# Adaptive lag synchronization of uncertain dynamical systems with time delays via simple transmission lag feedback *

Gu Wei-Dong(顾卫东), Sun Zhi-Yong(孙志勇)†, Wu Xiao-Ming(吴晓明), and Yu Chang-Bin(于长斌)

*Shandong Provincial Key Laboratory of Computer Network, Shandong Computer Science Center, Jinan* 250014, *China*

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In this paper we present an adaptive scheme to achieve lag synchronization for uncertain dynamical systems with time delays and unknown parameters. In contrast to the nonlinear feedback scheme reported in the previous literature, the proposed controller is a linear one which only involves simple feedback information from the drive system with signal propagation lags. Besides, the unknown parameters can also be identified via the proposed updating laws in spite of the existence of model delays and transmission lags, as long as the linear independence condition between the related function elements is satisfied. Two examples, i.e., the Mackey–Glass model with single delay and the Lorenz system with multiple delays, are employed to show the effectiveness of this approach. Some robustness issues are also discussed, which shows that the proposed scheme is quite robust in switching and noisy environment.

**Keywords:** adaptive feedback, time delay, lag synchronization, parameter identification, noise perturbation

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## 1. Introduction

Synchronization is a common and widespread phenomenon in nature and it also plays an important role in many engineering fields.[1] The synchronization control of dynamical systems has been a hot topic which attracted growing attention during the last two decades due to its numerous potential applications in many areas. Up to now, many types of synchronization notations have been observed and developed. The standard form is called complete synchronization, which means that the states in the response system track the exact trajectories of the driving states. In fact, the synchronous motion that Huygens observed in his famous clock experiment is a typical example of complete synchronization.[2] Some recent results on complete synchronization include the controller design of a noise-perturbed hyperchaotic system[3] and the synchronization control of stochastic dynamical networks.[4] Other synchronization forms include phase synchronization, lag synchronization, generalized synchronization, etc.[5] Amongst various kinds of synchronizations, lag synchronization, first proposed in Ref. [6], is of particular interest. Due to the presence of signal propagation delays, it is not reasonable to assume that the states in the response system should synchronize with the driving signals at exactly the same time. Hence, in designing a synchronization scheme, the propagation lags should be taken into consideration. In such a case, the response state $y(t)$ is expected to synchronize with the driving signal with certain transmission lag $\tau$, that is, the synchronization goal takes the form as $y(t) \to x(t-\tau)$. In recent years, several results have been reported on this lag-synchronization design problem. In Ref. [7] the authors proposed a single-controller-based method to achieve the lag synchronization between two identical Chen systems. In Ref. [8], the lag synchronization for a unified chaotic system was investigated. In Ref. [9], Lü and Li discussed the lag synchronization problem in uncertain autocatalytic reaction–diffusion chaotic systems. A recent article by Liu *et al.*[10] studied the synchronization design in a multi-scroll chaotic system with unknown disturbance by using the transmission lag information. Moreover, in recent years, several novel methods such as adaptive approach,[11] noise-induced coupling method,[12] and active sliding mode control[13] have been introduced in the controller design for lag synchronization.

Model uncertainties such as unknown parameters and system delays are ubiquitous in natural and synthetic systems. Typical examples include delayed optical system,[14] neuron delay models,[15] complex networks with node delays or coupling delays,[16] etc. It is well known that uncertainties and time delays are regarded as two of the main reasons for causing system instability or degenerating controller performance. Hence, they should also be considered in the controller design for lag synchronization. In the literature, there are already some attempts towards this direction. For example, Shahverdiev *et al.*[17] discussed the lag synchronization problem between time-delayed systems by showing some specific examples. Later, Li *et al.*[18] reported on a detailed investigation of the lag synchronization of time-delayed chaotic systems and further discussed its application in secure communication. The simultaneous task of achieving lag synchroniza-

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†Corresponding author. E-mail: kfmuzik@126.com, sun.zhiyong.cn@gmail.com, sunzhy@sdas.org

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tion and parameter identification in dynamical systems without delays has been discussed in Refs. [19] and [20] for uncertain integer-order systems and in Ref. [21] for fractional-order systems, respectively. However, if transmission lags, system delays, and parameter uncertainties are all taken into consideration, the design problem becomes even more challenging. This topic has been slightly covered in only a few previous articles.[22,23] Yu and Cao[22] proposed a nonlinear feedback controller to achieve the lag synchronization between delayed systems with unknown model parameters. Later this nonlinear approach has been extended in a recent paper[23] for the controller design of non-identical delayed systems. However, the synchronization schemes studied in Refs. [22] and [23] are based on the active control concept. As a result, the designed controllers take a complicated form. In view of engineering applications, a complicated controller is not favored as it is hard for practical implementation and the controller complexity may also result in vulnerable performance. Motivated by the above discussion, in this paper we propose a simple and effective approach to achieving lag synchronization between coupled dynamical systems with both time delays and unknown parameters. In contrast to Refs. [22] and [23], it is assumed in this paper that the only information about constructing the synchronization controller is the lag-transmitted signal from the drive system, while the control goal is to simultaneously achieve the lag synchronization and parameter estimations as well.

The rest of this paper is organized as follows. In Section 2, we introduce the model description and problem formulation together with some essential assumptions and lemmas. In Section 3, a general and simple method is proposed for achieving the desired lag synchronization in the presence of time delays, unknown parameters, and transmission lags. The theoretical result is proved by Lyapunov–Krasovskii functional method and LaSalle's Invariant Theorem. In addition, the sufficient and necessary condition for successful parameter estimation has also been derived. In Section 4 we present two examples to illustrate the effectiveness of the proposed method. In Section 5, we further discuss some robustness issues of the proposed scheme. We show, by detailed analysis and several numerical examples, that our result is simpler than those in previous articles and is more robust against noisy perturbations. Finally, in Section 6 we draw some conclusions from the present study.

## 2. Model description and preliminaries

Consider the following general expression of a dynamical system with unknown parameters and multiple time delays:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \bar{\boldsymbol{f}}(\boldsymbol{x}(t),\boldsymbol{x}(t-\boldsymbol{d}_1))+\bar{\boldsymbol{F}}(\boldsymbol{x}(t),\boldsymbol{x}(t-\boldsymbol{d}_2),\boldsymbol{\theta})\\ &= \boldsymbol{f}_1(\boldsymbol{x}(t))+\boldsymbol{f}_2(\boldsymbol{x}(t-\boldsymbol{d}_1))+\boldsymbol{F}_1(x(t))\boldsymbol{\theta}_1\\ &\quad +\boldsymbol{F}_2(\boldsymbol{x}(t-\boldsymbol{d}_2))\boldsymbol{\theta}_2, \qquad (1)\end{aligned}$$

where $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}\in R^n$ is the state vector, $\boldsymbol{f}_1,\boldsymbol{f}_2: R^n\to R^n$ are the continuous vector functions, $\boldsymbol{F}_1: R^n\to R^{n\times p}$ and $\boldsymbol{F}_2: R^n\to R^{n\times q}$ are the function matrices associated with the system parameters, $\boldsymbol{\theta}_1\in R^p$ and $\boldsymbol{\theta}_2\in R^q$ are the constant vectors denoting unknown system parameters which need to be identified, $\boldsymbol{d}_1,\boldsymbol{d}_2>0$ denote the time delays of system (1). In this paper, we consider constant delays. We should note that the system model described by Eq. (1) is quite general, which includes many well-known time-delayed dynamical systems or chaotic systems.

According to the drive-response configuration of the synchronization design, the response system is given as follows:

$$\begin{aligned}\dot{\boldsymbol{y}}(t) &= \boldsymbol{f}_1(\boldsymbol{y}(t))+\boldsymbol{f}_2(\boldsymbol{y}(t-\boldsymbol{d}_1))+\boldsymbol{F}_1(\boldsymbol{y}(t))\hat{\boldsymbol{\theta}}_1\\ &\quad +\boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2))\hat{\boldsymbol{\theta}}_2+\boldsymbol{u}, \qquad (2)\end{aligned}$$

where $\boldsymbol{y}=(y_1,y_2,\ldots,y_n)^{\mathrm{T}}\in R^n$ is the state vector in the response system, $\hat{\boldsymbol{\theta}}_1$ and $\hat{\boldsymbol{\theta}}_2$ are the time-varying estimates of the unknown parameters $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, respectively, $\boldsymbol{u}$ is the control input vector to be designed.

Due to the existence of transmission lags, the information received by the response system is $\boldsymbol{x}(t-\tau)$, a state at the past time $t-\tau$. According to the concept of lag synchronization, the synchronization error is defined as $\boldsymbol{e}(t)=\boldsymbol{y}(t)-\boldsymbol{x}(t-\tau)$. The problem is to design the control input $u$ and parameter estimation laws $\dot{\hat{\boldsymbol{\theta}}}_1$ and $\dot{\hat{\boldsymbol{\theta}}}_2$ such that $\boldsymbol{e}(t)\to 0$, $\hat{\boldsymbol{\theta}}_1(t)\to\boldsymbol{\theta}_1, \hat{\boldsymbol{\theta}}_2(t)\to\boldsymbol{\theta}_2$ for $t\to\infty$. In contrast to Refs. [22] and [23], we assume that the only information that can be used to construct the controllers $u$ and updating laws is the driving signals with some constant transmission lags $\tau$. In other words, the aim is to design a simple feedback controller scheme by using only transmission lag information $\boldsymbol{x}(t-\tau)$ but without seeking the help of other driving information in terms of nonlinear functions such as $\boldsymbol{f}_1(\boldsymbol{x}(t-\tau))$, $\boldsymbol{f}_2(\boldsymbol{x}(t-\boldsymbol{d}_1-\tau))$, etc.

Before presenting the main result of this paper, we first introduce the following useful assumptions and Lemmas.[24–28]

**Assumption 1** (A1, see Refs. [24] and [25]) There exist nonnegative constants $M$ and $N$ such that

$$\begin{aligned}&||\bar{\boldsymbol{f}}(\boldsymbol{x}(t),\boldsymbol{x}(t-\boldsymbol{d}_1))-\bar{\boldsymbol{f}}(\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_1))||^2\\ &\le M(||\boldsymbol{x}(t)-\boldsymbol{y}(t)||^2+||\boldsymbol{x}(t-\boldsymbol{d}_1)-\boldsymbol{y}(t-\boldsymbol{d}_1)||^2), \qquad (3)\end{aligned}$$

and

$$\begin{aligned}&||\bar{\boldsymbol{F}}(\boldsymbol{x}(t),\boldsymbol{x}(t-\boldsymbol{d}_2),\boldsymbol{\theta})-\bar{\boldsymbol{F}}(\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_2),\boldsymbol{\theta})||^2\\ &\le N(||\boldsymbol{x}(t)-\boldsymbol{y}(t)||^2+||\boldsymbol{x}(t-\boldsymbol{d}_2)-\boldsymbol{y}(t-\boldsymbol{d}_2)||^2), \qquad (4)\end{aligned}$$

where the norm $||\cdot||$ of vector $x$ is defined as $||x||=(x^{\mathrm{T}}x)^{1/2}$. This assumption is actually based on the commonly used Lipchitz condition, but it is more convenient in practical usage.

**Assumption 2** (A2, see Refs. [24] and [26]) Denote that

$$\begin{cases} \boldsymbol{F}_1(\boldsymbol{y}(t))=\{\boldsymbol{F}_1^{(1)}(\boldsymbol{y}(t)),\boldsymbol{F}_1^{(2)}(\boldsymbol{y}(t)),\ldots,\boldsymbol{F}_1^{(p)}(\boldsymbol{y}(t))\}, \\ \boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2))=\{\boldsymbol{F}_2^{(1)}(\boldsymbol{y}(t-\boldsymbol{d}_2)), \\ \quad \boldsymbol{F}_2^{(2)}(\boldsymbol{y}(t-\boldsymbol{d}_2)),\ldots,\boldsymbol{F}_2^{(q)}(\boldsymbol{y}(t-\boldsymbol{d}_2))\}, \end{cases} \tag{5}$$

where $\boldsymbol{F}_1^{(i)}(\boldsymbol{y}(t))|_{i=1,2,\ldots p} \in R^n$ and $\boldsymbol{F}_2^{(j)}(\boldsymbol{y}(t-\boldsymbol{d}_2))|_{j=1,2,\ldots,q} \in R^n$. Assume that $\boldsymbol{F}_1^{(i)}(\boldsymbol{y}(t))$ and $\boldsymbol{F}_2^{(j)}(\boldsymbol{y}(t-\boldsymbol{d}_2))$ are linearly independent of the orbit $\{\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_2)\}$.

**Lemma 1** (see Ref. [27])

For any vectors $x,y \in R^n$, the matrix inequality $2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} \le \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}$ holds.

**Lemma 2** (see Ref. [28])

If $f,\dot{f} \in L_\infty$, and $f \in L_p$ for some $p \in [1,\infty)$, then $f(t) \to 0$ as $t \to \infty$.

The result of Lemma 2 is essentially a special case of the well-known Barbalat's Lemma.

## 3. Main result

According to the control goal of lag synchronization, the error dynamical system is obtained as

$$\begin{aligned} \dot{\boldsymbol{e}} &= \dot{\boldsymbol{y}}(t)-\dot{\boldsymbol{x}}(t-\tau) \\ &= \boldsymbol{f}_1(\boldsymbol{y}(t))+\boldsymbol{f}_2(\boldsymbol{y}(t-\boldsymbol{d}_1))+\boldsymbol{F}_1(y(t))\hat{\boldsymbol{\theta}}_1+\boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2))\hat{\boldsymbol{\theta}}_2 \\ &\quad +\boldsymbol{u}-(\boldsymbol{f}_1(\boldsymbol{x}(t-\tau))+\boldsymbol{f}_2(\boldsymbol{x}(t-\boldsymbol{d}_1-\tau)) \\ &\quad +\boldsymbol{F}_1(\boldsymbol{x}(t-\tau))\boldsymbol{\theta}_1+\boldsymbol{F}_2(\boldsymbol{x}(t-\boldsymbol{d}_2-\tau))\boldsymbol{\theta}_2). \end{aligned} \tag{6}$$

The main challenge lies in how to design effective controller $\boldsymbol{u}$ with simple feedback form to achieve the synchronization and identification in the presence of system delays and transmission lags. In this section, based on the adaptive control concept[28] and linear feedback approach, we propose the following simple but feasible feedback scheme.

**Theorem 1** Drive system (1) with multi-time delays and unknown parameters will synchronize with response system (2), if one designs the following simple feedback controller, adaptive feedback gains, and parameter updating laws:

$$\begin{cases} \boldsymbol{u}=-k\boldsymbol{e}(t)=-\boldsymbol{k}(\boldsymbol{y}(t)-\boldsymbol{x}(t-\tau)), \\ \boldsymbol{k}=\mathrm{diag}(k_1,k_2,\ldots,k_n), \quad \dot{k}_i=\delta e_i^2, \end{cases} \tag{7a}$$

$$\begin{cases} \dot{\hat{\boldsymbol{\theta}}}_1=-\alpha\boldsymbol{F}_1^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{e}, \\ \dot{\hat{\boldsymbol{\theta}}}_2=-\beta\boldsymbol{F}_2^{\mathrm{T}}(\boldsymbol{y}(t-\boldsymbol{d}_2))\boldsymbol{e}, \end{cases} \tag{7b}$$

where $\delta$, $\alpha$, and $\beta$ are arbitrary positive constants. Furthermore, if Assumption A2 holds, then the unknown parameters can also be successfully identified by using only the transmission lag information $\boldsymbol{x}(t-\tau)$.

**Proof** The proof of the above result is partially based on the stability theorem of time-delayed dynamical system, the Barbalat's-like lemma (Lemma 2) and the LaSalle's invariant principle. Denote the parameter estimation errors as $\tilde{\boldsymbol{\theta}}_1=\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_1$ and $\tilde{\boldsymbol{\theta}}_2=\hat{\boldsymbol{\theta}}_2-\boldsymbol{\theta}_2$. To start with, we construct the following Lyapunov–Krasovskii functional:

$$\begin{aligned} &2V(t,\boldsymbol{e}(t)) \\ &= \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)+\sum_{i=1}^{n}\frac{1}{\delta}(k_i-k^*)^2+\frac{1}{\alpha}\tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_1+\frac{1}{\beta}\tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_2 \\ &\quad +M\int_{t-\boldsymbol{d}_1}^{t}\boldsymbol{e}^{\mathrm{T}}(s)\boldsymbol{e}(s)\mathrm{d}s+N\int_{t-\boldsymbol{d}_2}^{t}\boldsymbol{e}^{\mathrm{T}}(s)\boldsymbol{e}(s)\mathrm{d}s, \end{aligned} \tag{8}$$

where $k^*$ is a sufficiently large positive number which will be determined later in order to ensure the negative semi-definiteness of $\dot{V}$. By differentiating Eq. (8) with respect to time along the trajectory of error dynamics (6), one has

$$\begin{aligned} &\dot{V}(t,\boldsymbol{e}(t)) \\ &= \boldsymbol{e}^{\mathrm{T}}(t)\dot{\boldsymbol{e}}(t)+\sum_{i=1}^{n}\frac{1}{\delta}(k_i-k^*)\dot{k}_i+\frac{1}{\alpha}\tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}\dot{\tilde{\boldsymbol{\theta}}}_1+\frac{1}{\beta}\tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}\dot{\tilde{\boldsymbol{\theta}}}_2 \\ &\quad +\frac{M}{2}\{\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)-\boldsymbol{e}^{\mathrm{T}}(t-\boldsymbol{d}_1)\boldsymbol{e}(t-\boldsymbol{d}_1)\} \\ &\quad +\frac{N}{2}\{\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)-\boldsymbol{e}^{\mathrm{T}}(t-\boldsymbol{d}_2)\boldsymbol{e}(t-\boldsymbol{d}_2)\} \\ &= \boldsymbol{e}^{\mathrm{T}}(t)\{\bar{\boldsymbol{f}}(\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_1))-\bar{\boldsymbol{f}}(\boldsymbol{x}(t-\tau),\boldsymbol{x}(t-\boldsymbol{d}_1-\tau))\} \\ &\quad +\boldsymbol{e}^{\mathrm{T}}(t)\{\boldsymbol{F}_1(\boldsymbol{y}(t))\hat{\boldsymbol{\theta}}_1-\boldsymbol{F}_1(\boldsymbol{x}(t-\tau))\boldsymbol{\theta}_1 \\ &\quad +\boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2))\hat{\boldsymbol{\theta}}_2-\boldsymbol{F}_2(\boldsymbol{x}(t-\boldsymbol{d}_2-\tau))\boldsymbol{\theta}_2\} \\ &\quad -\left(k^*-\frac{M}{2}-\frac{N}{2}\right)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)+\tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}(-\boldsymbol{F}_1^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{e}) \\ &\quad +\tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}(-\boldsymbol{F}_2^{\mathrm{T}}(\boldsymbol{y}(t-\boldsymbol{d}_2))\boldsymbol{e}) \\ &\quad -\frac{M}{2}\boldsymbol{e}^{\mathrm{T}}(t-\boldsymbol{d}_1)\boldsymbol{e}(t-\boldsymbol{d}_1)-\frac{N}{2}\boldsymbol{e}^{\mathrm{T}}(t-\boldsymbol{d}_2)\boldsymbol{e}(t-\boldsymbol{d}_2). \end{aligned} \tag{9}$$

Notice that

$$\begin{aligned} &\boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}_1(\boldsymbol{y}(t))\hat{\boldsymbol{\theta}}_1+\tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}(-\boldsymbol{F}_1^{\mathrm{T}}(\boldsymbol{y}(t))\boldsymbol{e}) \\ &= \boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}_1(\boldsymbol{y}(t))\hat{\boldsymbol{\theta}}_1-\boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}_1(\boldsymbol{y}(t))\tilde{\boldsymbol{\theta}}_1 \\ &= \boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}_1(\boldsymbol{y}(t))\boldsymbol{\theta}_1 \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2))\hat{\boldsymbol{\theta}}_2+\tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}(-\boldsymbol{F}_2^{\mathrm{T}}(\boldsymbol{y}(t-\boldsymbol{d}_2))\boldsymbol{e} \\ &= \boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2))\hat{\boldsymbol{\theta}}_2-\boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2)\tilde{\boldsymbol{\theta}}_2 \\ &= \boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2))\boldsymbol{\theta}_2. \end{aligned} \tag{11}$$

Thus the expression of $\dot{V}(t,\boldsymbol{e}(t))$ can be simplified as

$$\begin{aligned} &\dot{V}(t,\boldsymbol{e}(t)) \\ &= \boldsymbol{e}^{\mathrm{T}}(t)\{\bar{\boldsymbol{f}}(\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_1))-\bar{\boldsymbol{f}}(\boldsymbol{x}(t-\tau),\boldsymbol{x}(t-\boldsymbol{d}_1-\tau))\} \\ &\quad +\boldsymbol{e}^{\mathrm{T}}(t)\{\bar{\boldsymbol{F}}(\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_2),\boldsymbol{\theta}) \\ &\quad -\bar{\boldsymbol{F}}(\boldsymbol{x}(t-\tau),\boldsymbol{x}(t-\boldsymbol{d}_2-\tau),\boldsymbol{\theta})\} \\ &\quad -\left(k^*-\frac{M}{2}-\frac{N}{2}\right)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) \\ &\quad -\frac{M}{2}\boldsymbol{e}^{\mathrm{T}}(t-\boldsymbol{d}_1)\boldsymbol{e}(t-\boldsymbol{d}_1)-\frac{N}{2}\boldsymbol{e}^{\mathrm{T}}(t-\boldsymbol{d}_2)\boldsymbol{e}(t-\boldsymbol{d}_2). \end{aligned} \tag{12}$$

According to Lemma 1 and Assumption A1, one has

$$\boldsymbol{e}^{\mathrm{T}}(t)\{\bar{\boldsymbol{f}}(\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_1))-\bar{\boldsymbol{f}}(x(t-\tau),\boldsymbol{x}(t-\boldsymbol{d}_1-\tau))\}$$

$$\leq \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)+\frac{1}{2}||\bar{\boldsymbol{f}}(\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_1)) - \bar{\boldsymbol{f}}(\boldsymbol{x}(t-\tau),\boldsymbol{x}(t-\boldsymbol{d}_1-\tau))||^2$$
$$\leq \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)+\frac{M}{2}(||\boldsymbol{e}(t)||^2+||\boldsymbol{e}(t-\boldsymbol{d}_1)||^2) \quad (13)$$

and similarly

$$\boldsymbol{e}^{\mathrm{T}}(t)\{\bar{\boldsymbol{F}}(\boldsymbol{y}(t),\boldsymbol{y}(t-\boldsymbol{d}_2),\boldsymbol{\theta}) - \bar{F}(\boldsymbol{x}(t-\tau),\boldsymbol{x}(t-\boldsymbol{d}_1-\tau),\boldsymbol{\theta})\}$$
$$\leq \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)+\frac{N}{2}(||\boldsymbol{e}(t)||^2+||\boldsymbol{e}(t-\boldsymbol{d}_2)||^2). \quad (14)$$

By inserting the above inequalities into Eq. (12), one can obtain

$$\dot{V}(t,\boldsymbol{e}(t)) \leq -(k^*-M-N-1)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t). \quad (15)$$

Obviously by choosing $k^* > M+N+2$, we can obtain the following inequality $\dot{V}(t,\boldsymbol{e}(t)) \leq -\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)$. Hence we have $\sup_{t\geq 0} V(t) \leq V(0)$. Since $V(0)$ is bounded, $V(t)$ is also bounded. From $\dot{V}(t,\boldsymbol{e}(t)) \leq -\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)$, one further has

$$\int_0^t (\boldsymbol{e}^{\mathrm{T}}(s)\boldsymbol{e}(s))\,\mathrm{d}s \leq -\int_0^t \dot{V}(s,\boldsymbol{e}(s))\,\mathrm{d}s = V(0)-V(t) < V(0). \quad (16)$$

The above inequality (16) indicates that $\boldsymbol{e}(t) \in L_2$. From Eq. (8), the boundedness of $V(t)$ also implies the boundedness of $\boldsymbol{e}(t)$. That is, $\boldsymbol{e}(t) \in L_2 \cap L_\infty$. From the error system (6) one knows that $\dot{\boldsymbol{e}}(t)$ exists and is also bounded for $t \in [0,+\infty)$. Then according to Lemma 2, one concludes that $\boldsymbol{e}(t) \to 0$ as $t \to \infty$. In addition, this in turn also implies that $\dot{\hat{\boldsymbol{\theta}}}_1(t), \dot{\hat{\boldsymbol{\theta}}}_2(t) \to 0$ as $t \to \infty$.

Moreover, since the dynamical model (1) and its delayed coupled counterpart (2), together with the adaptive controllers and updating laws (7a) and (7b), constitute the autonomous systems which have bounded trajectories as assumed above, the limit set for the convergence is nonempty and invariant. According to the LaSalle's invariant principle and Eq. (6), the largest invariant set $\Omega$ can be obtained as

$$\Omega = \{e \in R^n, \hat{\boldsymbol{\theta}}_1 \in R^p, \hat{\boldsymbol{\theta}}_1 \in R^q | e = 0, \boldsymbol{F}_1(\boldsymbol{y}(t))\tilde{\boldsymbol{\theta}}_1 + \boldsymbol{F}_2(\boldsymbol{y}(t-\boldsymbol{d}_2))\tilde{\boldsymbol{\theta}}_2 = 0\}. \quad (17)$$

From Assumption A2, the linear independence of the related function elements $\boldsymbol{F}_1^{(i)}(\boldsymbol{y}(t))|_{i=1,2,\ldots p}$ and $\boldsymbol{F}_2^{(j)}(\boldsymbol{y}(t-\boldsymbol{d}_2))|_{j=1,2,\ldots,q}$ will ensure that the unique solution to the equation described in set (17) is $\tilde{\boldsymbol{\theta}}_1 = 0$ and $\tilde{\boldsymbol{\theta}}_2 = 0$, i.e., the unknown parameters will be successfully identified. This completes the proof.

Several remarks and comments on the theoretical result are presented below.

**Remark 1** The negative semi-definiteness of $\dot{V}(t,\boldsymbol{e}(t))$ together with Lemma 2 ensures the asymptotical stability of the synchronization error dynamics (6). However, we cannot conclude that the unknown parameters would be safely estimated to obtain their true values. The correct identification of the unknown parameters is guaranteed by the linear independence condition from Assumption A2. In the literature, several methods are available for testing the linear independence of the function vectors, and we refer interested readers to Ref. [29] for a Gram matrix method and Ref. [30] for the persistent excitation concept. We should mention that in a recent article[25] which proposed a lag synchronization scheme for estimating the unknown topologies in complex networks, the proving steps are incomplete and insufficient. The authors in Ref. [25] misused the LaSalle's invariant principle and obtained an incorrect largest invariant set. Thus the important linear independence condition was neglected in that article.

**Remark 2** In Ref. [22], the authors studied a similar problem and proposed an effective nonlinear controller to achieve lag synchronization in uncertain time-delayed systems. However, the controller designed in Ref. [22] is rather complicated which contains several nonlinear terms such as $\boldsymbol{f}_1(\boldsymbol{x}(t-\tau))$, $\boldsymbol{F}_1(\boldsymbol{x}(t-\tau))$, and $\boldsymbol{F}_2(\boldsymbol{x}(t-\boldsymbol{d}_2-\tau))$. In the present paper, we propose a simple linear feedback controller which uses the sole knowledge of the driving information $\boldsymbol{x}(t-\tau)$. Some features and advantages of the proposed approach will be elaborated in Section 5. However, we should mention that the proposed scheme is only applicable between two systems with identical structures and the transmission lag here is assumed to be constant. For the lag synchronization design problem between two different systems, we recommend other schemes like the one studied in Section 3 of Ref. [23].

**Remark 3** In the proposed linear feedback controller (7a), the controlling gains can be adaptively adjusted to some optimal values without much energy consumed in the synchronization process. This means that we do not need to preset sufficiently large constant gains for the controller. In the above proof, the positive constant $k^*$ is chosen to be $k^* > M+N+2$ such that the negative semi-definiteness of the derivative of the Lyapunov functional is satisfied. One may argue that it is difficult to choose desired values for $k^*$ because it is a nontrivial task (or even quite a challenging task) to determine the parameters $M$ and $N$ in Assumption A1, especially in the case that the system parameters are unknown. However, we do not need to explicitly compute the values (or the upper bounds) for $M$ and $N$, as there always exist desired values for $k^*$ which satisfies the inequality. An obvious illustration for this feature can be found in Fig. 2(d) in Section 4. Furthermore, we will show that the adaptability of the feedback gains is beneficial in suppressing the adverse effect of noise. Our choice for the adaptive gain is partially inspired by Huang's series articles[31,32] in which the authors presented a detailed investigation on the adaptive mechanism of linear

feedback control for chaotic systems. Accordingly, the Lyapunov candidate given by Eq. (8) is a valid one which confirms the asymptotic convergence property of the augmented error dynamics described by Eqs. (6), (7a), and (7b).

**Remark 4** We note that the result in this paper is more general than those in several previous articles. For instance, if the dynamical systems (1) and (2) contain no time delays, then it becomes the parameter estimation and lag synchronization problem studied in Refs. [19] and [20]. Furthermore, if we suppose that system parameters are known in advance, then the problem degenerates into the one studied in Ref. [11]. The controllers designed in Refs. [11], [19], and [20] are actually the special cases of this article, and our result is still applicable to those special cases. In fact, this paper is devoted to investigating a more general problem, and we present quite a simple method.

## 4. Illustrative examples

In this section, we will present two typical examples to illustrate the effectiveness of the proposed method. Two systems are considered: one is the Mackey–Glass system with single time delay and the other is the Lorenz system with multiple time delays.

**Example 1** Consider the Mackey–Glass model, which is described by the following deferential delay equation:[33]

$$\dot{x} = \frac{\beta x(t-d)}{1+[x(t-d)]^n} - x, \tag{18}$$

where $d$ is the non-negative time delay and $\beta$ is the unknown parameter. The Mackey–Glass model is a typical differential delay system for describing the physiological dynamics in circulating blood cells. In the research area of physiology, the synchronous oscillatory and rhythmic process is quite common and significant. The Mackey–Glass model serves as a benchmark model for studying the time-delayed synchronization process of physiological dynamics. In the following, we choose $\beta = 2$, $n = 10$, and $d = 2$, which leads to the complex chaotic behavior as depicted in Fig. 1.

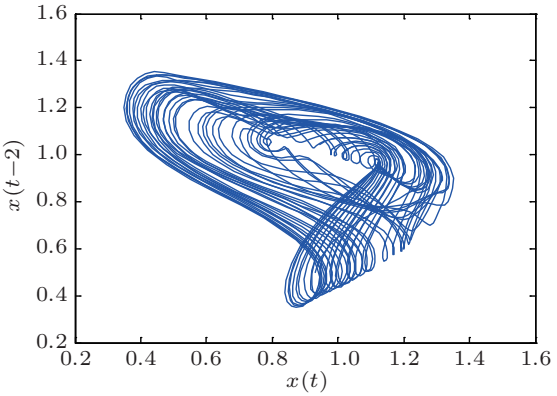



**Fig. 1.** (color online) Phase portrait of the Mackey–Glass model: $x(t-2)$ versus $x(t)$.

We construct the response system as

$$\dot{y} = \frac{\hat{\beta} y(t-d)}{1+[y(t-d)]^n} - y + u, \tag{19}$$

where the controller is designed as $u = -ke = -k(y(t) - x(t-\tau))$ with $\dot{k} = e^2$. The updating law for $\beta$ is constructed as $\dot{\hat{\beta}} = -y(t-d)e/\{1+[y(t-d)]^n\}$.

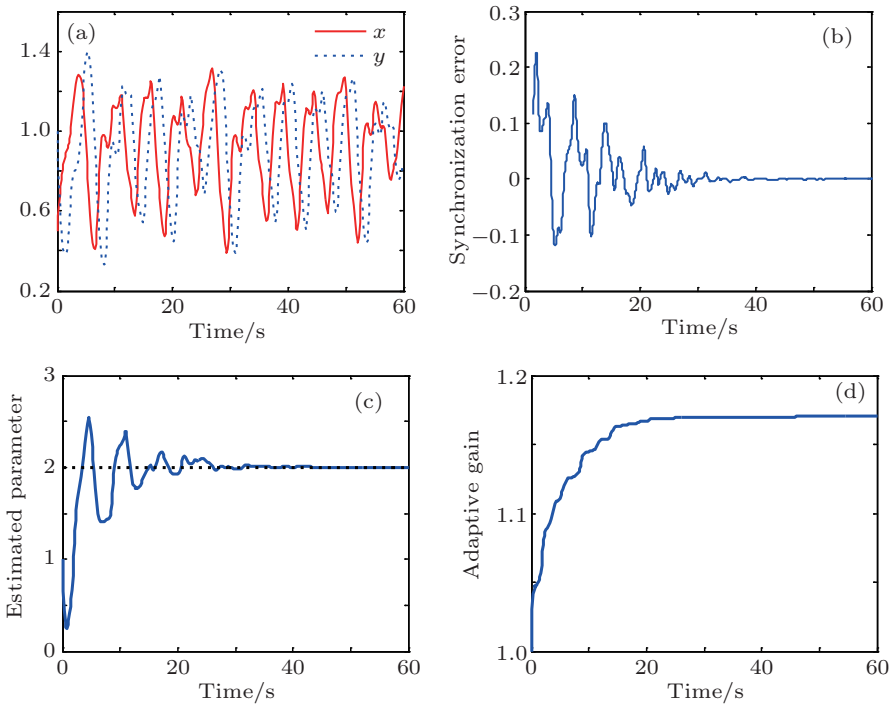



**Fig. 2.** (color online) Adaptive lag synchronizations of the Mackey–Glass model. (a) Time response of the driving signal and response signal; (b) lag synchronization error; (c) time evolution of the estimated parameter $\hat{\beta}$; (d) time evolution of the adaptive feedback gain.

In the simulation, the initial conditions are set as $x(s) = 0.5$, $y(s) = 1$, $k(s) = 0$, and $\hat{\beta}(s) = 1$ for $\forall s = [-2, 0]$. The transmission time lag is assumed to be $\tau = 1.5$. The numerical results are presented in Fig. 2.

As can be seen from Fig. 2, the response signal successfully synchronizes with the driving signal with the desired transmission lag $\tau = 1.5$ and the synchronization error converges to zero as expected. In addition, the unknown parameter $\hat{\beta}$ has been successfully identified to be its true value $\beta = 2$ and the controlling gain has been adaptively adjusted to the constant value as well.

**Example 2** Consider the following Lorenz system with multiple time delays:

$$\begin{cases} \dot{x}_1(t) = m(x_2(t) - x_1(t)), \\ \dot{x}_2(t) = rx_1(t) - x_2(t - d_1) - x_1(t)x_3(t), \\ \dot{x}_3(t) = x_1(t)x_2(t) - bx_3(t - d_2). \end{cases} \tag{20}$$

The research on chaotic synchronization is motivated by its potential application in secure communication. The Lorenz model is a typical one for testing the performance of a synchronization controller. By choosing the parameters to be $m = 10$, $r = 28$, and $b = 8/3$ and setting the model time delays to be $d_1 = 1/10$ and $d_2 = 1/3$, the system will display the typical butterfly-shaped chaotic attractor, which is shown in Fig. 3.

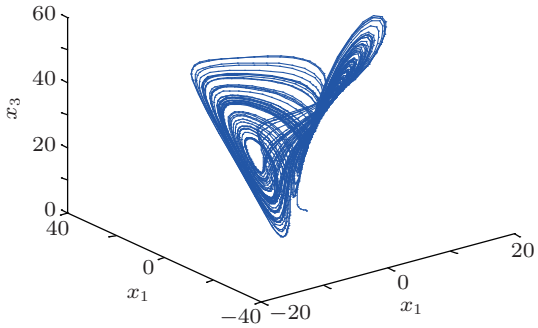



**Fig. 3.** (color online) Chaotic attractor of the Lorenz system with multiple time delays.

According to the drive-response configuration, the response Lorenz system with the same structure is defined as

$$\begin{cases} \dot{y}_1(t) = \hat{m}(y_2(t) - y_1(t)) + u_1, \\ \dot{y}_2(t) = \hat{r}y_1(t) - y_2(t - d_1) - y_1(t)y_3(t) + u_2, \\ \dot{y}_3(t) = y_1(t)y_2(t) - \hat{b}y_3(t - d_2) + u_3. \end{cases} \tag{21}$$

Then according to the main result in Section 3, the adaptive controller and the parameter updating laws are designed in the following simple forms:

$$\begin{cases} u_i(t) = -k_i(t)e_i(t), \\ \dot{k}_i(t) = \delta e_i^2(t), \quad i = 1, 2, 3, \end{cases} \tag{22a}$$

$$\begin{cases} \dot{\hat{m}} = -\alpha[y_2(t) - y_1(t)]e_1, \\ \dot{\hat{r}} = -\alpha y_1(t)e_2, \\ \dot{\hat{b}} = -\beta y_3(t - d_2)e_3. \end{cases} \tag{22b}$$

In the simulations, the initial conditions for the states in drive system (20) and response system (21) are chosen to be $[x_1(s), x_2(s), x_3(s)]^{\mathrm{T}} = [-3, 5, 6]^{\mathrm{T}}$ and $[y_1(s), y_2(s), y_3(s)]^{\mathrm{T}} = [3, 1, 2]^{\mathrm{T}}$, the initial conditions for the parameter estimations and adaptive gains are set to be $[\hat{m}(s), \hat{r}(s), \hat{b}(s)]^{\mathrm{T}} = [5, 20, 5]^{\mathrm{T}}$ and $k_i(s) = 1$ for $\forall s \in [-\tau', 0]$ where $\tau' = \max(d_1, d_2, \tau)$. We assume that the time lag caused by the transmission is $\tau = 0.5$. The positive constants in Eqs. (22a) and (22b) are set to be $\delta = \alpha = \beta = 1$. Through numerical simulation, the time responses of state variables in both drive system and response system are depicted in Fig. 4, which shows the success in the lag synchronization. The synchronization errors, in which all the errors converge to the origin in a short time, are illustrated in Fig. 5. By examining the property of the driving function elements from Eq. (22b), it is obvious that the function vectors are linearly independent and thus the parameter estimation can be correctly attained as shown in Fig. 6. From the simulation results in Figs. 4–6, one can conclude that the synchronization and the parameter identification are successfully achieved via the simple feedback controller in which only the transmission lag information $x_i(t - \tau)$ is used.

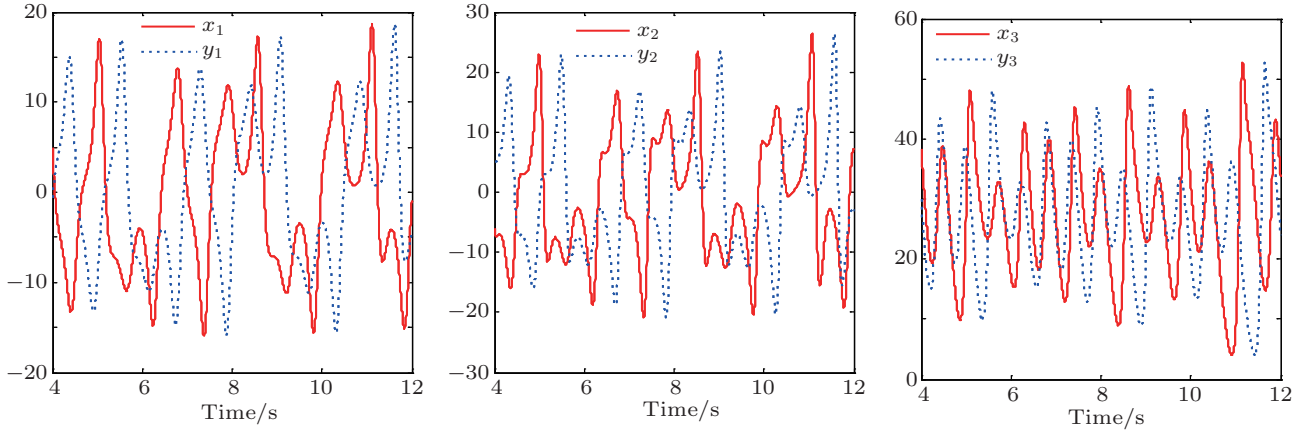



**Fig. 4.** (color online) Time evolutions of drive states $x = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ and the corresponding response states $y = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$.

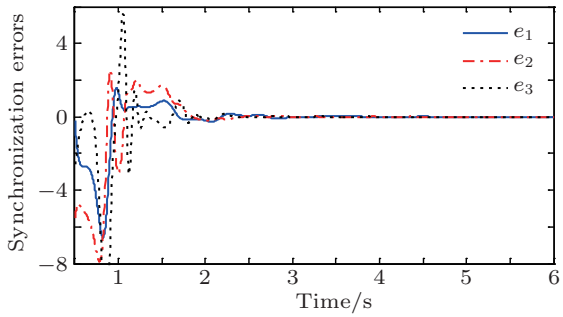



**Fig. 5.** (color online) Time evolutions of the lag synchronization errors.

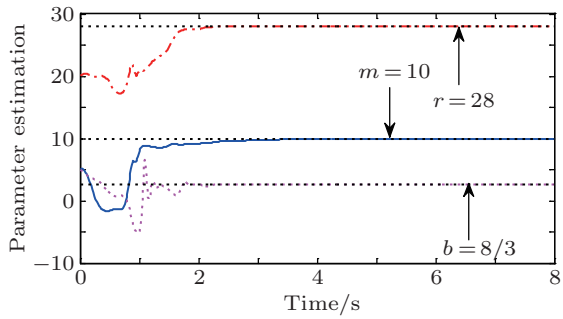



**Fig. 6.** (color online) Time evolutions of the parameter estimations.

**Remark 5** In Ref. [22], the authors also studied the synchronization controller design for uncertain dynamical systems with time delays and channel lags. The proposed method in Ref. [22] is based on the active control technique. That is, in order to obtain a linear error dynamic system, nonlinear parts in the error system are eliminated via the nonlinear controller. Hence, this designing strategy produces a complicated synchronization controller which is difficult for practical implementation. To see this point clearly, we consider again the lag synchronization problem for the time-delayed Lorenz system discussed in Example 2. According to the scheme in Ref. [22], the controller should be designed as

$$\begin{cases} u_1(t) = -\hat{m}(y_2(t) - y_1(t)) \\ \quad + \hat{m}(x_2(t-\tau) - x_1(t-\tau)) - e_1(t), \\ u_2(t) = -(\hat{r}y_1(t) - y_2(t-d_1) - y_1(t)y_3(t)) + \hat{r}x_1(t-\tau) \\ \quad - x_2(t-d_1-\tau) - x_1(t-\tau)x_3(t-\tau) - e_2(t), \\ u_3(t) = -(y_1(t)y_2(t) - \hat{b}y_3(t-d_2)) + x_1(t-\tau)x_2(t-\tau) \\ \quad - bx_3(t-d_2-\tau) - e_3(t). \end{cases} \tag{23}$$

One can find that the above controllers are much more complicated than the one designed in Eq. (22a). Another issue that should be mentioned when implementing Eq. (23) is that the nonlinear controller involves driving signals at the past moments $t - d_1 - \tau$ and $t - d_2 - \tau$ (i.e. the driving states $x_2(t - d_1 - \tau)$ and $x_3(t - d_2 - \tau)$). This means that one requires extra instruments to obtain the past information in practice. The above analysis indicates that the controller scheme designed in Ref. [22] is not convenient for practical use. Furthermore, as will be discussed in Subsection 5.3, the complexity in the controllers may lead to less robustness in a noisy environment.

## 5. Discussion and comparisons: Some robustness issues

### 5.1. Identification of unknown parameters with switching values

Switching property is very common in real-life plants and models. Examples include switching power converters, telecommunication systems, and switching biological models such as gene regulatory networks. In many cases the system parameters do not remain constant at some fixed values but instead they may switch between a set of values. Hence, it is of high significance to study how the controller would react under switching conditions. In this subsection we will examine the performance of the proposed method of synchronizing and identifying time-delayed dynamical systems with unknown parameters with switching values. Again, for convenience we use the above time-delayed Lorenz systems in our numerical verifications.

Suppose that the system parameters are switching at time $t = 6$ s and the values are reset to be $m' = 15$, $r' = 20$, and $b' = 5$. We employ the same initial settings for the simulation, and the results are shown in Figs. 7 and 8.

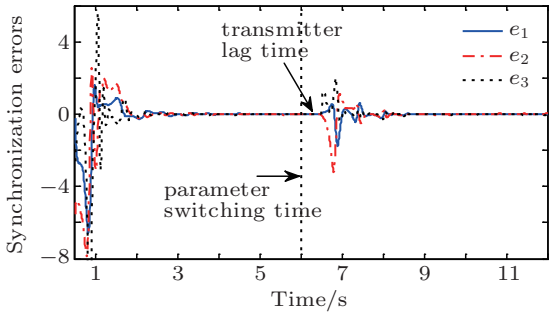



**Fig. 7.** (color online) Time evolutions of the synchronization errors in the switching parameter case.

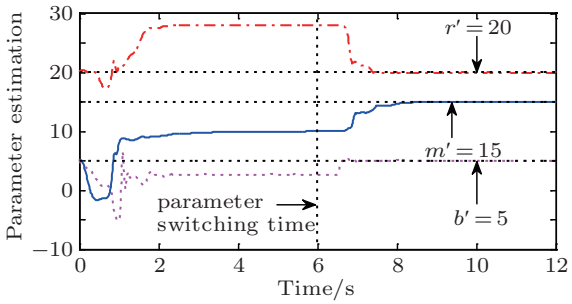



**Fig. 8.** (color online) Estimations of the unknown parameters with switching values.

It can be seen from Figs. 7 and 8 that after experiencing some short periods of transmission lag, the synchronization is successfully achieved in spite of the abrupt changing of the system parameters. Moreover, the parameter estimations converge to their new values after a short transient process. Hence, the proposed method is applicable for the online monitoring of the changing parameters. In fact, the response system is constructed to detect the changes of the drive systems by using the driving signals with transmission delays. The designed adaptive controller, though very simple, is very effective in achieving lag synchronization as well as in identifying the unknown switching parameters.

### 5.2. Performance analysis in the presence of noise perturbations

In the information transmission process, the communication signals will inevitably be exposed to noises or perturbations. Hence, it is an essential task to examine the performance of the synchronization controller in a noisy environment. We note that in the literature, however, relatively less attention has been paid to this topic.

We assume that noise is additive and that the transmission signals are disturbed by some uniformly distributed random noises, i.e., the contaminated driving signals are expressed as $\boldsymbol{x}'(t-\tau) = \boldsymbol{x}(t-\tau) + l_n\boldsymbol{\eta}$, where $x(t-\tau)$ is an original time series with transmission lags from the drive system, $\boldsymbol{\eta}$ denotes the random noise ranging from $-1$ to 1, and $l_n$ is a constant representing the noise intensity. In our numerical experiment, we further suppose that without loss of generality and for easy comparisons, the second information channel (i.e. the time series $x_2$) is contaminated. Three levels of noise intensities, $l_n = 1, 2, 5$, are considered in the simulation. The synchronization error $e_2$ and the estimated parameter $r$ (with the updating law $\dot{\hat{r}} = -\alpha y_1(t) e_2$) directly affect the noisy measurement $x_2'(t-\tau)$, hence we plot the numerical results for $r$ and $e_2$ in Figs. 9 and 10 respectively.

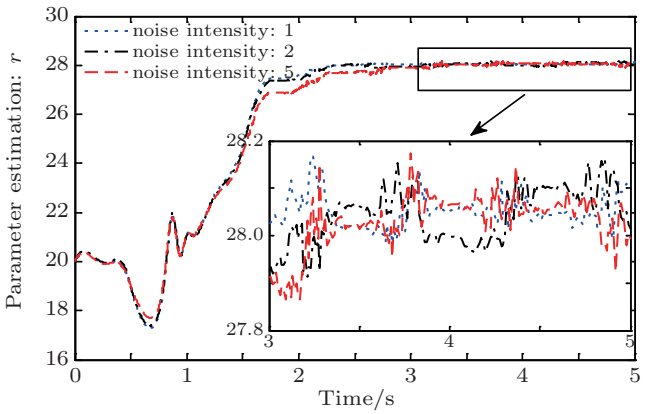



**Fig. 9.** (color online) Estimations of the unknown parameter $r$ under different levels of noise intensities.

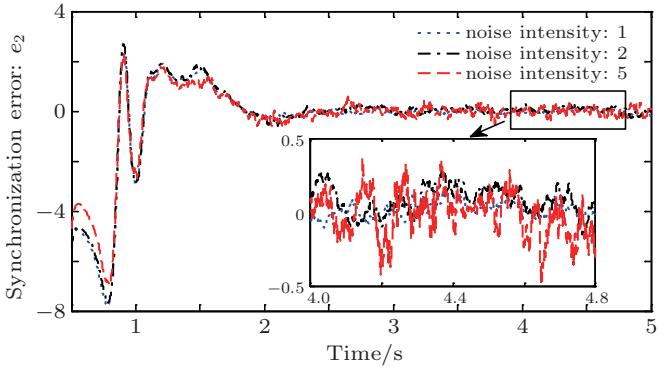



**Fig. 10.** (color online) Time evolutions of the synchronization error $e_2$ under different levels of noise intensities.

As can be seen from the above figures, despite the presence of different intensities of noise, the unknown parameter $r$ still converges to its true value with minor fluctuations and the synchronization error approaches the minor vicinity around the origin as well.

It is worthwhile mentioning that when we choose $l_n = 5$, the amplitude of the noise accounts for approximately 20% of the signal amplitude. However, even with such a high level of noise intensity, the synchronization error and the parameter estimation error are still satisfactorily confined in an acceptably bounded region. In fact, this anti-noise property is partially contributed to by the adaptive mechanism of the feedback gain $k_i$. The feedback gain $k_i$ is adaptively adjusted online according to Eq. (22a) which helps to suppress the noise in the synchronization process. In this section, via numerical examples, we have further demonstrated that the proposed controller is quite robust to a certain level of noise perturbations.

### 5.3. Comparison study with existing methods

In Remark 5 we have reviewed the nonlinear controller scheme studied in Refs. [22] and [23]. By introducing nonlinear feedback functions into the synchronization scheme, the expressions of the controllers (23) are indeed much more complicated than the linear feedback one proposed in this paper. In this section, through concrete numerical examples, we will further evaluate their respective performances when they are subjected to the same noise disturbances.

For simplicity and without loss of generality, we again assume that the transmission signal $x_2$ is disturbed by the noise $\boldsymbol{\eta}$ and the noise intensity is set at a high level of $l_n = 5$. All other simulation settings are the same for both control schemes. The numerical results for the linear feedback scheme are shown in Figs. 11 and 12, which illustrate the time evolutions of the parameter estimations and synchronization errors respectively. Note that the time evolutions for $r$ and $e_2$ have been already shown in Figs. 9 and 10, respectively; hence, they do not appear in Figs. 11 and 12. For the nonlinear controller scheme, the numerical results are depicted in Figs. 13 and 14, which

present some intuitive observations of the synchronization and identification process.

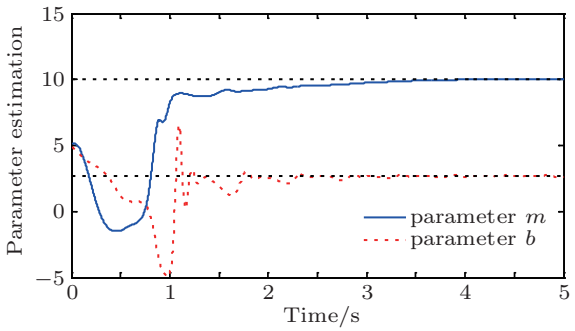



**Fig. 11.** (color online) Time evolutions of the parameter estimations of $m$ and $b$ with the noisy measurement $x_2'(t-\tau)$ (noise intensity: 5).

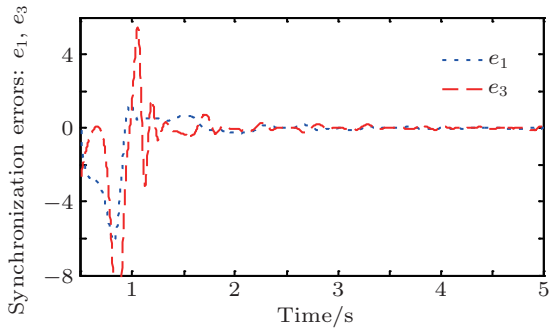



**Fig. 12.** (color online) Synchronization errors $e_1$ and $e_3$ with the measurement $x_2'(t-\tau)$ contaminated by the highest level of noise (noise intensity: 5 ).

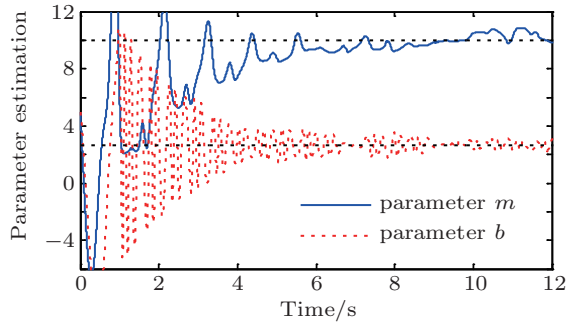



**Fig. 13.** (color online) Time evolutions of the parameter estimation of $m$ and $b$ with the noisy measurement $x_2'(t-\tau)$ by using the nonlinear controller Eq. (23) (a comparison with Fig. 11).

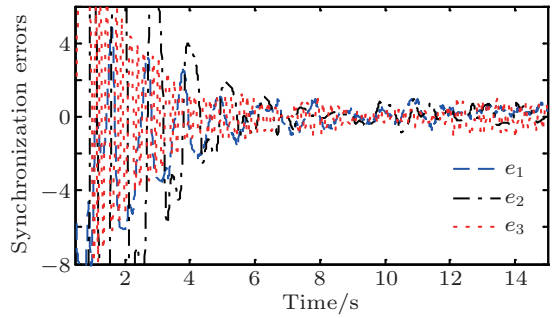



**Fig. 14.** (color online) Time evolutions of the synchronization errors with the noisy measurement $x_2'(t-\tau)$ by using the nonlinear controller Eq. (23) (a comparison with Fig. 12).

From Figs. 11 and 12 it follows that despite the fact that the signal $x_2$ is severely disturbed by a high level of noise, the synchronization and identification process in the other two channels, $x_1$ and $x_3$, are much less affected. In contrast, when the nonlinear controller approach is adopted, the estimated parameters will fluctuate drastically around the true values. Moreover, the steady-state synchronization errors oscillate around the origin with quite large fluctuations (see Figs. 13 and 14). The reason for the obvious performance deterioration of the nonlinear scheme can be explained as follows. As can be seen in Eq. (23), the controllers are highly coupled; thus, the noise from one single channel (here $x_2$) will be multiplied and amplified into the other controllers of the other channels. On the contrary, in the linear feedback scheme, each state in the response system uses the transmission lag information from its *own* channel to attain the synchronization and identification. Thus, they will still preserve relatively good performance even if other channel signals are contaminated by heavy noises. It is worth noting that this feature is very desirable in the secure communication applications. In short, the simple feedback scheme proposed in this paper outperforms its nonlinear counterpart not only in terms of simplified expressions, but also in the sense of higher robustness against noise perturbations.

## 6. Conclusions

For the practical consideration of synchronization design, the information transmission process always experiences certain propagation lags; hence, it is more realistic to design synchronization controllers by using the information with transmission lags. When model delays and unknown system parameters are taken into account, this problem becomes even more challenging. In this paper we propose an adaptive and simple method to deal with this problem. By utilizing only the linear feedback of the state information with transmission lags from the drive system, the lag synchronization can be swiftly achieved and the unknown parameters can also be successfully identified, provided that the important linear independence condition is satisfied. The effectiveness of this method has been demonstrated by two illustrative examples. In contrast to previous results, this scheme is general, simple, and easy for practical application. We have also numerically demonstrated that the proposed scheme has a good performance for identifying unknown parameters with switching values. Furthermore, this method is quite robust against a certain level of noise perturbations.

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