Comment on 'Mathematical structure of the three-dimensional (3D) Ising model'*

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The review paper by Zhang Zhi-Dong (Zhang Z D 2013 *Chin. Phys. B* **22** 030513, arXiv:1305.2956) contains many errors and is based on several earlier works that are equally wrong.

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1. Introduction

In 2007 Zhang Zhi-Dong published the very long paper,[1] claiming to present complete exact solutions for the free energy and spontaneous magnetization per site of the Ising model on a three-dimensional orthorhombic lattice. He also claimed results for the pair correlation function. That his claims are false is clearly stated in Refs. [2] and [3]. Zhang's formulae (74) and (102) with his choice of the weights $w_x = 1$, $w_y = w_z = 0$, see pages 5339 and 5370, do not reproduce the well-established high- and low-temperature series results, as required by rigorous theorems to the contrary. Also his paper[1] opens with an incorrect application of the Jordan–Wigner transformation.[3]

To this criticism Zhang brings up "the possibility of the occurrence of a phase transition at infinite temperature according to the Yang–Lee theorems" on page 3097 of Ref. [4], see also pages 5369–5371 of Ref. [1]. This is a clear error, as Zhang's picture means that the partition function zeros in the large system limit are to pinch the real β axis for $\beta = 0$ and β_c, but not in between. Thus he claims a singularity at $\beta = 0$, even for zero magnetic field, such that one cannot apply the series test. This violates rigorous theorems. The absurdity of this argument is immediately clear as Zhang's formula (49) in Ref. [1] can be expanded in β with a finite radius of convergence, so that he contradicts himself. Pinching of Yang–Lee zeros at $\beta = 0$ would require zero radius of convergence.

In spite of the fact that Zhang's magnum opus is clearly in error, he published many more papers, mostly with Norman H. March, adding more errors. The last paper,[5] to which this comment applies, contains all relevant references, so that the referees of this work are without excuse for failing to reject it.

In the next few sections we shall discuss several of the errors in Zhang's work in more detail.

2. Zhang's results violate established series results, even in first nontrivial orders

To explicitly see what the series test can reveal, it is sufficient to restrict ourselves to the isotropic cubic Ising model ($J_1 = J_2 = J_3 = J$, $K = \beta J = J/k_B T$). For that case Zhang expands his "putative" exact partition function per site in Eq. (A13) on page 5406 of Ref. [1] as

$$Z^{1/N} = 2\cosh^3 K\left[1+\frac{7}{2}\kappa^2+\frac{87}{8}\kappa^4+\frac{3613}{48}\kappa^6+\cdots\right], \quad \kappa=\tanh K. \tag{1}$$

This differs from the well-known high-temperature series given in Eq. (A12) of Ref. [1] as

$$Z^{1/N} = 2\cosh^3 K\left[1+0\kappa^2+3\kappa^4+22\kappa^6+\cdots\right]. \tag{2}$$

Zhang then suggests that both results are correct, the first for finite temperatures, the second only for an infinitesimal neighborhood of $\beta \equiv 1/k_B T = 0$, see e.g. pages 5382–5384, 5394, 5400 of Ref. [1]. This makes no sense as both series have a finite radius of convergence. It can be rigorously shown that the first result (1) is wrong, whereas the first so many terms of series (2), known for over 60 years,[6] are correct in the thermodynamic limit. We shall discuss that in a later section.

From Eq. (103) on page 5342 of Ref. [1] we obtain the "putative" low-temperature expansion of the spontaneous magnetization,

$$I = 1 - 6x^8 - 12x^{10} - 18x^{12} + \cdots, \quad x = \mathrm{e}^{-2K}. \tag{3}$$

Some coefficients related to the well-known expansion, existing for over six decades in the literature,[6] are listed in Table 2 on page 5380 of Ref. [1], implying

$$I = 1 - 2x^6 - 12x^{10} + 14x^{12} - \cdots. \tag{4}$$

The textbook derivation starts with a d-dimensional hypercubic lattice of N sites, each site having $2d$ neighbors. There is one state with all spins up and energy E_+, N states with only

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one spin down and energy $E_+ + 4dJ$, etc. Thus,

$$I = \frac{Z_\sigma}{Z} = \frac{1+(N-2)x^{2d}+\cdots}{1+Nx^{2d}+\cdots} = 1-2x^{2d}+\cdots, \quad 2d=6. \quad (5)$$

Only if the down spin is at the position of the spin operator we get a minus sign, or $N-2=(N-1)(+1)+(-1)$. It is well known that in the linked cluster expansion[6] the N dependence cancels in all orders. Again, Zhang's result (3) is clearly wrong, displaying $d=4$ behavior.

Much more can be said on the spontaneous magnetization and the pair correlation. However, here it suffices to discuss errors in the treatment of the free energy in Refs. [1] and [5], as the rest of Zhang's work is built on the same erroneous foundation.

2.1. Two different expansions for the same?

It has been shown already in the 1960s that the high-temperature series of βf, ($\beta f = -\lim \ln Z^{1/N}$), and all correlation functions on the cubic lattice have finite nonzero radius of convergence. By duality with the Ising model with 4-spin interactions on all faces of the cubic lattice and at high temperatures, the low-temperature series for the spontaneous magnetization I should also converge. Therefore, one cannot have two different expansions as given by Zhang.

To get out of this dilemma, Zhang (on pages 5381, 5383, and 5394 of Ref. [1]) comes with a mathematically absurd suggestion implying that the old series (2) and (4) are asymptotic with zero radius of convergence, whereas his "putative solution" is analytic with finite radius of convergence. On pages 12 and 13 of Ref. [5] (and elsewhere) he claims that there is an (essential?) singularity at $\beta = h = 0$ and that this is due to the zeros of Z^{-1} and that Perk "went on perpetrating the fraud, discussing the singularity of βf_N" [Ref. [5], page 12] and [Ref. [7], page 63], not f. To use the word "fraud" is clearly highly unprofessional. We discuss this next.

2.2. Irrelevant pole of f

It has been rigorously proved (one proof discussed in the next section) that βf has an absolutely convergent series, uniformly convergent in the thermodynamic limit, so that

$$\beta f = \sum_{i=0}^{\infty} a_i \beta^i, \quad |\beta| < r. \quad (6)$$

Therefore,

$$f = \frac{a_0}{\beta} + \sum_{i=1}^{\infty} a_i \beta^{i-1}, \quad 0 < |\beta| < r, \quad (7)$$

is a convergent Laurent series, totally equivalent to Eq. (6).

The pole at $\beta = 0$ has no significance, as βf is the relevant quantity from statistical mechanics point of view, entering the normalization $Z = \mathrm{e}^{-N\beta f}$ for the Boltzmann–Gibbs canonical distribution. Also, note that in the Appendix A of Ref. [1] Zhang expanded $Z^{1/N} = \mathrm{e}^{-\beta f}$. This makes his objection to expanding βf rather out of place.

2.3. Zeros of 1/Z are irrelevant

Zhang and March repeatedly claim the importance of the zeros of Z^{-1}, with $Z = z^N$ the total partition function, see [Ref. [7], pages 64 and 65], [Ref. [8], page 87], and [Ref. [5], pages 12 and 13]. However, unlike the complex Yang–Lee zeros of Z, the zeros of Z^{-1} are irrelevant:

(i) For a finite number N of sites, Z is a finite Laurent polynomial in e^K, $K \equiv \beta J$, and only can become infinite when $\mathrm{Re}\, K = \pm\infty$, i.e. zero-temperature type limits.

(ii) For the infinite system, $N \to \infty$, and finite K, the infinity of Z should be seen as just a manifestation of the thermodynamic limit, in which $z = Z^{1/N}$ remains finite.

One can easily see that $\beta f < 0$ for K real, so that $Z = \mathrm{e}^{-N\beta f} = \infty$ for all real K when $N = \infty$. There is nothing special about the $N = \infty$ zeros of Z^{-1}. Hence, Zhang and March cannot claim that the zeros of Z^{-1} play any role. They are there the same way for the one-dimensional Ising model, $Z \approx (2\cosh K)^N$, and free Ising spins in field B, $Z = (2\cosh \beta B)^N$, for $N \to \infty$.

3. Analyticity properties of the correlation functions at high temperatures

There are several rigorous proofs in the literature showing that the convergence toward the thermodynamic limit is uniform and that the resulting high-temperature series has a nonzero radius of convergence, which then rigorously shows that the main "putative" results of Ref. [1] are wrong.[2,3] The search of such proofs was initiated by Groeneveld,[9] who found such a proof for the Mayer expansion as part of his thesis research under professor Jan de Boer, the father of professor Frank de Boer — one of the supervisors of Zhang Zhi-Dong's thesis research.

Such proofs of analyticity mostly belong to two classes, i.e. starting from (linked) cluster expansions,[6] like the Mayer expansion, or applying linear correlation-function identities.[10] In the next few subsections we shall present an outline of the proof in Ref. [11], which belongs to the second class. The intermediate steps in this proof can be used to pinpoint other errors in the works of Zhang and March.

We first prove that the correlation functions of the Ising model on the periodic cubic lattice of size $N = n^3$, have high-temperature series absolutely convergent when $|\beta J| < r_0$, uniform in n. The corresponding analyticity statement for βf_N then follows using the elementary identity

$$\frac{\partial(\beta f_N)}{\partial \beta} = u_N = 3J\langle \sigma\sigma' \rangle_N,$$

with σ and σ' nearest-neighbor spins. As more and more coefficients become independent of n, the analyticity then carries over in the thermodynamic limit $N \to \infty$.

3.1. Laurent polynomial lemma

The partition function Z_N of the Ising model on a periodic cubic lattice of sides n lattice spacings, $N = n^3$, is a Laurent polynomial in $e^{\beta J}$. This means that it is a polynomial in both $e^{\beta J}$ and $e^{-\beta J}$. Therefore, $\beta f_N = -N^{-1} \ln Z_N$ is singular only for the zeros of this Laurent polynomial and for some cases with $|e^{\pm\beta J}| = \infty$. As Z_N is a sum of positive terms for real βJ, Z_N cannot have zeros on the real βJ-axis.

It follows that all correlation functions $\langle \prod_l \sigma_l \rangle_N$ are meromorphic functions with poles only at the zeros of Z_N. Hence,

$$\left\langle \prod_l \sigma_l \right\rangle_N = \sum_{i=1}^{\infty} c_i (\beta J)^i, \quad |c_i| < C_N r^{-i}, \tag{8}$$

with r the absolute value of the zero closest to $\beta J = 0$ and C_N some positive constants.

3.2. Lemma (M. Suzuki, 1965) for $B = 0$

$$\left\langle \prod_{i=1}^{m} \sigma_{j_i} \right\rangle_N = \frac{1}{m} \sum_{k=1}^{m} \left\langle \left(\prod_{i=1, i\neq k}^{m} \sigma_{j_i} \right) \tanh\left(\beta J \sum_{l \text{ nn } j_k} \sigma_l \right) \right\rangle_N, \tag{9}$$

where $j_1, \ldots, j_m$ are the labels of m spins and l runs through the labels of the six spins that are nearest neighbors of σ_{j_k}.[11–13] The averaging over k treats the m spins σ_{j_k} symmetrically. The lemma without the averaging over k was presented by Suzuki and can be used instead, selecting one of the m spins at random.

3.3. Expanding tanh lemma

It is straightforward[11] to prove the following or to verify it using Mathematica or Maple:

$$\begin{aligned} \tanh\left(\beta J \sum_{l=1}^{6} \sigma_l \right) &= a_1 \sum_{(6)} \sigma_l + a_3 \sum_{(20)} \sigma_{l_1} \sigma_{l_2} \sigma_{l_3} \\ &\quad + a_5 \sum_{(6)} \sigma_{l_1} \sigma_{l_2} \sigma_{l_3} \sigma_{l_4} \sigma_{l_5}, \end{aligned} \tag{10}$$

where the sums are over the 6, 20, or 6 choices of choosing 1, 3, or 5 spins from the given $\sigma_1, \ldots, \sigma_6$. The coefficients a_i are

$$\begin{aligned} a_{1,5} &= \frac{1}{24} \left(\frac{p_1 t}{1+(p_1 t)^2} + \frac{p_2 t}{1+(p_2 t)^2} \right) \\ &\quad \pm \frac{\sqrt{2}}{8} \left(\frac{p_3 t}{1+(p_3 t)^2} + \frac{p_4 t}{1+(p_4 t)^2} \right) + \frac{1}{3} \frac{p_5 t}{1+(p_5 t)^2}, \\ a_3 &= \frac{1}{24} \left(\frac{p_1 t}{1+(p_1 t)^2} + \frac{p_2 t}{1+(p_2 t)^2} \right) - \frac{1}{6} \frac{p_5 t}{1+(p_5 t)^2}, \\ p_{1,2} &= 2 \pm \sqrt{3}, \quad p_{3,4} = \sqrt{2} \pm 1, \quad p_5 = 1, \\ &\quad (p_1 p_2 = p_3 p_4 = 1). \end{aligned} \tag{11}$$

The poles of the a_i are at $t = \pm(2 \pm \sqrt{3})\mathrm{i}$, $t = \pm(\sqrt{2} \pm 1)\mathrm{i}$, and $t = \pm\mathrm{i}$. The a_i have series expansions in terms of odd powers of t alternating in sign and converging absolutely as long as $|\beta J| < \arctan(2 - \sqrt{3}) = \pi/12$.

3.4. Uniform convergence for finite N

Expanding the tanh in Suzuki's lemma replaces any even correlation by a linear combination of even correlations, with coefficients given by the a_i. As for fixed $|t| \equiv x$, each $|a_i|$ is maximal for imaginary $t = \mathrm{i}x$, we find for the sum s of the absolute values of all coefficients:

$$s \equiv 6|a_1| + 20|a_3| + 6|a_5| \leq \frac{2x(3-x^2)(1-3x^2)}{(1-x^2)(1-14x^2+x^4)}. \tag{12}$$

It easily follows that $s < 1$ for

$$\begin{aligned} &|t| < (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1) = t_0 = 0.131652497\cdots, \quad \text{or} \\ &|\beta J| < \arctan[(\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)] = 0.130899693\cdots. \end{aligned} \tag{13}$$

This determines a lower bound on the radius of convergence: Keep repeatedly applying Suzuki's lemma on all correlations that are not the "zero-point correlation function" $\langle 1 \rangle = 1$. Terms with $\langle 1 \rangle$ are explicitly known and terms that are not are at least one order higher in t or βJ. After one iteration we have the very generous upper bound

$$\left| \left\langle \prod_{i=1}^{m} \sigma_{j_i} \right\rangle_N \right| < s + s \max \left| \left\langle \prod_{i=1}^{m'} \sigma_{j'_i} \right\rangle_N \right|, \tag{14}$$

with maximum over all correlations generated by Suzuki's lemma. After I iterations, a given even correlation is then split into an explicitly given sum bounded by $s + s^2 + \cdots + s^I$ and a remainder sum with correlations left for further iteration, higher order in t or βJ. If we increase the right-hand side of expression (14) by taking the maximum M over all $2^N - 1$ correlation functions that are not $\langle 1 \rangle$ ($m \neq 0$), then it immediately follows that $M < s + sM$, so that

$$\left| \left\langle \prod_{i=1}^{m} \sigma_{j_i} \right\rangle_N \right| < \frac{s}{1-s}, \quad \text{if} \quad m \neq 0, \quad |t| < t_0. \tag{15}$$

This implies that the remainder term is bounded by the term $s^{I+1}/(1-s)$. This is consistent with iterating ad infinitum, which gives $s^{I+1} + s^{I+2} + \cdots = s^{I+1}/(1-s)$. It is also consistent with the fact that each correlation function can have at most poles given by the complex Yang–Lee zeros of Z_N.

As the bounds are independent of N, we have shown absolute and uniform convergence of the high-temperature series for any correlation function with finite radius of convergence bounded below by expression (13).

3.5. Uniform convergence for infinite N

As a consequence, $\langle \prod_{i=1}^{m} \sigma_{j_i} \rangle_N$ converges to a unique limit as $N \to \infty$ for $|t| < 2 - \sqrt{3}$, defined by its series. Let d be the largest edge of the minimal parallelepiped containing all sites $j_1, \ldots, j_m$. Then the coefficient of t^k with $k < n - d$, for the cubic lattice with $N = n^3$ sites and periodic boundary conditions, equals the corresponding coefficient for larger N. It takes at least $n - d$ iteration steps to notice the finite size of the lattice. Increasing N by one step makes one more coefficient independent of N. As $N \to \infty$ all coefficients take their

thermodynamic limit value and the remainder term tends to zero uniformly, according to the previous subsection.

3.6. Theorem for reduced free energy and its thermodynamic limit

The reduced free energy βf_N for arbitrary N and its thermodynamic limit βf are analytic in βJ for sufficiently high temperatures. They have series expansions in t or βJ with radius of convergence bounded below by expression (13) and uniformly convergent for all N including $N=\infty$. The first $n-1$ coefficients of these series for $N=n^3$ equal their limiting values for $N=\infty$.

This is easily proved using

$$u_N = \frac{1}{N}\langle \mathcal{H}_N\rangle_N = \frac{\partial(\beta f_N)}{\partial\beta} = -3J\langle \sigma_{0,0,0}\sigma_{1,0,0}\rangle_N, \qquad (16)$$

with $\sigma_{0,0,0}$ and $\sigma_{1,0,0}$ a nearest-neighbor pair of spins. The proof then follows from a well-known theorem in complex calculus integrating the series for u_N and using $Z_N|_{\beta=0} = 2^N$, implying $\lim_{\beta\to 0}\beta f_N = -\log 2$.

4. The first nontrivial coefficient

Applying Suzuki's lemma once to $\langle\sigma_{0,0,0}\sigma_{1,0,0}\rangle$, we find

$$\langle\sigma_{0,0,0}\sigma_{1,0,0}\rangle = a_1\langle 1\rangle + O(t^2) = t + O(t^2) = \beta J + O(\beta^2). \quad (17)$$

Hence,

$$\begin{aligned}\beta f &= -\log 2 - \int_0^\beta 3J\langle\sigma_{0,0,0}\sigma_{1,0,0}\rangle \mathrm{d}\beta \\ &= -\log 2 - \frac{3}{2}(\beta J)^2 + O(\beta^3). \end{aligned} \qquad (18)$$

This and the next few terms so obtained agree with the usual series expansion (2), but disagree with Zhang's "putative" exact result.

The only possible conclusion is that the conjectured answers of Zhang[1,5] are wrong and we shall next see more reasons why.

5. No obvious choice of weight functions

One problem with conjecture 2 (Refs. [1] and [5]), is that there is no obvious choice for the weight functions. In Eq. (49) on page 5325 of Ref. [1] Zhang writes

$$\begin{aligned} N^{-1}\ln Z = \ln 2 + \frac{1}{2(2\pi)^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \ln\big[\cosh 2K \\ \times \cosh 2(K'+K''+K''') - \sinh 2K\cos\omega' \\ - \sinh 2(K'+K''+K''')(w_x\cos\omega_x + w_y\cos\omega_y \\ + w_z\cos\omega_z)\big]\,\mathrm{d}\omega'\,\mathrm{d}\omega_x\,\mathrm{d}\omega_y\,\mathrm{d}\omega_z, \end{aligned} \qquad (19)$$

with $K = K' = K'' = K'''$ for the isotropic cubic lattice.

Even though this is a result of flawed assumptions, it is an integral transform that could give the right answer with a suitable choice of weight functions w_x, w_y, w_z. Zhang's wrong "putative" result comes from the choice $w_x = 1$, $w_y = w_z = 0$. In Appendix A of Ref. [1] Zhang reversely engineers an other (truncated) series choice derived from known coefficients of the Domb–Sykes high-temperature series. There is no more information than is in the limited series results provided by others, so that this is not an exact result.

6. Conjecture 1 is manifestly wrong

The original paper[1] has an error in the application of the Jordan–Wigner transformation pointed out in Ref. [3]. This error has only been corrected explicitly in Ref. [5], which makes it easy to pinpoint the error with Conjecture 1: Zhang and March violate the property that Lie groups are closed under commutation and that the product of fermionic Gaussians is another fermionic Gaussian.[14]

It is well known that in the spinor representation of the orthogonal groups each element g can be written as a "fermionic Gaussian" of the form

$$g = \exp\left(\frac{1}{2}\sum_i\sum_j A_{ij}\boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j\right), \quad A_{ji} = -A_{ij}, \qquad (20)$$

with Clifford algebra elements satisfying $\boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j + \boldsymbol{\Gamma}_j\boldsymbol{\Gamma}_i = 2\delta_{ij}$ and antisymmetric complex coefficients A_{ij}. The spinor representation has been used in the Ising context first by Kaufman[15] in 1949.

The closure property of Lie groups and the Baker–Campbell–Hausdorff formula require that any product, commutator or inverse of elements of this form is again of the same fermionic Gaussian form. Equivalently, the sum, commutator, or inverse of Lie algebra elements can only produce Lie algebra elements.

6.1. Remark: Lie group and Lie algebra

The group elements g act on the $\boldsymbol{\Gamma}$'s as

$$\boldsymbol{\Gamma}_k \longrightarrow g\boldsymbol{\Gamma}_k g^{-1}. \qquad (21)$$

Choosing infinitesimal

$$g = \mathbf{1} + \frac{\varepsilon}{2}\sum_i\sum_j A_{ij}\boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j + O(\varepsilon^2), \qquad (22)$$

we find from Eq. (22) in $O(\varepsilon^2)$, that the corresponding Lie algebra action is

$$\left[\frac{1}{2}\sum_i\sum_j A_{ij}\boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_j\,,\,\boldsymbol{\Gamma}_k\right] = \sum_i A_{ik}\boldsymbol{\Gamma}_i\,, \qquad (23)$$

showing that the infinitesimal action is through multiplication by antisymmetric matrices, the generators of rotations.

Therefore, the g's indeed form a representation of a rotation group.

6.2. Transfer matrix

It is easily proved that the free energy per site of the ferromagnetic Ising model in the thermodynamic limit does not depend on boundary conditions. Therefore, the Hamiltonian (1)

in Ref. [1] can be rewritten using the scew boundary conditions of Kramers and Wannier[16] as

$$-\beta\hat{H}=\sum_{\tau=1}^{n}\sum_{j=1}^{ml}\left[Ks_j^{(\tau)}s_j^{(\tau+1)}+K's_j^{(\tau)}s_{j+1}^{(\tau)}+K''s_j^{(\tau)}s_{j+m}^{(\tau)}\right]. \quad (24)$$

For this purpose we have made the change of notation

$$s_{\rho,\delta}^{(\tau)}\equiv s_j^{(\tau)}, \quad j\equiv\rho+(m-1)\delta, \quad \sum_{\rho=1}^{m}\sum_{\delta=1}^{l}=\sum_{j=1}^{ml}, \quad (25)$$

where $\tau=1,\ldots,n$; $\rho=1,\ldots,m$; $\delta=1,\ldots,l$.

This leads to the transfer matrix $\boldsymbol{T}=\boldsymbol{V}_3\boldsymbol{V}_2\boldsymbol{V}_1$ in Ref. [5], with

$$\boldsymbol{V}_3=\exp\left(-\mathrm{i}K''\sum_{j=1}^{ml}\boldsymbol{\Gamma}_{2j}\left[\prod_{k=j+1}^{j+m-1}\mathrm{i}\boldsymbol{\Gamma}_{2k-1}\boldsymbol{\Gamma}_{2k}\right]\boldsymbol{\Gamma}_{2j+2m-1}\right), \quad (26)$$

$$\boldsymbol{V}_2=\exp\left(-\mathrm{i}K'\sum_{j=1}^{ml}\boldsymbol{\Gamma}_{2j}\boldsymbol{\Gamma}_{2j+1}\right),$$

$$\boldsymbol{V}_1=\exp\left(\mathrm{i}K^*\sum_{j=1}^{ml}\boldsymbol{\Gamma}_{2j-1}\boldsymbol{\Gamma}_{2j}\right), \quad (27)$$

compare Eqs. (15), (16), and (17) of Ref. [5], with n replaced by m. Clearly, equation (26) is not of the fermionic Gaussian form and, therefore, not an element of the group.

At this point, Zhang introduces a fourth dimension, stacking o copies of the model. Without changing the free energy per site in the large system limit, one can connect the copies to give Eqs. (26) and (27) with the upper bounds of the sums ml replaced by mlo.

Next, Zhang made the absurd conjecture that multipying $\boldsymbol{V}_3'\boldsymbol{V}_2'\boldsymbol{V}_1'$ so obtained by

$$\boldsymbol{V}_4'=\exp\left\{-\mathrm{i}K'''\sum_{j=1}^{mlo}\boldsymbol{\Gamma}_{2j}\boldsymbol{\Gamma}_{2j+1}\right\}, \quad K'''=\frac{K'K''}{K}, \quad (28)$$

as given in Eqs. (18) and (19) in Ref. [5], miraculously produces a rotation group element. This is in violation of the Baker–Campbell–Hausdorff formula. The argumentation in Ref. [1] for the form of K''' is ad hoc and also makes no sense.

7. Some other issues out of many more

Zhang and March also falsely claim that setting $\beta=1$ in 1960's references is a loss of generality, losing $T=\infty$. On the contrary,[17] as βf is only a function of $K=\beta J$, this is no problem. Having $J\equiv K$ and choosing a fixed new $\bar{J}$ and a new $\bar{\beta}=J/\bar{J}\not\equiv 1$, we can write $J=K=\bar{\beta}\bar{J}$, recovering the full two-variable case depending on β (including $\beta=0$) and a new J (omitting the bars).[14]

Next, Zhang and March fail to realize that $K_{\beta\phi'}(X,T)$ in Ref. [18] vanishes for $\beta=0$. The cited inequality does not fail for $\beta=0$.[14]

Also, the three-dimensional Virasoro algebra approach in Refs. [5] and [19] is based on an erroneous solution of the 3D Ising model and twice writing $\mathrm{Re}|\mathrm{e}^{\mathrm{i}\phi_i}|$, the real part of a positive real number [Ref. [19], pages 39 and 40] is another error.[14]

8. Conclusion

In conclusion, much of what Zhang and March wrote about the three-dimensional Ising model is either misleading or completely wrong. All their main results are in error. It is said in Ref. [5] that everything is based on two conjectures. This is also false, as a careful reading of Ref. [1] reveals that many steps there lack mathematical logic and should be considered unfounded assumptions. However, there should be no need to go through these other issues in lengthy detail, after all that is already said.

References

[1] Zhang Z D 2007 *Philos. Mag.* **87** 5309, arXiv: 0705.1045
[2] Wu F Y, McCoy B M, Fisher M E and Chayes L 2008 *Philos. Mag.* **88** 3093, arXiv: 0811.3876
[3] Perk J H H 2009 *Philos. Mag.* **89** 761, arXiv: 0811.1802
[4] Zhang Z D 2008 *Philos. Mag.* **88** 3097, arXiv: 0812.2330
[5] Zhang Z D 2013 *Chin. Phys. B* **22** 030513, arXiv:1305.2956
[6] Domb C and Green M S editors 1974 *Series Expansions for Lattice Models, Phase Transitions and Critical Phenomena* Vol. 3 (London: Academic Press)
[7] Zhang Z D and March N H 2012 *Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform.* **62**:3 61, arXiv: 1209.3247
[8] Zhang Z D and March N H 2013 *Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform.* **63**:1 85, arXiv: 1209.3247
[9] Groeneveld J 1962 *Phys. Lett.* **3** 50
[10] Ruelle D 1969 *Statistical Mechanics, Rigorous Results* (New York: Benjamin)
[11] Perk J H H 2012 *Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform.* **62**:3 45, arXiv: 1209.0731
[12] Suzuki M 1965 *Phys. Lett.* **19** 267
[13] Suzuki M 2002 *Int. J. Mod. Phys. B* **16** 1749
[14] Perk J H H 2013 *Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform.* **63**:1 89, arXiv: 1209.0731
[15] Kaufman B 1949 *Phys. Rev.* **76** 1232
[16] Kramers H A and Wannier G H 1941 *Phys. Rev.* **60** 252
[17] Perk J H H 2012 *Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform.* **62**:3 71, arXiv: 1209.0731
[18] Gallavotti G, Miracle-Solé S and Robinson D W 1967 *Phys. Lett. A* **25** 493
[19] Zhang Z D and March N H 2012 *Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform.* **62**:3 35, arXiv: 1110.5527