Studying the intermittent stable theorem and the synchronization of a delayed fractional nonlinear system*

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In this paper, an intermittent synchronizing delayed fractional nonlinear system is studied. We propose a novel intermittent stable theorem for the delayed fractional system and derive a new synchronization criterion for delayed fractional systems by means of fractional stable theorem and the differential inequality method. Intermittent synchronizing fractional delayed Newton–Leipnik system is taken as an illustrative example and numerical simulation of this example is presented to show the feasibility and effectiveness of the proposed theorem.

Keywords: fractional, delay, stability, intermittent synchronization

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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary (non-integer) order. Real-world processes generally or most likely are fractional-order systems, for example, dielectric polarization,[1] electromagnetic waves,[2] quantitative finance,[3] and viscoelastic systems.[4] It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives.[5–7] Compared with the integer models, fractional models provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. It would be far better if many practical problems were described by fractional-order dynamical systems rather than integer-order ones. For example, local fractional calculus is revealed as one of the useful tools to deal with everywhere continuous but nowhere differentiable functions in areas ranging from fundamental science to engineering.[8] Research about the theory of fractional calculus has attracted much interest[9–15] due to the exact description of nonlinear phenomena.[16–19]

On the other hand, time delay is an inherent property of many dynamical systems in nature and in practical systems. This intrinsic time delay makes a system infinitely dimensional and more complex. In addition, a coupling delay may appear during a transmission of signal or information in natural systems such as long axons of neurons practical systems, communication channel via optical link, and electrical transmission lines. It is very difficult to achieve satisfactory performance for delay systems. Synchronization of fractional order delayed systems becomes a challenging and interesting problem due to the potential applications in secure communication and control processing.[20–24]

Since Pecora L and Carroll T realized chaos synchronization in 1990,[25] numerous methods have been developed for controlling and synchronizing various chaotic systems which include linear and nonlinear feedback control,[26] time-delay feedback control,[27] adaptive control,[28] impulsive control,[29,30] intermittent control,[31,32] etc.

In comparison with continuous control of chaos synchronization, the discontinuous control methods, which include impulsive control and intermittent control, have attracted more interest recently due to their easy implementation in engineering control and validity in real-life applications. Intermittent control is different from impulsive control. Intermittent control has a nonzero control width, while impulsive control is activated only at some isolated instants. Recently, much effort has been devoted to studying the synchronization of integer chaotic systems and dynamical networks by using intermittent control, and many interesting results have been obtained. However, the intermittent synchronizing fractional chaotic system is rarely reported, especially the synchronizing delayed fractional system owning to a lack of relevant stable theorems.

Motivated by the above discussions, we will make a contribution to intermittent synchronizing fractional systems with delay in this paper. Based on the stability theory of fractional system, an intermittent stable theorem for fractional system with delay is proposed, which presents a sufficient condition for the intermittent control of a fractional delayed system. The proposed approach is applied to synchronize the delayed fractional Newton–Leipnik system to explain the proposed theorem, and the effectiveness is also illustrated by computer simulations.

The rest of the paper is organized as follows: Some definitions and lemmas are given in Section 2. In Section 3, the main

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results are proposed. In Section 4, the proposed method is utilized to synchronize the fractional delayed Newton–Leipnik system. In Section 5, numerical simulations are presented and conclusions are finally drawn in Section 6.

2. Fractional calculus

There are many definitions of fractional derivatives such as the Riemann–Liouville derivative[33] and the Caputo derivative[34] that are mostly used in fractional systems. In this paper, we adopt the Caputo definition for fractional derivative. The important reason for choosing Caputo derivatives for solving initial-value fractional-order differential equations is that the Riemann–Liouville initial value problems require homogeneous initial conditions, though the Caputo initial-value problem holds for both homogeneous and non-homogeneous conditions. The Caputo definition can be expressed as:

$$ {}_a^{\mathrm{C}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \times \int_a^t (t-\tau)^{-\alpha+n-1} f^{(n)}(\tau)\,\mathrm{d}\tau, \quad (1) $$

where n is the first integer which is not less than α, i.e. $n-1 \le \alpha \le n$ and $\Gamma(\cdot)$ is gamma function.

In general, a common fractional linear differential equation can be expressed as:

$$ {}_a^{\mathrm{C}}D_t^{\alpha} x(t) = Ax(t). \quad (2) $$

The solution of formula (2) generally can be expressed as:

$$ x(t) = x(0)E_{\alpha}(At^{\alpha}), \quad (3) $$

where Mittag–Leffter function is

$$ E_{\alpha}(Z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}. $$

Lemma 1 (Rayleigh principle) Define $\lambda_{\min}(\lambda_{\max})$ as the minimum (maximum) eigenvalue of matrix Q, for any $x(t) \in R^n$, $\lambda_{\min} x^{\mathrm{T}}(t)x(t) \le x^{\mathrm{T}}(t)Qx(t) \le \lambda_{\max} x^{\mathrm{T}}(t)x(t)$.

A common fractional nonlinear system can be depicted as:

$$ {}_a^{\mathrm{C}}D_t^{\alpha} x(t) = f(x(t)), \quad (4) $$

where $0 < \alpha < 1, x(t) \in R^n$, and $f(\cdot)$ is a nonlinear function.

Zhao *et al.* proposed a stable theorem about a fractional nonlinear system, which is expressed as[35]

Lemma 2 When fractional order $\alpha \le 1$, if

$$ \frac{(x(t))^{\mathrm{T}} {}_a^{\mathrm{C}}D_t^{\alpha} x(t)}{(x(t))^{\mathrm{T}}x(t)} = \frac{(x(t))^{\mathrm{T}} f(x(t))}{(x(t))^{\mathrm{T}}x(t)} \le \vartheta $$

for any $x(t) \in R^n$ and $x(t) \ne 0$, then $|\, x(t) \,| \le |\, x_0 \,| E_{\alpha}(\vartheta t^{\alpha})$.

3. The main result

Consider a delayed fractional system via intermittent control governed by the following equation:

$$ {}_a^{\mathrm{C}}D_t^{\alpha} x(t) = f(x(t), x(t-\tau)) + u(t), \quad (5) $$

where $x(t) = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$ is the state vector.

$$ f(x(t), x(t-\tau)) = [f_1(x(t), x(t-\tau)), f_2(x(t), x(t-\tau)), \ldots, f_n(x(t), x(t-\tau))] $$

is nonlinear vector function satisfying Leibniz condition, and $u(t)$ is the intermittent controller vector defined as

$$ u(t) = \begin{cases} 0, & nT \le t < nT+\sigma, \\ u, & nT+\sigma \le t < (n+1)T, \end{cases} \quad n = 0, 1, 2, \ldots, \quad (6) $$

where $u(t)$ is the intermittent controller, a set of control period $T > 0$, the control width $\sigma : 0 < \sigma < T$.

Under the controller (6), we can obtain a nonlinear system with delay as follows:

$$ \begin{cases} {}_a^{\mathrm{C}}D_t^{\alpha} x(t) = f(x(t), x(t-\tau)), & nT \le t < nT+\sigma, \\ {}_a^{\mathrm{C}}D_t^{\alpha} x(t) = f(x(t), x(t-\tau)) + u, & nT+\sigma \le t < (n+1)T, \end{cases} \quad n = 0, 1, 2, \ldots \quad (7) $$

The main aim of this paper is to design suitable control strength, control period, and control width to realize controlling fractional system (5) under the controller (6).

We define a special function $g(x(t))$ as:

$$ g(x(t)) = \frac{x^{\mathrm{T}}(t) {}_a^{\mathrm{C}}D_t^{\alpha} x(t)}{x^{\mathrm{T}}(t)x(t)}. \quad (8) $$

When $nT \le t < nT+\sigma, n = 0, 1, 2, \ldots$, we define the maximum of $g(x(t))$ as

$$ \max(g(x(t))) = \max\left(\frac{x^{\mathrm{T}}(t) {}_a^{\mathrm{C}}D_t^{\alpha} x(t)}{x^{\mathrm{T}}(t)x(t)} \right) = k_1. \quad (9) $$

When $nT+\sigma \le t < (n+1)T, n = 0, 1, 2, \ldots$, we similarly define the maximum of $g(x(t))$

$$ \max(g(x(t))) = \max\left(\frac{x^{\mathrm{T}}(t) {}_a^{\mathrm{C}}D_t^{\alpha} x(t)}{x^{\mathrm{T}}(t)x(t)} \right) = k_2. \quad (10) $$

Theorem 1 If intermittent control law (6) makes k_2 in function (10) satisfy

$$ E_{\alpha}(k_1\sigma^{\alpha}) E_{\alpha}(k_2(T-\sigma)^{\alpha}) < 1. \quad (11) $$

Then, the system (5) is asymptotically stable to zero.

Proof

When

$$t \in [0,\sigma), g(x(t)) = \frac{x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t)}{x^{\mathrm{T}}(t)x(t)} \le k_1,$$

according to Lemma 2

$$|\,x(t)\,| \le |\,x(0)\,|\,E_{\alpha}(k_1 t^{\alpha}), \tag{12}$$

when

$$t \in [\sigma,T), g(x(t)) = \frac{x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t)}{x^{\mathrm{T}}(t)x(t)} \le k_2,$$

we can obtain

$$\begin{aligned} |\,x(t)\,| &\le |\,x(\sigma)\,|\,E_{\alpha}(k_2 t^{\alpha}) \\ &\le |\,x(0)\,|\,E_{\alpha}(k_1\sigma^{\alpha})E_{\alpha}(k_2(t-\sigma)^{\alpha}). \end{aligned} \tag{13}$$

By inducing, it follows that

(i) If $t \in [nT, nT+\sigma)$, we get

$$\begin{aligned} |\,x(t)\,| &\le |\,x(nT)\,|\,E_{\alpha}(k_2(t-nT)^{\alpha}) \\ &\le |\,x(0)\,|\,(E_{\alpha}(k_1\sigma^{\alpha})E_{\alpha}(k_2(t-\sigma)^{\alpha}))^{n-1} \\ &\quad \times E_{\alpha}(k_2(t-nT)^{\alpha}). \end{aligned} \tag{14}$$

(ii) If $t \in [nT+\sigma, (n+1)T)$, we get

$$\begin{aligned} |\,x(t)\,| &\le |\,x(nT)\,|\,E_{\alpha}(k_1\sigma^{\alpha})E_{\alpha}(k_2(t-nT-\sigma)^{\alpha}) \\ &\le |\,x(0)\,|\,(E_{\alpha}(k_1\sigma^{\alpha})E_{\alpha}(k_2(t-\sigma)^{\alpha}))^{n-1} \\ &\quad \times E_{\alpha}(k_1\sigma^{\alpha})E_{\alpha}(k_2(t-nT-\sigma)^{\alpha}). \end{aligned} \tag{15}$$

Then from expressions (14) and (15), if $E_{\alpha}(k_1\delta^{\alpha})E_{\alpha}(k_2(T-\delta)^{\alpha}) < 1$, we can get

$$\lim_{n\to\infty} |\,x(0)\,|\,(E_{\alpha}(k_1\sigma^{\alpha})E_{\alpha}(k_2(t-\sigma)^{\alpha}))^{n-1} = 0. \tag{16}$$

We obtain $\lim\limits_{t\to\infty} |\,x(0)\,| = 0$. Thus, the system (5) with delay is asymptotically stable to zero under the intermittent controller (6). This proof of Theorem 1 is completed.

In general, the goal of controlling a nonlinear system is to make an unstable system stable. According to Lemma 1, $k_1 > 0$ is certain in unstable system (5). How to calculate the value of k_1 is one of the key problems of designing the suitable intermittent controller $u(t)$ to make k_2 satisfy the condition (11). In this paper, we propose a new approach to ascertain the upper bound of k_1.

Theorem 2 If there is a semi-positive matrix P, for any $x(t) \in R^n$, satisfying

$$\frac{x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t) + (x^{\mathrm{T}}(t)Px(t) - x^{\mathrm{T}}(t-\tau)Px(t-\tau))}{x^{\mathrm{T}}(t)x(t)} \le k,$$

where $k \ge 0$, then

$$\frac{x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t)}{x^{\mathrm{T}}(t)x(t)} \le k.$$

Proof

Suppose the function $x^{\mathrm{T}}(\varepsilon)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(\varepsilon)/x^{\mathrm{T}}(\varepsilon)x(\varepsilon)$ can take the maximum ζ when $\varepsilon \in (t-\tau, t]$ and ${}_a^{\mathrm{C}}D_t^{\alpha}x(\varepsilon) = \zeta x(\varepsilon)$. According to Lemma 2, $|\,x(t)\,| = |\,x(t-\tau)\,|\,E_{\alpha}(\zeta\tau^{\alpha})$, and we can get

$$x^{\mathrm{T}}(t)Px(t) = x^{\mathrm{T}}(t-\tau)Px(t-\tau)(E_{\alpha}(\zeta\tau^{\alpha}))^2. \tag{17}$$

Then

$$\begin{aligned} &x^{\mathrm{T}}(t)Px(t) - x^{\mathrm{T}}(t-\tau)Px(t-\tau) \\ &= \left(1 - \frac{1}{(E_{\alpha}(\zeta\tau^{\alpha}))^2}\right)x^{\mathrm{T}}(t)Px(t). \end{aligned} \tag{18}$$

Define $x^{\mathrm{T}}(t)Px(t) = \lambda x^{\mathrm{T}}(t)x(t)$. Because P is a semi-positive matrix, it is obviously $0 \le \lambda_{\min}(P) \le \lambda \le \lambda_{\max}(P)$ according to Lemma 1.

Suppose the conclusion $x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t)/x^{\mathrm{T}}(t)x(t) \le k$ is not correct for any $x(t) \in R^n$, that is to say

$$\max\left(\frac{x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t)}{x^{\mathrm{T}}(t)x(t)}\right) = \zeta > k \ge 0.$$

We can obtain inequalities $(1 - 1/(E_{\alpha}(\zeta\tau^{\alpha}))^2) > 0$ and $(1 - 1/(E_{\alpha}(\zeta\tau^{\alpha}))^2)\lambda \ge 0$. Then it can be gained

$$\zeta x^{\mathrm{T}}(t)x(t) + \left(1 - \frac{1}{(E_{\alpha}(\zeta\tau^{\alpha}))^2}\right)\lambda x^{\mathrm{T}}(t)x(t) > kx^{\mathrm{T}}(t)x(t), \tag{19}$$

which contradict with

$$\frac{x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t) + (x^{\mathrm{T}}(t)Px(t) - x^{\mathrm{T}}(t-\tau)Px(t-\tau))}{x^{\mathrm{T}}(t)x(t)} \le k,$$

for any $x(t) \in R^n$. Thus the supposition $\zeta > k$ cannot be established. Then ζ is not more than k, that is to say, $x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t))/x^{\mathrm{T}}(t)x(t) \le k$ for any $x(t) \in R^n$. The proof of Theorem 2 is completed.

Theorem 3 For a certain delay time τ, if there is a semi-positive matrix Q satisfying

$$\begin{aligned} &\frac{x^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}x(t) + (x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau))}{x^{\mathrm{T}}(t)x(t)} \\ &= \frac{x^{\mathrm{T}}(t)(f(x(t),x(t-\tau)) + u) + (x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau))}{x^{\mathrm{T}}(t)x(t)} \\ &= l_2 + \left(1 - \frac{1}{(E_{\alpha}(l_2\tau^{\alpha}))^2}\right)\lambda \le l_2 + \left(1 - \frac{1}{(E_{\alpha}(l_2\tau^{\alpha}))^2}\right)\lambda_{\max}, \quad \lambda_{\min} \le \lambda \le \lambda_{\max}, \end{aligned} \tag{20}$$

where $l_2 \leq 0$, and $\lambda_{\min}(\lambda_{\max})$ is the minimum (maximum) eigenvalue of the matrix Q. Then $x^{\mathrm{T}}(t){}_a^{\mathrm{C}}D_t^{\alpha}x(t)/x^{\mathrm{T}}(t)x(t) \leq l_2$.

Proof

We first suppose

$$\max\left(\frac{x^{\mathrm{T}}(t){}_a^{\mathrm{C}}D_t^{\alpha}x(t)}{x^{\mathrm{T}}(t)x(t)}\right) = k_3. \tag{21}$$

We need to prove $k_3 \leq k_2$.

We first consider a special case, that is, when $\varepsilon \in (t-\tau, t)$, ${}_a^{\mathrm{C}}D_t^{\alpha}x(\varepsilon) = k_3 x(\varepsilon)$, which satisfies

$$\max\left(\frac{x^{\mathrm{T}}(t){}_a^{\mathrm{C}}D_t^{\alpha}x(t)}{x^{\mathrm{T}}(t)x(t)}\right) = k_3.$$

According to Lemma 2

$$x(t) = x(t-\tau)E_{\alpha}(k_3\tau^{\alpha}), \tag{22}$$

$$x^{\mathrm{T}}(t)Qx(t) = x^{\mathrm{T}}(t-\tau)Qx(t-\tau)E_{\alpha}^2(k_3\tau^{\alpha}). \tag{23}$$

We suppose that the conclusion $k_3 \leq k_2$ is not correct, that is to say $k_3 > k_2$.

When $k_3 > k_2$, k_3 may be bigger or not more than zero. We need to consider two cases: $k_3 > 0$ or $k_3 \leq 0$.

Firstly, we consider $k_3 > 0$.

Because $k_3 > 0$, $E_{\alpha}^2(k_3\tau^{\alpha}) > 1$, according to Rayleigh principle

$$\begin{aligned}
&\left(1 - \frac{1}{E_{\alpha}^2(k_3\tau^{\alpha})}\right)\lambda_{\max}x^{\mathrm{T}}(t)Qx(t) \\
&\geq x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau) \\
&= \left(1 - \frac{1}{E_{\alpha}^2(k_3\tau^{\alpha})}\right)x^{\mathrm{T}}(t)Qx(t) \\
&\geq (1 - \frac{1}{E_{\alpha}^2(k_3\tau^{\alpha})})\lambda_{\min}x^{\mathrm{T}}(t)Qx(t) > 0,
\end{aligned} \tag{24}$$

which contracts with

$$\frac{x^{\mathrm{T}}(t)(f(x(t),x(t-\tau))+u) + x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau)}{x^{\mathrm{T}}(t)x(t)} \leq k_2 < 0, \tag{25}$$

then k_3 cannot be greater than 0.

Secondly, we need to prove the conclusion that $k_3 > k_2$ cannot be correct when $k_3 \leq 0$.

When $k_3 \leq 0$, $E_{\alpha}^2(k_3\tau^{\alpha}) \leq 1$ and $(1 - 1/E_{\alpha}^2(k_3\tau^{\alpha})) \leq 0$, we can get

$$0 \geq \lambda_{\min}\left(1 - \frac{1}{E_{\alpha}^2(k_3\tau^{\alpha})}\right) \geq \lambda\left(1 - \frac{1}{E_{\alpha}^2(k_3\tau^{\alpha})}\right) \geq \lambda_{\max}\left(1 - \frac{1}{E_{\alpha}^2(k_3\tau^{\alpha})}\right). \tag{26}$$

Then

$$\begin{aligned}
&\frac{x^{\mathrm{T}}(t)(f(x(t),x(t-\tau))+u) + x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau)}{x^{\mathrm{T}}(t)x(t)} \\
&\geq k_3 + \lambda_{\max}\left(1 - \frac{1}{E_{\alpha}^2(k_3\tau^{\alpha})}\right) > k_2 + \lambda_{\max}\left(1 - \frac{1}{E_{\alpha}^2(k_2\tau^{\alpha})}\right),
\end{aligned} \tag{27}$$

which contracts with

$$\frac{x^{\mathrm{T}}(t)(f(x(t),x(t-\tau))+u) + x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau)}{x^{\mathrm{T}}(t)x(t)} \leq k_2 + \lambda_{\max}\left(1 - \frac{1}{E_{\alpha}^2(k_2\tau^{\alpha})}\right).$$

Then when $k_3 \leq 0$, the conclusion $k_3 > k_2$ also is not correct.

From the two steps of the proof process, we can see that k_3 cannot be greater than k_2, then $k_3 \leq k_2$. The proof of Theorem 3 is completed.

4. Intermittent synchronizing fractional Newton–Leipnik system with time delay

The fractional-order Newton–Leipnik chaotic system is proposed in Ref. [36], which can be depicted as

$$\begin{cases}
{}_a^{\mathrm{C}}D_t^{\alpha}x_1(t) = -ax_1(t) + y_1(t) + 10y_1(t)z_1(t), \\
{}_a^{\mathrm{C}}D_t^{\alpha}y_1(t) = -x_1(t) - 0.4y_1(t) + 5x_1(t)z_1(t), \\
{}_a^{\mathrm{C}}D_t^{\alpha}z_1(t) = bz_1(t) - 5x_1(t)y_1(t),
\end{cases} \tag{28}$$

where α is fractional order. It has been shown that the system will exhibit chaotic behavior when $0.94 \leq \alpha < 1$ and $a = 0.4$, $b = 0.175$. The delayed fractional-order Newton–Leipnik system can be expressed as

$$\begin{cases}
{}_a^{\mathrm{C}}D_t^{\alpha}x_1(t) = -ax_1(t) + y_1(t) + 10y_1(t)z_1(t), \\
{}_a^{\mathrm{C}}D_t^{\alpha}y_1(t) = -x_1(t) - 0.4y_1(t-\tau) + 5x_1(t)z_1(t), \\
{}_a^{\mathrm{C}}D_t^{\alpha}z_1(t) = bz_1(t-\tau) - 5x_1(t)y_1(t),
\end{cases} \tag{29}$$

where $\tau > 0$ is the delay time. Taking parameter $b = 0.175$, fractional order $\alpha = 0.97$ and delay time $\tau = 0.03$ s, we get the bifurcation diagram of system (29) by varying parameters

a as shown in Fig. 1. From the result, we take $a = 0.4$ and get the attractor of system (29) as shown in Figs. 2 and 3.

From Figs. 1–3, we can see clearly that the fractional Newton–Leipnik system with delay still exhibits the chaotic behavior when parameter a, b, and delay τ are taken in a certain range.

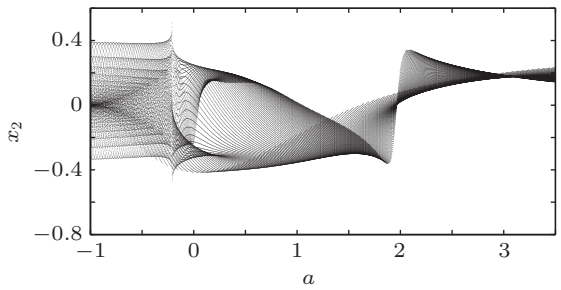


Fig. 1. Bifurcation diagram of fractional system (29) versus parameter a.

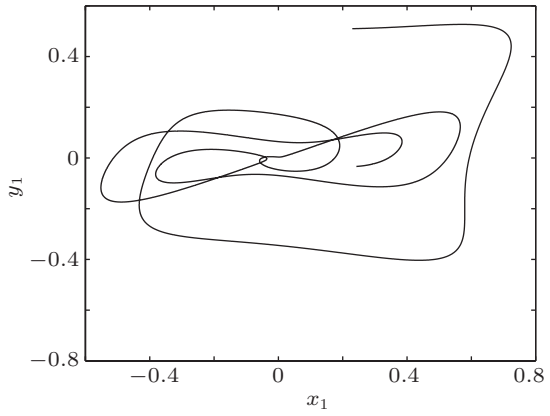


Fig. 2. The chaotic attractors x_1–y_1 of system (29).

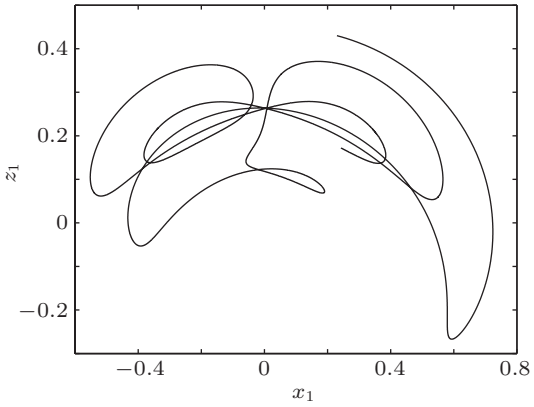


Fig. 3. The chaotic attractors x_1–z_1 of system (29).

Taking system (29) as the drive system, then the response system via intermittent control can be defined as

$$\begin{cases} {}_a^{\mathrm{C}}D_t^{\alpha}x_2(t) = -ax_2(t) + y_2(t) + 10y_2(t)z_2(t), \\ {}_a^{\mathrm{C}}D_t^{\alpha}y_2(t) = -x_2(t) - 0.4y_2(t-\tau) + 5x_2(t)z_2(t), \\ {}_a^{\mathrm{C}}D_t^{\alpha}z_2(t) = bz_2(t-\tau) - 5x_2(t)y_2(t), \\ nT \le t < nT + \sigma, \\ {}_a^{\mathrm{C}}D_t^{\alpha}x_2(t) = -ax_2(t) + y_2(t) + 10y_2(t)z_2(t) + u_1(t), \\ {}_a^{\mathrm{C}}D_t^{\alpha}y_2(t) = -x_2(t) - 0.4y_2(t-\tau) + 5x_2(t)z_2(t) + u_2(t), \\ {}_a^{\mathrm{C}}D_t^{\alpha}z_2(t) = bz_2(t-\tau) - 5x_2(t)y_2(t) + u_3(t), \\ nT + \sigma \le t < (n+1)T, \end{cases} \tag{30}$$

where $n = 0, 1, 2, \ldots$. Define the synchronizing error between the drive system (29) and the response system (30) as

$$\begin{aligned} &e_1(t) = x_2(t) - x_1(t), \quad e_2(t) = y_2(t) - y_1(t), \\ &e_3(t) = z_2(t) - z_1(t). \end{aligned} \tag{31}$$

We can get the synchronizing error as:

$$\begin{cases} {}_a^{\mathrm{C}}D_t^{\alpha}e_1(t) = -ae_1(t) + e_2(t) + 10(z_2(t)e_2(t) + y_1(t)e_3(t)), \\ {}_a^{\mathrm{C}}D_t^{\alpha}e_2(t) = -e_1(t) - 0.4e_2(t-\tau) + 5(x_2(t)e_3(t) + z_1(t)e_1(t)), \quad nT \le t < nT + \sigma, \\ {}_a^{\mathrm{C}}D_t^{\alpha}e_3(t) = be_3(t-\tau) - 5(x_2(t)e_2(t) + y_1(t)e_1(t)), \\ {}_a^{\mathrm{C}}D_t^{\alpha}e_1(t) = -ae_1(t) + e_2(t) + 10(z_2(t)e_2(t) + y_1(t)e_3(t)) + u_1(t), \\ {}_a^{\mathrm{C}}D_t^{\alpha}e_2(t) = -e_1(t) - 0.4e_2(t-\tau) + 5(x_2(t)e_3(t) + z_1(t)e_1(t)) + u_2(t), \quad nT + \sigma \le t < (n+1)T. \\ {}_a^{\mathrm{C}}D_t^{\alpha}e_3(t) = be_3(t-\tau) - 5(x_2(t)e_2(t) + y_1(t)e_1(t)) + u_3(t), \end{cases} \tag{32}$$

When $nT \le t < nT + \sigma$, we can obtain

$$\begin{aligned} & e^{\mathrm{T}}(t){}_a^{\mathrm{C}}D_t^{\alpha}e(t) \\ = \; & -ae_1^2(t) + e_1(t)e_2(t) + 10e_1(t)z_2(t)e_2(t) \\ & - e_1(t)e_2(t) - 0.4e_2(t-\tau)e_2(t) - 0.4e_2(t-\tau)e_2(t) \\ & + 5x_2(t)e_3(t)e_2(t) + 5z_1(t)e_1(t)e_2(t) \\ & + be_3(t-\tau)e_3(t) - 5x_2(t)e_2(t)e_3(t) - 5y_1(t)e_1(t)e_3(t) \\ \le \; & -ae_1^2(t) + (\mid 10z_2(t) + 5z_1(t) \mid)\frac{e_1^2(t) + e_2^2(t)}{2} \\ & + 5 \mid y_1(t) \mid \frac{e_1^2(t) + e_3^2(t)}{2} \\ & + 0.4\frac{e_2^2(t-\tau) + e_2^2(t)}{2} + \mid b \mid \frac{e_3^2(t-\tau) + e_3^2(t)}{2} \\ = \; & \left(-a + \frac{\mid 10z_2(t) + 5z_1(t) \mid}{2} + \frac{5 \mid y_1(t) \mid}{2}\right)e_1^2(t) \\ & + \left(\frac{\mid 10z_2(t) + 5z_1(t) \mid}{2} + 0.2\right)e_2^2(t) \\ & + \left(\frac{5 \mid y_1(t) \mid}{2} + \frac{\mid b \mid}{2}\right)e_3^2(t) \\ & + 0.2e_2^2(t-\tau) + \frac{\mid b \mid}{2}e_3^2(t-\tau). \end{aligned} \tag{33}$$

Then we can define semi-positive matrix P as

$$P = \begin{bmatrix} 0 & & \\ & 0.2 & \\ & & \frac{|\,b\,|}{2} \end{bmatrix}. \tag{34}$$

Then

$$\begin{aligned} & e^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}e(t) + e^{\mathrm{T}}(t-\tau)Pe(t-\tau) \\ \le & \left(-a + \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + \frac{5\,|\,y_1(t)\,|}{2}\right)e_1^2(t) \\ & + \left(\frac{|\,10z_2(t)+5z_1(t)\,|}{2} + 0.4\right)e_2^2(t) \\ & + \left(\frac{5\,|\,y_1(t)\,|}{2} + |\,b\,|\right)e_3^2(t), \end{aligned} \tag{35}$$

then

$$\begin{aligned} & \frac{e^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}e(t) + e^{\mathrm{T}}(t-\tau)Pe(t-\tau)}{e^{\mathrm{T}}(t)e(t)} \\ \le & \max\left(-a + \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + \frac{5\,|\,y_1(t)\,|}{2}, \right. \\ & \left. \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + 0.4, \frac{5\,|\,y_1(t)\,|}{2} + |\,b\,|\right). \end{aligned} \tag{36}$$

According to Theorem 1, define:

$$\begin{aligned} k_1 = & \max\left(-a + \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + \frac{5\,|\,y_1(t)\,|}{2}, \right. \\ & \left. \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + 0.4, \frac{5\,|\,y_1(t)\,|}{2} + |\,b\,|\right). \end{aligned}$$

When $nT+\sigma \le t < (n+1)T$, we design the controller as

$$\begin{aligned} u_1(t) &= le_1(t), \\ u_2(t) &= le_2(t), \\ u_3(t) &= le_3(t). \end{aligned} \tag{37}$$

According to expressions (33), (34), and (35), we can obtain

$$\begin{aligned} & e^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}e(t) + e^{\mathrm{T}}(t-\tau)Pe(t-\tau) \\ \le & \left(-a + \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + \frac{5\,|\,y_1(t)\,|}{2} + l\right)e_1^2(t) \\ & + \left(\frac{|\,10z_2(t)+5z_1(t)\,|}{2} + 0.4 + l\right)e_2^2(t) \\ & + \left(\frac{5\,|\,y_1(t)\,|}{2} + |\,b\,| + l\right)e_3^2(t). \end{aligned} \tag{38}$$

Then

$$\begin{aligned} & \frac{e^{\mathrm{T}}(t)\,{}_a^{\mathrm{C}}D_t^{\alpha}e(t) + e^{\mathrm{T}}(t-\tau)Pe(t-\tau)}{e^{\mathrm{T}}(t)e(t)} \\ \le & \max\left(-a + \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + \frac{5\,|\,y_1(t)\,|}{2} + l, \right. \\ & \left. \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + 0.4 + l, \frac{5\,|\,y_1(t)\,|}{2} + |\,b\,| + l\right). \end{aligned} \tag{39}$$

Define

$$\begin{aligned} k_3 = & \max\left(-a + \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + \frac{5\,|\,y_1(t)\,|}{2} + l, \right. \\ & \left. \frac{|\,10z_2(t)+5z_1(t)\,|}{2} + 0.4 + l, \frac{5\,|\,y_1(t)\,|}{2} + |\,b\,| + l\right). \end{aligned}$$

Then $k_3 = k_1 + l$. Suppose

$$E_{\alpha}(k_1\sigma^{\alpha})E_{\alpha}(k_2(T-\sigma)^{\alpha}) = 1. \tag{40}$$

If we can select a suitable l to make k_3 satisfy

$$k_3 \le k_2 + \lambda_{\max}\left(1 - \frac{1}{E_{\alpha}^2(k_2\tau^{\alpha})}\right). \tag{41}$$

According to Theorem 1, Theorem 2, and Theorem 3, the error system (32) is stable to zero via intermittent control (37).

5. Numerical simulation

In this section, we will show the numerical results for intermittent synchronization of the fractional delayed Newton–Leipnik system to verify and illustrate the effectiveness of the proposed theorems.

From the numerical simulations result in Figs. 2 and 3, we can see:

$$\begin{aligned} & x_1 \in [-0.5569,\ 0.7229], \quad y_1 \in [-0.4038,\ 0.5267], \\ & z_1 \in [-0.27,\ 0.43]. \end{aligned}$$

In function (9), we can get

$$k_1 = \max(4.14175, 3.625, 1.49175) = 4.14175. \tag{42}$$

In numerical simulation, the parameters of the drive system (29) are taken as $\tau = 0.03$ and fractional order $\alpha = 0.97$ and the initial values are selected as $x_1(0) = 0.23, y_1(0) = 0.51, z_1(0) = 0.43$, and the control period as $T = 0.1$, the control width $\delta = 0.06$. From $E_{\alpha}(k_1\sigma^{\alpha})E_{\alpha}(k_2(T-\sigma)^{\alpha}) = 1$, $\lambda_{\max} = 0.2$, we can get: $k_2 = -6.163$, $k_3 \le -6.163 + 0.2(1 - 1/E_{\alpha}^2(-0.198)) = -6.207$, then we can get $l \ge 10.31$ from Section 4. Here we select $l = 11$ and the initial values of the response system (17) are selected as $x_2(0) = -7.23, y_2(0) = -8.41, z_2(0) = -10.2$ and the intermittent synchronization errors are shown in Fig. 4. Figure 4 shows that the intermittent synchronization errors gradually converge to zero with time t, which implies that the systems achieve intermittent synchronization. It also shows the effectiveness of the controller in the response system designed according to the proposed method.

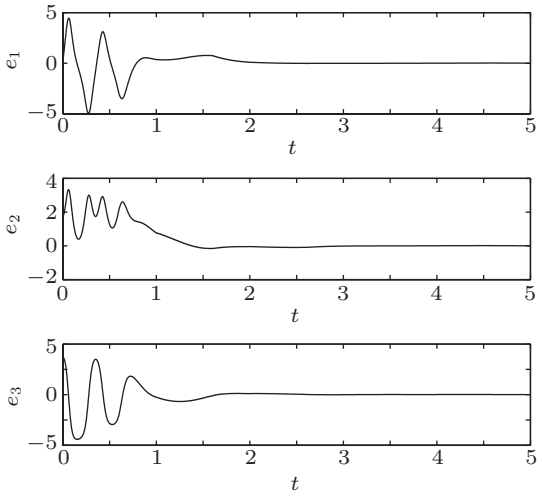


Fig. 4. The synchronization errors $e_1(t)$, $e_2(t)$, $e_3(t)$ with time t.

6. Conclusion

In this paper, we proposed a novel stability method and an intermittent stable theorem for a fractional nonlinear system with delay. The proposed methods have been applied to synchronize fractional Newton–Leipnik system with time delay. The numerical simulations have shown the effectiveness of the proposed methods. The two methods also pave the way for controlling and synchronizing a fractional system with delay by more ways.

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