}. \tag{41}$$

According to our above results, the operators $M_\pm$ and M_0 justify the following algebraic relations

$$[M_-,M_+]=2M_0,\quad [M_0,M_\pm]=\pm M_\pm. \tag{42}$$

As is well known, according to Eq. (42), the operators $M_\pm$ and M_0 correspond to an $su(1,1)$ Lie algebra. In addition, the system Hamiltonian can be simply expressed as

$$H|n\rangle=\Big(2n+s+\frac{3}{2}\Big)|n\rangle=[M_-,M_+]|n\rangle. \tag{43}$$

The form of the Hamiltonian in Eq. (43) satisfies the eigenvalue equation $H|n\rangle=E_n|n\rangle$ with E_n determined by Eq. (23). It may be interesting to notice that while for the harmonic oscillator one has $H_{\mathrm{HO}}|n\rangle_{\mathrm{HO}}=(n+1/2)|n\rangle_{\mathrm{HO}}=\{a,a^\dagger\}|n\rangle_{\mathrm{HO}}$, where a, $a^\dagger$ are the standard bosonic annihilation and creation operators, the anti-commutation relation for the harmonic oscillator gives its place to the commutation relation for the pseudoharmonic oscillator.

5. Coherent states for the pseudoharmonic oscillator: Barut–Girardello and Gilmore–Perelomov

We are now able to produce two distinct classes of coherent states for the considered system. Usually, the Barut–Girardello type of the coherent state is defined as the right eigenstate of the annihilation operator, i.e.,

$$M_-|z\rangle_{\rm BG} = z|z\rangle_{\rm BG}. \tag{44}$$

To calculate the explicit form of $|z\rangle_{\rm BG}$ with the traditional method, we write

$$|z\rangle_{\rm BG} = \sum_{n=0}^{\infty} c_n|n\rangle. \tag{45}$$

By substituting it into Eq. (44) and making use of Eq. (37), we finally arrive at the explicit form of the coherent state

$$|z\rangle_{\rm BG} = N_{\rm BG}(|z|^2)\sum_{n=0}^{\infty}\left[\frac{\Gamma(s+3/2)}{\Gamma(n+1)\Gamma(n+s+3/2)}\right]^{1/2} z^n|n\rangle, \tag{46}$$

where the factor $N_{\rm BG}(|z|^2)$ can be calculated by the normalization condition $\langle z|z\rangle = 1$, i.e.,

$$N_{\rm BG}(|z|^2) = \left({}_0{\rm F}_1(s+\frac{3}{2},|z|^2)\right)^{-1/2}. \tag{47}$$

where ${}_0{\rm F}_1(a,x)$ denotes the hypergeometric function.

On the other hand, referring the $su(1,1)$ symmetry algebra of the generators in Eq. (42), we are able to construct the Gilmore–Perelomov coherent state associated with the pseudoharmonic potential as follows:[55]

$$\begin{aligned}|z\rangle_{\rm GP} &= D(z)|0\rangle = \exp(\xi M_+ - \xi^* M_-)|0\rangle \\ &= \exp(zM_+)(1-|z|^2)^{M_0}\exp(-z^*M_-)|0\rangle\end{aligned} \tag{48}$$

with

$$z = \frac{\xi}{|\xi|}\tanh(\xi).$$

Straightforward calculations leads one to the corresponding normalized Gilmore–Perelomov coherent state, i.e.,

$$|z\rangle_{\rm GP} = N_{\rm GP}(|z|^2)\sum_{n=0}^{\infty}\left[\frac{\Gamma(n+s+3/2)}{\Gamma(n+1)\Gamma(s+3/2)}\right]^{1/2} z^n|n\rangle, \tag{49}$$

with the normalization factor

$$N_{\rm GP}(|z^2|) = \left(1-|z|^2\right)^{s/2+3/4}. \tag{50}$$

Now, after finding out the two distinct classes of coherent states, we are ready to investigate some of the nonclassical properties which are of the most importance in the content of coherent states in quantum optics studies. But, before paying attention to this subject, we should investigate the resolution of the identities for the above introduced states. If we are successful, the introduced states can be truly named as coherent states.

Resolution of the identity: any coherent state $|z\rangle$ should satisfy the resolution of the identity which is defined as

$$\int {\rm d}^2 z w(|z|^2)|z\rangle\langle z| = \hat{I} = \sum_{n=0}^{\infty}|n\rangle\langle n|, \tag{51}$$

where $w(|z|^2)$ is a positive weight function that we search for obtaining it. For the first class of states which is introduced in this section, substituting Eq. (46) into Eq. (51) and using the change of variables $z = r{\rm e}^{{\rm i}\theta}$ and $r^2 = x$ lead us to the following integral relation

$$\int_0^{\infty}\tilde{w}_{\rm BG}(x)x^n\,{\rm d}x = \frac{\Gamma(n+1)\Gamma(n+s+3/2)}{\Gamma(s+3/2)}, \tag{52}$$

where we have used the abbreviation

$$\tilde{w}_{\rm BG}(x) = \pi w_{\rm BG}(x)N^2_{\rm BG}(x).$$

This is indeed an inverse moment problem which can be solved by using well-known methods such as the Mellin transform technique.[56] However, with the definition of Meijer's G-function, it follows that[57]

$$\begin{aligned}&\int_0^{\infty}{\rm d}x x^{k-1}G^{m,n}_{p,q}\left(\beta x\left|\begin{matrix}a_1,\ \ldots,\ a_n,\ a_{n+1},\ \ldots,\ a_p\\ b_1,\ \ldots,\ b_m,\ b_{m+1},\ \ldots,\ b_q\end{matrix}\right.\right)\\ &= \frac{1}{\beta^k}\frac{\prod_{j=1}^m\Gamma(b_j+k)\prod_{j=1}^n\Gamma(1-a_j-k)}{\prod_{j=m+1}^q\Gamma(1-b_j-k)\prod_{j=n+1}^p\Gamma(a_j+k)}.\end{aligned} \tag{53}$$

Comparing Eqs. (52) and (53), it can be easily checked that the required weight function which can satisfy the integral equation (51) reads as

$$w_{\rm BG}(x) = \frac{{}_0{\rm F}_1(s+\frac{3}{2},x^2)}{\pi\Gamma(s+\frac{3}{2})}G^{4,0}_{2,4}\left(x\left|\begin{matrix}(0,s+1/2)\\ (0,0,s+1/2,s+1/2)\end{matrix}\right.\right), \tag{54}$$

where G is the well-known Meijer function and ${}_0{\rm F}_1(a;x)$ is the generalized hypergeometric function.

For the Gilmore–Perelomov states, by substituting Eq. (49) into Eq. (51) we will have

$$\int_0^1\tilde{w}_{\rm GP}(x)x^n\,{\rm d}x = \frac{\Gamma(n+1)\Gamma\left(s+3/2\right)}{\Gamma\left(n+s+3/2\right)}, \tag{55}$$

where we have used $\tilde{w}_{\rm GP}(x) = \pi w_{\rm GP}(x)N^2_{\rm GP}(x)$. Similarly, in this case it is easy to check that the required weight function has the following form

$$w_{\rm GP}(x) = \frac{\Gamma(s+3/2)}{\pi\Gamma(s+1/2)}(1-x)^{-2}{}_2{\rm F}_1\left(0,0;s+\frac{1}{2};1-\frac{1}{x}\right), \tag{56}$$

where ${}_2{\rm F}_1(a_1,a_2;x)$ is the generalized hypergeometric function.

A few words seem to be necessary about the limits of the integrals in Eq. (52) and Eq. (55). Indeed, any quantum state belongs legally to a Hilbert space if its normalization constant satisfies the inequality $0 < N < \infty$. This leads us to the unit disk and the whole of the complex plane for the Barut–Girardello and Gilmore–Perelomov states respectively.

6. Nonclassical features of the introduced states

In this section, we briefly discuss some of the criteria of nonclassicality which are usually used in the relevant literature,[58] and this will be helpful for investigating the nonclassicality exhibition of our introduced states in the continuation of the paper. To achieve this aim, we refer to the normal squeezing, amplitude-squared squeezing, and sub-Poissonian statistics as the most popular nonclassicality signs. To mention the common feature of the above criteria we should indicate that if a state is squeezed or its quantum statistics is sub-Poissonian in a region of the allowed space, the corresponding Glauber–Sudarshan P-function will not be positive-definite, i.e., it has no classical analogue. But, we would like to imply that finding this function is usually a hard task to do. Altogether, each of the above effects is in fact a sufficient (not necessary) condition for a quantum state to belong to the family of nonclassical states.

6.1. Normal squeezing

In order to examine the quantum fluctuations of the quadratures of the field,[59] we introduce the Hermitian operators

$$X_1 = \frac{M_- + M_+}{2}, \quad P_1 = \frac{M_- - M_+}{2i}. \tag{57}$$

The squeezing parameters can be defined as

$$S_\gamma = \frac{(\Delta\gamma)^2}{\sqrt{\frac{1}{4}|\langle[X_1, P_1]\rangle|^2}} - 1, \quad \gamma = X_1, P_1, \tag{58}$$

where $(\Delta\gamma)^2 = \langle\gamma^2\rangle - \langle\gamma\rangle^2$. Using Eq. (57) in Eq. (58) we obtain the following first-order squeezing parameters

$$\begin{aligned} S_{X_1} &= \frac{1}{2\langle M_0\rangle}(\langle M_-^2\rangle + \langle M_- M_+\rangle + \langle M_+ M_-\rangle + \langle M_+^2\rangle \\ &\quad - \langle M_-^2\rangle - \langle M_+^2\rangle - \langle M_-\rangle\langle M_+\rangle - \langle M_+\rangle\langle M_-\rangle) - 1, \quad (59) \\ S_{P_1} &= \frac{1}{2\langle M_0\rangle}(-\langle M_-^2\rangle + \langle M_- M_+\rangle + \langle M_+ M_-\rangle - \langle M_+^2\rangle \\ &\quad + \langle M_-^2\rangle + \langle M_+^2\rangle - \langle M_-\rangle\langle M_+\rangle - \langle M_+\rangle\langle M_-\rangle) - 1, \quad (60) \end{aligned}$$

where all expectation values can be straightforwardly obtained with respect to both classes of states. A state is squeezed in X_1 (P_1) if it satisfies the inequality $-1 < S_{X_1} < 0$ ($-1 < S_{P_1} < 0$).

By comparing the two displayed diagrams in Fig. 1, it can be realized that the squeezing of Gilmore–Perelomov coherent states occurs only in the P_1 direction in the whole real space. Our calculations about the squeezing for Barut–Girardello coherent states in X_1 and P_1 show that this parameter is approximately 10^{-15} ($\cong 0$) for both operators. Thus, in view of squeezing property, it is deduced that the Barut–Girardello states associated with the considered system behave similarly to the linear harmonic oscillator.

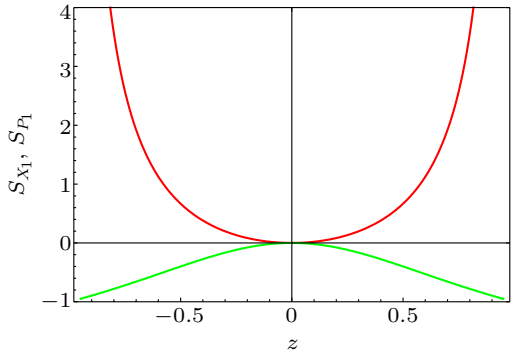


Fig. 1. (color online) Plot of squeezing parameters, S_{X_1} (up diagram) and S_{P_1} (down diagram) against real z with $s = 1$ for Gilmore–Perelomov coherent states.

6.2. Amplitude-squared squeezing

Following the proposal of Hillery,[60] to evaluate this parameter we define the Hermitian operators

$$X_2 = \frac{M_-^2 + M_+^2}{2}, \quad P_2 = \frac{M_-^2 - M_+^2}{2i}. \tag{61}$$

The squeezing conditions in X_2 and P_2 are respectively given by $-1 < S_{X_2} < 0$ and $-1 < S_{P_2} < 0$, where S_{X_2} and S_{P_2} are defined as follows:

$$S_\ell = \frac{(\Delta\ell)^2}{\sqrt{\frac{1}{4}|\langle[X_2, P_2]\rangle|^2}} - 1, \quad \ell = X_2, P_2 \tag{62}$$

with the definition $(\Delta\ell)^2 = \langle\ell^2\rangle - \langle\ell\rangle^2$. The explicit form of the latter parameters for the second-order squeezing can also be obtained in a similar manner to what is done in Eqs. (59) and (60) for the normal squeezing. In Fig. 2 the amplitude-squared squeezing has been plotted against real z for Gilmore–Perelomov coherent states, in X_2 and P_2 directions. According to the diagrams placed in Fig. 2, it appears that, like the first-order squeezing, for this state again amplitude-squared squeezing occurs only in P_2 direction. Also, this parameter for Barut–Girardello states in the two directions X_2 and P_2 takes approximately the value of zero. This latter observation is also the same as the situation which is likely to occur in ordinary coherent states.

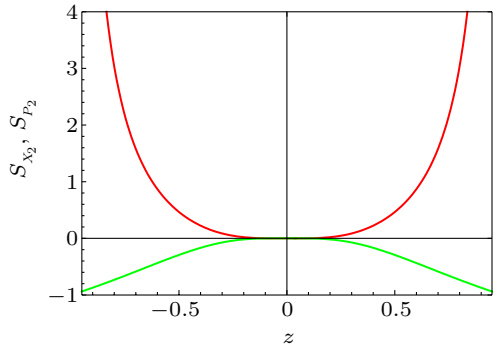


Fig. 2. (color online) Plot of amplitude-squared squeezing parameters, S_{X_2} (up diagram) and S_{P_2} (down diagram) against real z with $s = 1$ for Gilmore–Perelomov coherent states.

6.3. Mandel parameter

To establish the quantum statistical properties and as another nonclassicality sign, we calculate the Mandel parameter,[61] which can be generalized as

$$Q = \frac{\langle M_+^2 M_-^2\rangle - \langle M_+ M_-\rangle^2}{\langle M_+ M_-\rangle} - 1. \tag{63}$$

The states for which $Q = 0$, $Q < 0$, and $Q > 0$ respectively correspond to the Poissonian (standard coherent states), sub-Poissonian (nonclassical states), and super-Poissonian (classical states) statistics. In Fig. 3 the Mandel parameter has been plotted versus real z for Gilmore–Perelomov coherent states. As is shown, these states behave as nonclassical features in a finite region of space. Our numerical calculation of this parameter for Barut–Girardello states takes the fixed value $Q = -1$. Therefore, their quantum statistics is that the Mandel parameter gets the lowest possible negative value, like the number states of the harmonic oscillator which sometimes have been called as the most nonclassical states.

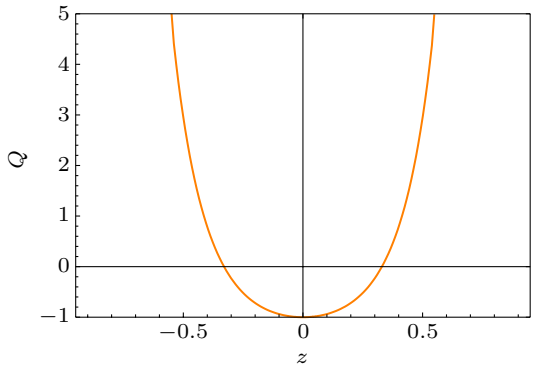


Fig. 3. (color online) Plot of Mandel parameter with $s = 1$ for Gilmore–Perelomov coherent states versus real z.

Adding the above observations, we may conclude that, from the point of view of squeezing properties, the pseudoharmonic oscillator Barut–Girardello coherent state which is a particular superposition of number states (of the pseudoharmonic oscillator) behaves like the canonical coherent state

$$|z\rangle = \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle_{\rm HO}.$$

However, taking into account the quantum statistical properties of the field, the pseudoharmonic oscillator Barut–Girardello coherent state behaves like the number state of the harmonic oscillator (note that $|n\rangle_{\rm HO} \neq |n\rangle$, where $|n\rangle$ in the present paper is related to the eigenkets of the pseudoharmonic oscillator potential). So, in this view, the latter state can also be named as the most nonclassical state, too.

7. Summary and conclusion

In summary, in this paper we obtained the eigenvalues and eigenfunctions of the one-dimensional pseudoharmonic oscillator by using the factorization method. The presented approach allows us to find out appropriate generalized creation and annihilation operators by which one can raise and lower arbitrary eigenkets of the considered system. Then, we have explored that the system under consideration possesses a symmetrical realization of a particular dynamical group and it is shown that the constructed ladder operators can constitute the generators of the $su(1,1)$ Lie algebra. Next, we have produced the explicit form of Barut–Girardello and Gilmore–Perelomov coherent states for the considered system. The resolution of the identity, as the most important task for establishing any state as a class of "coherent states", is established analytically so the over-completeness relations for both states are demonstrated. Finally, we could highlight some of the nonclassical properties, such as first-order, amplitude-squared squeezing, and sub-Poissonian statistics for the introduced states, numerically. Interestingly, we have also found that, in particular, considering squeezing properties, the pseudoharmonic oscillator Barut–Girardello coherent state behaves like the canonical coherent state, however taking into account the quantum statistical properties, the latter state behaves like the number state of the harmonic oscillator. Although we have applied the method to the pseudoharmonic oscillator in this paper, the approach can be used for other systems of interest too. As another example, we have also done the same procedure to a well-known problem, i.e., a quantum particle in an infinite square well potential, successfully with the new and novel results, which will appear in near future.

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