The fractional coupled KdV equations: Exact solutions and white noise functional approach

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(Received 21 October 2012; revised manuscript received 7 March 2013)

Variable coefficients and Wick-type stochastic fractional coupled KdV equations are investigated. By using the modified fractional sub-equation method, Hermite transform, and white noise theory the exact travelling wave solutions and white noise functional solutions are obtained, including the generalized exponential, hyperbolic, and trigonometric types.

Keywords: coupled KdV equations, fractional calculus, white noise, Hermite transform

PACS: 05.40.−a, 02.30.Jr **DOI:** 10.1088/1674-1056/22/8/080501

1. Introduction

In this paper, we will investigate the fractional coupled KdV equations:

$$\begin{cases} \mathrm{D}_t^{\alpha}u + p(t)u\mathrm{D}_x^{\alpha}u + q(t)v\mathrm{D}_x^{\alpha}v + r(t)\mathrm{D}_x^{3\alpha}u = 0, \\ \mathrm{D}_t^{\alpha}v + \mathrm{D}_x^{3\alpha}v - 3u\mathrm{D}_x^{\alpha}v = 0, \end{cases} \tag{1}$$

where $(x,t) \in \mathbb{R}\times\mathbb{R}_+$, $0<\alpha\le 1$, $\mathrm{D}_t^{\alpha}u$ and $\mathrm{D}_x^{\alpha}u$ are the modified Riemann–Liouville derivatives, while $p(t)$, $q(t)$, and $r(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. The stochastic version of Eq. (1) in the Wick sense can be written in the following way

$$\begin{cases} \mathrm{D}_t^{\alpha}U + P(t)\diamond U\diamond\mathrm{D}_x^{\alpha}U + Q(t)\diamond V\diamond\mathrm{D}_x^{\alpha}V \\ \qquad + R(t)\diamond\mathrm{D}_x^{3\alpha}U = 0, \\ \mathrm{D}_t^{\alpha}V + \mathrm{D}_x^{3\alpha}V - 3U\diamond\mathrm{D}_x^{\alpha}V = 0, \end{cases} \tag{2}$$

where "$\diamond$" is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$, $P(t)$, $Q(t)$, and $R(t)$ are $(\mathcal{S})_{-1}$-valued functions.[1]

The analytical and approximate solutions of deterministic fractional partial differential equations attracted great attention and became a considerably interesting subject in mathematical physics. There are many methods for calculating the approximate solutions for nonlinear fractional partial differential equations such as the variational iterations method,[2] Adomian decomposition method,[3,4] the homotopy perturbation method,[5,6] and the expansion-function method.[7–11]

The exact solutions for nonlinear fractional partial differential equations are still of great interest. Li and He[12] introduced complex transform for reducing fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus. Since Wadati first introduced and studied stochastic KdV equation,[13] many authors, e.g., Xie,[14–17] Chen and Xie,[18–20] Ghany,[21] Ghany and Saad,[22] and many other authors,[23,24] have investigated more intensively the stochastic partial differential equations (SPDE).

The present paper is motivated by the desire to propose the modified fractional sub-equation method to construct exact analytical solutions for the variable coefficients Wick-type stochastic fractional coupled KdV equations with the modified Riemann–Liouville derivative. As pointed out in Refs. [25] and [26] the anomalous motion of two weakly nonlinear water wave modes with nearly coincident linear long-wave speeds can be described by Eq. (1). Moreover, if the problem is considered in a random environment, we can get fractional random coupled KdV equations. In order to give the exact solutions of random coupled KdV equations, we only consider this problem in white noise environment, that is, we will study the variable coefficients Wick-type stochastic fractional coupled KdV equations (2).

Our first interest in this work is to implement new strategies that give white noise functional solutions of the variable coefficients Wick-type stochastic fractional coupled KdV equations. The strategies that will be pursued in this work rest mainly on Hermite transform, white noise theory, and modified fractional sub-equation method, all of which are employed to find white noise functional solutions of Eq. (2). The proposed schemes, as we believe, are entirely new and introduce new solutions in addition to the well-known traditional solutions. The ease of using these methods shows their power to determine shock or solitary type of solutions.

The rest of this paper is organized as follows: In Section 2, we recall the definition and some properties of the modified Riemann–Liouville derivative and introduce the modified fractional sub-equation method. In Section 3, we apply Her-

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mite transform, white noise theory, and modified fractional sub-equation method to explore exact travelling wave solutions for Eq. (1). In Section 4, we use the inverse Hermite transform and Theorem 4.1.1 (Ref. [1]) to obtain white noise functional solutions for Eq. (2). The last section is devoted to a summary and discussion.

2. Preliminaries

There are different definitions for fractional derivatives; for more details see Ref. [27]. In our paper we use the modified Riemann–Liouville derivative which was defined by Jumarie[28]

$$\mathrm{D}_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-y)^{-\alpha-1}[f(y)-f(0)]\,\mathrm{d}y, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^x (x-y)^{-\alpha}[f(y)-f(0)]\,\mathrm{d}y, & 0 < \alpha < 1, \\ \left[f^{(\alpha-n)}(x)\right]^{(n)}, & n \le \alpha < n+1,\ n \in \mathbb{N}, \end{cases} \tag{3}$$

which has merits over the original one, for example, the α-order derivative of a constant is zero. The main properties of the modified Riemann–Liouville derivative were summarized in Ref. [28] and three useful formulas of them are given as follows:

$$\begin{cases} \mathrm{D}_x^\alpha x^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta > 0, \\ \mathrm{D}_x^\alpha (u(x)v(x)) = u(x)\mathrm{D}_x^\alpha v(x) + v(x)\mathrm{D}_x^\alpha u(x), \\ \mathrm{D}_x^\alpha [f(u(x)] = \dfrac{\mathrm{d}f}{\mathrm{d}u}\mathrm{D}_x^\alpha u(x) = \left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)^\alpha D_u^\alpha f(u). \end{cases} \tag{4}$$

Now, we will outline the main idea of the modified fractional sub-equation method. Many authors considered nonlinear fractional partial differential equation, say, in two variables

$$F(u, u_x, u_t, \mathrm{D}_x^\alpha u, \mathrm{D}_t^\alpha u, \ldots) = 0, \quad 0 < \alpha \le 1, \tag{5}$$

where F is a nonlinear function with respect to the indicated variables. To determine the solution $u = u(x,t)$ explicitly, we first introduce the following transformation

$$u = u(\xi), \quad \xi = \xi(x,t), \tag{6}$$

which converts Eq. (5) into a fractional ordinary differential equation

$$G(u, u', u'', D_\xi^\alpha u, D_\xi^{2\alpha} u, \ldots) = 0. \tag{7}$$

Next we introduce a new variable $Y = Y(\xi)$ which is a solution of the fractional Riccati equation

$$D_\xi^\alpha Y = \lambda + \mu Y + \nu Y^2, \quad 0 < \alpha \le 1, \tag{8}$$

where λ, μ, and ν are arbitrary constants. Equation (8) is the fractional Riccati differential equation, where α is a parameter describing the order of the fractional derivative. In the case of $\alpha = 1$, equation (8) is reduced to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems. The feedback gain of the linear quadratic optimal control depends on a solution of a Riccati differential equation which has to be found for the whole time horizon of the control process.[29,30] Then we propose the following series expansion as a solution of Eq. (5)

$$u(x,t) = u(\xi) = \sum_{k=-n}^{n} a_k(x,t) Y^k(\xi), \tag{9}$$

where $a_k (k = -n, -n+1, \ldots, 0, 1, \ldots, n)$ are functions to be determined later and n is a positive integer which can be determined via the balancing of the highest derivative term with the nonlinear term in Eq. (7). Inserting Eq. (9) into Eq. (7) and using Eq. (8) will give an algebraic equation in powers of Y. Since all coefficients of Y^k must vanish, this will give a system of algebraic equations with respect to a_k. With the aid of Mathematica, we can determine a_k. From the recent paper by Zhang *et al.*,[30] we can deduce the following set of solutions of Eq. (8)

$$\begin{cases} Y(\xi) = E_\alpha(\xi) - 1, & \lambda = \mu = 1, \nu = 0, \\ Y(\xi) = \coth_\alpha(\xi) \pm \mathrm{csch}_\alpha(\xi), Y(\xi) = \tanh_\alpha(\xi) \pm \mathrm{isech}_\alpha(\xi), & \lambda = \dfrac{1}{2}, \mu = 0, \nu = -\dfrac{1}{2}, \\ Y(\xi) = \dfrac{1}{2}\tan_\alpha(2\xi), Y(\xi) = \dfrac{1}{2}\cot_\alpha(2\xi), & \lambda = 1, \mu = 0, \nu = 4, \end{cases} \tag{10}$$

with the generalized hyperbolic and trigonometric functions

$$\tanh_\alpha(x) = \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)}, \quad \coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)},$$
$$\mathrm{csch}_\alpha(x) = \frac{1}{\sinh_\alpha(x)}, \quad \mathrm{sech}_\alpha(x) = \frac{1}{\cosh_\alpha(x)},$$
$$\sinh_\alpha(x) = \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2},$$
$$\cosh_\alpha(x) = \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2},$$
$$\tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)}, \quad \cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)},$$
$$\sin_\alpha(x) = \frac{E_\alpha(\mathrm{i}x^\alpha) - E_\alpha(-\mathrm{i}x^\alpha)}{2\mathrm{i}},$$
$$\cos_\alpha(x) = \frac{E_\alpha(\mathrm{i}x^\alpha) + E_\alpha(-\mathrm{i}x^\alpha)}{2},$$

defined by the Mittag–Leffler function

$$E_\alpha(y)=\sum_{j=0}^{\infty}\frac{y^j}{\Gamma(1+j\alpha)}.$$

For more details about the generalized exponential, hyperbolic and trigonometric functions, see Ref. [31].

3. Exact travelling wave solutions for Eq. (1)

In this section, we apply Hermite transform, white noise theory, and modified fractional sub-equation method to explore exact travelling wave solutions for Eq. (1). Taking the Hermite transform of Eq. (2), we get the deterministic equation

$$\begin{cases}\mathrm{D}_t^\alpha\tilde{U}(x,t,z)+\tilde{P}(t,z)\tilde{U}(x,t,z)\mathrm{D}_x^\alpha\tilde{U}(x,t,z)\\ +\tilde{Q}(t,z)\tilde{V}(x,t,z)\mathrm{D}_x^\alpha\tilde{V}(x,t,z)+\tilde{R}(t,z)\mathrm{D}_x^{3\alpha}\tilde{U}(x,t,z)=0,\\ \mathrm{D}_t^\alpha\tilde{V}(x,t,z)+\mathrm{D}_x^{3\alpha}\tilde{V}(x,t,z)-3\tilde{U}(x,t,z)\mathrm{D}_x^\alpha\tilde{V}(x,t,z)=0,\end{cases}\quad(11)$$

where $z=(z_1,z_2,\ldots)\in(\mathbb{C}^{\mathbb{N}})_c$ is a vector parameter. To look for the travelling wave solution of Eq. (11), we make the transformations $\tilde{U}(x,t,z)=u(x,t,z)=u(\xi(x,t,z))$, $\tilde{V}(x,t,z)=v(x,t,z)=v(\xi(x,t,z))$, with

$$\xi(x,t,z)=kx+s\int_0^t l(\tau,z)\,\mathrm{d}\tau+c,$$

where k, s, and c are arbitrary constants which satisfy $ks\neq0$ and $l(\tau,z)$ is nonzero function of the indicated variables to be determined later. Equation (11) can be changed to the form

$$\begin{cases}(sl)^\alpha\mathrm{D}_\xi^\alpha u+k^\alpha pu\mathrm{D}_\xi^\alpha u+k^\alpha qv\mathrm{D}_\xi^\alpha v+k^{3\alpha}r\mathrm{D}_\xi^{3\alpha}u=0,\\ (sl)^\alpha\mathrm{D}_\xi^\alpha v+k^{3\alpha}\mathrm{D}_\xi^{3\alpha}v-3k^\alpha u\mathrm{D}_\xi^\alpha v=0,\end{cases}\quad(12)$$

where $p(t,z)=\tilde{P}(t,z)$, $q(t,z)=\tilde{Q}(t,z)$, and $r(t,z)=\tilde{R}(t,z)$. Balancing $\mathrm{D}_\xi^{3\alpha}u$, $u\mathrm{D}_\xi^\alpha u$, and $\mathrm{D}_\xi^{3\alpha}v$, $v\mathrm{D}_\xi^\alpha v$, and $u\mathrm{D}_\xi^\alpha v$ in turn, gives $n=m=2$, hence we set the solution of Eq. (11) by

$$\begin{cases}u(x,t)=u(\xi)\\ \quad=a_0(t,z)+a_1(t,z)Y(\xi)+a_2(t,z)Y^2(\xi)\\ \quad\quad+a_{-1}(t,z)Y^{-1}(\xi)+a_{-2}(t,z)Y^{-2}(\xi),\\ v(x,t)=v(\xi)\\ \quad=b_0(t,z)+b_1(t,z)Y(\xi)+b_2(t,z)Y^2(\xi)\\ \quad\quad+b_{-1}(t,z)Y^{-1}(\xi)+b_{-2}(t,z)Y^{-2}(\xi),\end{cases}\quad(13)$$

where $Y(\xi)$ satisfies the fractional Riccati equation (8). By substituting Eq. (13) along with Eq. (8) into Eq. (12), collecting the coefficients of Y^k and setting them to be zero, we will obtain a system of algebraic equations in the unknowns a_k,b_k ($k=-2,-1,0,1,2$), and l of the form

$$\begin{cases}(sl)^\alpha G_i+k^\alpha p\zeta_i+k^\alpha q\zeta_i^*+k^{3\alpha}rK_i=0, & i=0,1,2,3,\\ (sl)^\alpha H_i+k^\alpha p\eta_i+k^\alpha q\eta_i^*+k^{3\alpha}rL_i=0, & i=1,2,3,\\ p\zeta_i+q\zeta_i^*+k^{2\alpha}rK_i=0, & i=4,5,\\ p\eta_i+q\eta_i^*+k^{2\alpha}rL_i=0, & i=4,5,\\ (sl)^\alpha G_i^*+k^{3\alpha}K_i^*-3k^\alpha\rho_i=0, & i=0,1,2,3,\\ (sl)^\alpha H_i^*+k^{3\alpha}L_i^*-3k^\alpha\omega_i=0, & i=1,2,3,\\ k^{2\alpha}K_i^*-3\rho_i=0, & i=4,5,\\ k^{2\alpha}L_i^*-3\omega_i=0, & i=4,5,\end{cases}\quad(14)$$

where,

$G_0=\lambda a_1-\nu a_{-1}$, $G_1=2\lambda a_2+\mu a_1$, $G_2=2\mu a_2+\nu a_1$,
$G_3=3\nu a_2$, $G_0^*=\lambda b_1-\nu b_{-1}$, $G_1^*=2\lambda b_2+\mu b_1$,
$G_2^*=2\lambda b_2+\nu b_1$, $G_3^*=2\nu b_2$, $H_1=-(2\nu a_{-2}+\mu a_{-1})$,
$H_2=-(2\mu a_{-2}+\lambda a_{-1})$, $H_3=-2\lambda a_{-2}$,
$H_1^*=-(2\nu b_{-2}+\mu b_{-1})$, $H_2^*=-(2\mu b_{-2}+\lambda b_{-1})$,
$H_3^*=-2\lambda b_{-2}$, $K_0=\lambda(2\lambda G_2+\mu G_1)+\nu(2\nu H_2+\mu H_1)$,
$K_1=2\lambda(3\lambda G_3+2G_2+G_1)+\mu(2\lambda G_2+\mu G_1)$,
$K_2=3\lambda(3\mu G_3+2\nu G_2)+2\mu(3\lambda G_3+2\mu G_2+\nu G_1)$
$\quad+\nu(2\lambda G_2+\mu G_1)$,
$K_3=2\nu(3\lambda G_3+2\mu G_2+\nu G_1)+3\mu(3\mu G_3+2\nu G_2)$
$\quad+12\lambda\nu G_3$,
$K_4=3\nu(3\mu G_3+2\nu G_2)+12\mu\nu G_3$, $K_5=12\nu^2G_3$,
$K_0^*=\lambda(2\lambda G_2^*+\mu G_2^*)+\nu(2\nu H_2^*+\mu H_1^*)$,
$K_1^*=2\lambda(3\lambda G_3^*+2\mu G_2^*+\nu G_1^*)+\mu(2\lambda G_2^*+\mu G_1^*)$,
$K_2^*=3\lambda(3\mu G_3^*+2\nu G_2^*)+2\mu(3\lambda G_3^*+2\mu G_2^*+\nu G_1^*)$
$\quad+\nu(2G_2^*+\mu G_1^*)$,
$K_3^*=2\nu(3\lambda G_3^*+2\mu G_2^*+\nu G_1^*)+3\mu(3\mu G_3^*+2\nu G_2^*)+12G_3^*$,
$K_4^*=3\nu(3\mu G_3^*+2\nu G_2^*)+12\mu\nu G_3^*$, $K_5^*=12\nu^2G_3^*$,
$L_1=2\nu(3\nu H_3+2\mu H_2+\lambda H_1)+\mu(2H_2+\mu H_1)$,
$L_2=3\nu(3\mu H_3+2\lambda H_2)+2\mu(3\nu H_3+2\mu H_2+\lambda H_1)$
$\quad+\lambda(2\nu H_2+\mu H_1)$,
$L_3=2\lambda(3\nu H_3+2\mu H_2+\lambda H_1)+3\mu(3\mu H_3+2\lambda H_2)$
$\quad+12\lambda\nu H_3$,
$L_4=3\lambda(3H_3+2\lambda H_2)+12\mu\lambda H_3$, $L_5=12\lambda^2H_3$,
$L_1^*=2\nu(3\nu H_3^*+2\mu H_2^*+\lambda H_1^*)+\mu(2H_2^*+\mu H_1^*)$,
$L_2^*=3\nu(3\mu H_3^*+2\lambda H_2^*)+2\mu(3\nu H_3^*+2\mu H_2^*+\lambda H_1^*)$
$\quad+\lambda(2\nu H_2^*+\mu H_1^*)$,
$L_3^*=2\lambda(3\nu H_3^*+2\mu H_2^*+\lambda H_1^*)+3\mu(3\mu H_3^*+2\lambda H_2^*)$
$\quad+12\lambda\nu H_3^*$,
$L_4^*=3\lambda(3H_3^*+2\lambda H_2^*)+12\mu\lambda H_3^*$, $L_5^*=12\lambda^2H_3^*$,
$\zeta_0=a_0G_0+a_1H_1+a_2H_2+a_{-1}G_1+a_{-2}G_2$,
$\zeta_1=a_0G_1+a_1G_0+a_2H_1+a_{-1}G_2+a_{-2}G_3$,
$\zeta_2=a_0G_2+a_1G_1+a_2G_0+a_{-1}G_3$,
$\zeta_3=a_0G_3+a_1G_2+a_2G_1$, $\zeta_4=a_1G_3+a_2G_2$, $\zeta_5=a_2G_3$,
$\zeta_0^*=b_0G_0^*+b_1H_1^*+b_2H_2^*+b_{-1}G_1^*+b_{-2}G_2^*$,
$\zeta_1^*=b_0G_1^*+b_1G_0^*+b_2H_1^*+b_{-1}G_2^*+b_{-2}G_3^*$,
$\zeta_2^*=b_0G_2^*+b_1G_1^*+b_2G_0^*+b_{-1}G_3^*$,
$\zeta_3^*=b_0G_3^*+b_1G_2^*+b_2G_1^*$,
$\zeta_4^*=b_1G_3^*+b_2G_2^*$, $\zeta_5^*=b_2G_3^*$,
$\eta_1=a_0H_1+a_1H_2+a_2H_3+a_{-1}G_0+a_{-2}G_1$,
$\eta_2=a_0H_2+a_1H_3+a_{-2}G_0+a_{-1}H_1$,

$$\eta_3 = a_0H_3 + a_{-1}H_2 + a_{-2}H_1,$$
$$\eta_4 = a_{-1}H_3 + a_{-2}H_2, \quad \eta_5 = a_{-2}H_3,$$
$$\eta_1^* = b_0H_1^* + b_1H_2^* + b_2H_3^* + b_{-1}G_0^* + b_{-2}G_1^*,$$
$$\eta_2^* = b_0H_2^* + b_1H_3^* + b_{-2}G_0^* + b_{-1}H_1^*,$$
$$\eta_3^* = b_0H_3^* + b_{-1}H_2^* + b_{-2}H_1^*,$$
$$\eta_4^* = b_{-1}H_3^* + b_{-2}H_2^*, \quad \eta_5^* = b_{-2}H_3^*,$$
$$\rho_0 = a_0G_0^* + a_1H_1^* + a_2H_2^* + a_{-1}G_1^* + a_{-2}G_2^*,$$
$$\rho_1 = a_0G_1^* + a_1G_0^* + a_2H_1^* + a_{-1}G_2^* + a_{-2}G_3^*,$$
$$\rho_2 = a_0G_2^* + a_1G_1^* + a_2G_0^* + a_{-1}G_3^*,$$
$$\rho_3 = a_0G_3^* + a_1G_2^* + a_2G_1^*, \quad \rho_4 = a_1G_1^* + a_2G_2^*,$$
$$\rho_5 = a_2G_3^*,$$
$$\omega_1 = a_0H_1^* + a_1H_2^* + a_2H_3^* + a_{-1}G_0^* + a_{-2}G_1^*,$$
$$\omega_2 = a_0H_2^* + a_1H_3^* + a_{-1}H_1^* + a_{-2}G_0^*,$$
$$\omega_3 = a_0H_3^* + a_{-1}H_2^* + a_{-2}H_1^*,$$
$$\omega_4 = a_{-1}H_3^* + a_{-2}H_2^*, \quad \text{and} \quad \omega_5 = a_{-2}H_3^*.$$

Case 1 If we set $\lambda = \mu = 1$ and $\nu = 0$ in Eq. (8), and use Mathematica to solve the resulting system, we will obtain the following set of solutions

$$a_0 = a_1 = a_2 = b_1 = b_2 = 0, \quad a_{-1} = \frac{7}{2}k^{2\alpha} - \frac{\theta_1\theta_2}{2k^{2\alpha}\sqrt{q}(p+3r)},$$
$$a_{-2} = 4k^{2\alpha}, \quad b_{-1} = \frac{2}{q}(-3\sqrt{q}\theta_1 + \theta_2), \quad b_{-2} = \frac{-4}{\sqrt{q}}\theta_1,$$
$$b_0 = \frac{11\sqrt{q}\theta_1(5+11p+(2r-p-7)r)+5\theta_2\,(5+p+(p-7)r)}{16((5+p)p+(15+p-6r)r)},$$
$$l = \frac{k^3}{s}\left(\frac{-p^2}{q}\right)^{1/\alpha},$$

where, $\theta_1 = \mathrm{i}k^{2\alpha}\sqrt{p+3r}$, $\theta_2 = \mathrm{i}\sqrt{q(p+3r)}$, $q > 0$, $p + 3r > 0$.

Substituting these values into Eq. (13) and using Eq. (10), we obtain travelling wave solutions of generalized exponential type

$$u_1(x,t,z) = \frac{7k^{4\alpha}\sqrt{q(t,z)}[p(t,z)+3r(t,z)] - \theta_1(t,z)\theta_2(t,z)}{2k^{2\alpha}\sqrt{q(t,z)}[p(t,z)+3r(t,z)][E_\alpha[\xi_1(x,t,z)]-1]} + \frac{4k^{2\alpha}}{[E_\alpha[\xi_1(x,t,z)]-1]^2}, \tag{15}$$

$$v_1(x,t,z) = \frac{11\sqrt{q(t,z)}\theta_1(t,z)[5+11p(t,z)+(2r(t,z)-p(t,z)-7)r(t,z)]}{16[(5+p(t,z))p(t,z)+(15+p(t,z)-6r(t,z))r(t,z)]} + \frac{5\theta_2(t,z)[5+p(t,z)+(p(t,z)-7)r(t,z)]}{16[(5+p(t,z))p(t,z)+(15+p(t,z)-6r(t,z))r(t,z)]}$$
$$+ \left(\frac{2}{q(t,z)}\left(-3\sqrt{q(t,z)}\theta_1(t,z)+\theta_2(t,z)\right)\left(E_\alpha[\xi_1(x,t,z)]-1\right) - \frac{4}{\sqrt{q(t,z)}}\theta_1(t,z)\right) \times \left(E_\alpha[\xi_1(x,t,z)]-1\right)^{-2}, \tag{16}$$

with

$$\xi_1(x,t,z) = kx + (-1)^{1/\alpha}k^3\int_0^t\left(\frac{p^2(\tau,z)}{q(\tau,z)}\right)^{1/\alpha}\mathrm{d}\tau + c_1.$$

Case 2 If we set $\lambda = 1/2, \mu = 0$ and $\nu = -1/2$ in Eq. (8), and use Mathematica to solve the resulting system, we will obtain the following set of solutions

$$a_1 = a_2 = a_{-1} = b_1 = b_2 = b_{-1} = 0,$$
$$a_0 = \frac{k^{2\alpha}r[(13-3k^\alpha)p+10k^\alpha q]}{2p[p+k^\alpha q]}, \quad b_0 = \frac{3k^{2\alpha}r}{q},$$
$$a_{-2} = -b_{-2} = \frac{-3k^{3\alpha}r}{p+k^\alpha q}, \quad l = \frac{3k^3}{2s}\left(\frac{(k^\alpha-1)rp}{p+k^\alpha q}\right)^{1/\alpha},$$

where $q \neq 0$.

Substituting these values into Eq. (13) and using Eq. (10), we obtain the travelling wave solutions of generalized hyperbolic type

$$u_2(x,t,z) = \frac{k^{2\alpha}r(t,z)[(13-3k^\alpha)p(t,z)+10k^\alpha q(t,z)]}{2p(t,z)[p(t,z)+k^\alpha q(t,z)]} - \frac{3k^{3\alpha}r(t,z)}{p(t,z)+k^\alpha q(t,z)} \times \left(\coth_\alpha[\xi_2(x,t,z)] \pm \mathrm{csch}_\alpha[\xi_2(x,t,z)]\right)^{-2}, \tag{17}$$

$$u_3(x,t,z) = \frac{k^{2\alpha}r(t,z)[(13-3k^\alpha)p(t,z)+10k^\alpha q(t,z)]}{2p(t,z)[p(t,z)+k^\alpha q(t,z)]} - \frac{3k^{3\alpha}r(t,z)}{p(t,z)+k^\alpha q(t,z)} \times \left(\tanh_\alpha[\xi_2(x,t,z)] \pm \mathrm{i}\ \mathrm{sech}_\alpha[\xi_2(x,t,z)]\right)^{-2}, \tag{18}$$

$$v_2(x,t,z) = \frac{3k^{2\alpha}r(t,z)}{q(t,z)} + \frac{3k^{3\alpha}r(t,z)}{p(t,z)+k^\alpha q(t,z)} \times \left(\coth_\alpha[\xi_2(x,t,z)] \pm \mathrm{csch}_\alpha[\xi_2(x,t,z)]\right)^{-2}, \tag{19}$$

$$v_3(x,t,z) = \frac{3k^{2\alpha}r(t,z)}{q(t,z)} + \frac{3k^{3\alpha}r(t,z)}{p(t,z)+k^\alpha q(t,z)} \times \left(\tanh_\alpha[\xi_2(x,t,z)] \pm \mathrm{i}\ \mathrm{sech}_\alpha[\xi_2(x,t,z)]\right)^{-2}, \tag{20}$$

with

$$\xi_2(x,t,z) = kx + \frac{3}{2}k^3(k^\alpha-1)^{1/\alpha} \times \int_0^t\left(\frac{r(\tau,z)p(\tau,z)}{p(\tau,z)+k^\alpha q(\tau,z)}\right)^{1/\alpha}\mathrm{d}\tau + c_2.$$

Case 3 If we set $\lambda = 1$, $\mu = 0$, and $\nu = 4$ in Eq. (8), and use Mathematica to solve the resulting system, we will obtain the following set of solutions

$$a_1 = a_{-1} = a_{-2} = b_0 = b_1 = b_{-1} = b_{-2} = 0, \quad a_2 = 64k^{2\alpha},$$
$$b_2 = \mathrm{i}k^{2\alpha}\sqrt{\frac{6(683p+2048r)}{q}},$$
$$a_0 = k^{2\alpha}\left[\frac{32}{3} + \frac{k^\alpha}{3+k^\alpha p}\left(16r - \frac{4093}{384}p\right)\right],$$
$$l = \frac{k^3}{s(k^\alpha+3)^{1/\alpha}}\left(48r - \frac{4093}{128}p\right)^{1/\alpha},$$

where $q \neq 0$, and $683p + 2048r/q > 0$.

Substituting these values into Eq. (3) and using Eq. (8), we obtain travelling wave solutions of generalized trigonometric type

$$u_4(x,t,z) = k^{2\alpha}\left[\frac{32}{3} + \frac{k^\alpha}{3+k^\alpha p(t,z)}\left(16r(t,z) - \frac{4093}{384}p(t,z)\right)\right] + 16k^{2\alpha}\cot_\alpha^2[2\xi_3(x,t,z)], \tag{21}$$

$$u_5(x,t,z) = k^{2\alpha}\left[\frac{32}{3} + \frac{k^\alpha}{3+k^\alpha p(t,z)}\left(16r(t,z) - \frac{4093}{384}p(t,z)\right)\right] + 16k^{2\alpha}\tan_\alpha^2[2\xi_3(x,t,z)], \tag{22}$$

$$v_4(x,t,z) = \frac{\mathrm{i}k^{2\alpha}}{4}\sqrt{\frac{6[683p(t,z)+2048r(t,z)]}{q(t,z)}} \times \cot_\alpha^2[2\xi_3(x,t,z)], \tag{23}$$

$$v_5(x,t,z) = \frac{\mathrm{i}k^{2\alpha}}{4}\sqrt{\frac{6[683p(t,z)+2048r(t,z)]}{q(t,z)}} \times \tan_\alpha^2[2\xi_3(x,t,z)], \tag{24}$$

with

$$\xi_3(x,t,z) = kx + \frac{k^3}{(k^\alpha+3)^{1/\alpha}} \times \int_0^t \left(48r(\tau,z) - \frac{4093}{128}p(\tau,z)\right)^{1/\alpha}\mathrm{d}\tau + c_3.$$

4. White noise functional solutions for Eq. (2)

In this section, we use the inverse Hermite transform and Theorem 4.1.1 in Ref. [1] to obtain white noise functional solutions for Eq. (2). The properties of generalized exponential, hyperbolic, and trigonometric functions yield that there exists a bounded open set $G \subset \mathbb{R}\times\mathbb{R}_+$, $m < \infty$, $n > 0$ such that the solution $\{u(x,t,z), v(x,t,z)\}$ of Eq. (11) and all its fractional derivatives which are involved in Eq. (11) are uniformly bounded for $(x,t,z) \in G \times K_m(n)$, continuous with respect to $(x,t) \in G$ for all $z \in K_m(n)$ and analytic with respect to $z \in K_m(n)$, for all $(x,t) \in G$. From Theorem 4.1.1 in Ref. [1], there exist $U(x,t,z)$, $V(x,t,z) \in (\mathcal{S})_{-1}$ such that $u(x,t,z) = \tilde{U}(x,t)(z)$ and $v(x,t,z) = \tilde{V}(x,t)(z)$ for all $(x,t,z) \in G \times K_m(n)$ and $\{U(x,t), V(x,t)\}$ solves Eq. (2) in $(\mathcal{S})_{-1}$. Hence, by applying the inverse Hermite transform to Eqs. (15)–(24), we will obtain the white noise functional solutions of Eq. (2) as given below.

4.1. White noise functional solutions of generalized exponential type

For $Q(t) > 0$, $P(t) + 3R(t) > 0$, the solutions of Eq. (2) are given by

$$U_1(x,t) = \frac{7k^{4\alpha}\sqrt{Q(t)}\diamond[P(t)+3R(t)] - \Theta_1(t)\diamond\Theta_2(t)}{2k^{2\alpha}\sqrt{Q(t)}\diamond[P(t)+3R(t)]\diamond[E_\alpha^\diamond[\Xi_1(x,t)]-1]} + \frac{4k^{2\alpha}}{[E_\alpha^\diamond[\Xi_1(x,t)]-1]^{\diamond 2}}, \tag{25}$$

$$V_1(x,t,z) = \frac{11\sqrt{Q(t)}\diamond\Theta_1(t)\diamond[5+11P(t)+(2R(t)-P(t)-7)\diamond R(t)]}{16[(5+P(t,z))\diamond P(t)+(15+P(t)-6R(t))\diamond R(t)]} + \frac{5\Theta_2(t)\diamond[5+P(t)+(P(t)-7)\diamond R(t)]}{16[(5+P(t,z))\diamond P(t)+(15+P(t)-6R(t))\diamond R(t)]} + \left(\frac{2}{Q(t)}\left(-3\sqrt{Q(t)}\diamond\Theta_1(t)+\Theta_2(t)\right)\diamond\left(E_\alpha^\diamond[\Xi_1(x,t)]-1\right) - \frac{4}{\sqrt{Q(t)}}\diamond\Theta_1(t)\right)\diamond\left(E_\alpha^\diamond[\Xi_1(x,t)]-1\right)^{\diamond(-2)}, \tag{26}$$

with,

$$\Theta_1(t) = \mathrm{i}k^{2\alpha}\sqrt{P(t)+3R(t)},$$
$$\Theta_2(t) = \mathrm{i}\sqrt{Q(t)\diamond(P(t)+3R(t))},$$

and

$$\Xi_1(x,t) = kx + (-1)^{1/\alpha}k^3\int_0^t\left(\frac{p^{\diamond 2}(\tau)}{Q(\tau)}\right)^{\diamond 1/\alpha}\mathrm{d}\tau + c_1.$$

4.2. White noise functional solutions of generalized hyperbolic type

For $P(t)\diamond Q(t) \neq 0$, the solutions of Eq. (2) are given by

$$U_2(x,t) = \frac{k^{2\alpha}R(t)\diamond[(13-3k^\alpha)P(t)+10k^\alpha Q(t)]}{2P(t)\diamond[P(t)+k^\alpha Q(t)]} - \frac{3k^{3\alpha}R(t)}{P(t)+k^\alpha Q(t)}\diamond\left(\coth_\alpha^\diamond[\Xi_2(x,t)] \pm \mathrm{csch}_\alpha^\diamond[\Xi_2(x,t)]\right)^{\diamond(-2)}, \tag{27}$$

$$U_3(x,t) = \frac{k^{2\alpha}R(t)\diamond[(13-3k^\alpha)P(t)+10k^\alpha Q(t)]}{2P(t)\diamond[P(t)+k^\alpha Q(t)]} - \frac{3k^{3\alpha}R(t)}{P(t)+k^\alpha Q(t)}\diamond\left(\tanh_\alpha^\diamond[\Xi_2(x,t)] \pm \mathrm{isech}_\alpha^\diamond[\Xi_2(x,t)]\right)^{\diamond(-2)}, \tag{28}$$

$$V_2(x,t) = \frac{3k^{2\alpha}R(t)}{Q(t)} + \frac{3k^{3\alpha}R(t)}{P(t)+k^\alpha Q(t)}\diamond\left(\coth_\alpha^\diamond[\Xi_2(x,t)] \pm \mathrm{csch}_\alpha^\diamond[\Xi_2(x,t)]\right)^{\diamond(-2)}, \tag{29}$$

$$V_3(x,t) = \frac{3k^{2\alpha}R(t)}{Q(t)} + \frac{3k^{3\alpha}R(t)}{P(t)+k^\alpha Q(t)}\diamond\left(\tanh_\alpha^\diamond[\Xi_2(x,t)] \pm \mathrm{isech}_\alpha^\diamond[\Xi_2(x,t)]\right)^{\diamond(-2)}, \tag{30}$$

with

$$\Xi_2(x,t) = kx + \frac{3}{2}k^3(k^\alpha-1)^{1/\alpha} \times \int_0^t\left(\frac{R(\tau)\diamond P(\tau)}{P(\tau)+k^\alpha Q(\tau)}\right)^{\diamond 1/\alpha}\mathrm{d}\tau + c_2.$$

4.3. White noise functional solutions of generalized trigonometric type

For $P(t) \neq 0$, $[683P(t)+2048R(t)]\diamond Q(t)^{\diamond(-1)} > 0$, the solutions of Eq. (2) are given by

$$U_4(x,t) = k^{2\alpha}\left[\frac{32}{3} + \frac{k^\alpha}{3+k^\alpha P(t)}\diamond\left(16R(t) - \frac{4093}{384}P(t)\right)\right] + 16k^{2\alpha}\cot_\alpha^{\diamond 2}[2\Xi_3(x,t)], \tag{31}$$

$$U_5(x,t)=k^{2\alpha}\Big[\frac{32}{3}+\frac{k^\alpha}{3+k^\alpha P(t)}\diamond\Big(16R(t)-\frac{4093}{384}P(t)\Big)\Big]+16k^{2\alpha}\tan_\alpha^{\diamond 2}[2\Xi_3(x,t)], \tag{32}$$

$$V_4(x,t)=\frac{\mathrm{i}k^{2\alpha}}{4}\sqrt{\frac{6[683P(t)+2048R(t)]}{Q(t)}}\diamond\cot_\alpha^{\diamond 2}[2\Xi_3(x,t)], \tag{33}$$

$$V_5(x,t)=\frac{\mathrm{i}k^{2\alpha}}{4}\sqrt{\frac{6[683P(t)+2048R(t)]}{Q(t)}}\diamond\tan_\alpha^{\diamond 2}[2\Xi_3(x,t)], \tag{34}$$

with

$$\Xi_3(x,t)=kx+\frac{k^3}{(k^\alpha+3)^{1/\alpha}}\times\int_0^t\Big(48R(\tau)-\frac{4093}{128}P(\tau)\Big)^{\diamond 1/\alpha}\mathrm{d}\tau+c_3.$$

We observe that for different forms of $P(t)$, $Q(t)$, and $R(t)$, we can get different solutions of Eq. (2) from formulas (25)–(34).

Example 1 Let $W(t)=\dot{B}(t)$ be the Gaussian white noise, where $B(t)$ is the Brownian motion. We have the Hermite transform $W(t,z)=\sum_{i=1}^\infty z_i\int_0^t\eta_i(s)\,\mathrm{d}s$. Since $E_\alpha^\diamond[B(t)]=E_\alpha[B(t)-t^2/2]$, we have $\tan_\alpha^\diamond[B(t)]=\tan_\alpha[B(t)-t^2/2]$, $\cot_\alpha^\diamond[B(t)]=\cot_\alpha[B(t)-t^2/2]$, $\tanh_\alpha^\diamond[B(t)]=\tanh_\alpha[B(t)-t^2/2]$, $\coth_\alpha^\diamond[B(t)]=\coth_\alpha[B(t)-t^2/2]$, $\mathrm{sech}_\alpha^\diamond[B(t)]=\mathrm{sech}_\alpha[B(t)-t^2/2]$, and $\mathrm{csch}_\alpha^\diamond[B(t)]=\mathrm{csch}_\alpha[B(t)-t^2/2]$. Suppose $P(t)=R(t)=\pi_1Q(t)$ and $Q(t)=[f(t)+\pi_2W(t)]^\alpha$, where π_1,π_2 are arbitrary constants and $f(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$. Then the white noise functional solutions of Eq. (2) are as follows:

$$U_6(x,t)=\frac{7k^{2\alpha}+1}{2(E_\alpha[\Psi_1(x,t)]-1)}+\frac{4k^{2\alpha}}{(E_\alpha[\Psi_1(x,t)]-1)^2}, \tag{35}$$

$$V_6(x,t)=-\frac{(11k^{2\alpha}+5)(5+\pi_1^2[f(t)+\pi_2W(t)]^{2\alpha})+5\pi_1(5k^{2\alpha}-6)[f(t)+\pi_2W(t)]^\alpha}{8\mathrm{i}\sqrt{\pi_1}\big(20-4\pi_1[f(t)+\pi_2W(t)]^\alpha\big)}+\frac{4\mathrm{i}\sqrt{\pi_1}(1-3k^{2\alpha})}{E_\alpha[\Psi_1(x,t)]-1}-\frac{8\mathrm{i}k^{2\alpha}\sqrt{\pi_1}}{(E_\alpha[\Psi_1(x,t)]-1)^2}, \tag{36}$$

with

$$\Psi_1(x,t)=kx+(-1)^{1/\alpha}k^3\pi_1^{2/\alpha}\times\Big(\int_0^t f(\tau)\,\mathrm{d}\tau+\pi_2B(t)-\frac{\pi_2t^2}{2}\Big)+c_1,$$

$$U_7(x,t)=\frac{k^{2\alpha}\pi_1(13-3k^\alpha)+10k^{3\alpha}}{2\pi_1+2k^\alpha}-\frac{3\pi_1k^{3\alpha}}{\pi_1+k^\alpha}\times\big(\coth_\alpha[\Psi_2(x,t)]\pm\mathrm{csch}_\alpha[\Psi_2(x,t)]\big)^{-2}, \tag{37}$$

$$U_8(x,t)=\frac{k^{2\alpha}\pi_1(13-3k^\alpha)+10k^{3\alpha}}{2\pi_1+2k^\alpha}-\frac{3\pi_1k^{3\alpha}}{\pi_1+k^\alpha}\times\big(\tanh_\alpha[\Psi_2(x,t)]\pm\mathrm{isech}_\alpha[\Psi_2(x,t)]\big)^{-2}, \tag{38}$$

$$V_7(x,t)=3k^{2\alpha}\pi_1+\frac{3k^{3\alpha}\pi_1}{\pi_1+k^\alpha}\big(\coth_\alpha[\Psi_2(x,t)]\pm\mathrm{csch}_\alpha[\Psi_2(x,t)]\big)^{-2}, \tag{39}$$

$$V_8(x,t)=3k^{2\alpha}\pi_1+\frac{3k^{3\alpha}\pi_1}{\pi_1+k^\alpha}\big(\tanh_\alpha[\Psi_2(x,t)]\pm\mathrm{isech}_\alpha[\Psi_2(x,t)]\big)^{-2}, \tag{40}$$

with

$$\Psi_2(x,t)=kx+\frac{3}{2}\Big(\frac{\pi_1^2(k^\alpha-1)}{\pi_1+k^\alpha}\Big)\times\Big(\int_0^t f(\tau)\,\mathrm{d}\tau+\pi_2B(t)-\frac{\pi_2t^2}{2}\Big)+c_2,$$

$$U_9(x,t)=k^{2\alpha}\Big(\frac{32}{3}+\frac{2051k^\alpha\pi_1[f(t)+\pi_2W(t)]^\alpha}{1152+384k^\alpha\pi_1[f(t)+\pi_2W(t)]^\alpha}\Big)+16k^{2\alpha}\cot_\alpha^2[2\Psi_3(x,t)], \tag{41}$$

$$U_{10}(x,t)=k^{2\alpha}\Big(\frac{32}{3}+\frac{2051k^\alpha\pi_1[f(t)+\pi_2W(t)]^\alpha}{1152+384k^\alpha\pi_1[f(t)+\pi_2W(t)]^\alpha}\Big)+16k^{2\alpha}\tan_\alpha^2[2\Psi_3(x,t)], \tag{42}$$

$$V_9(x,t)=\frac{\mathrm{i}k^{2\alpha}}{4}\sqrt{16386\pi_1}\cot_\alpha^2[2\Psi_3(x,t)], \tag{43}$$

$$V_{10}(x,t)=\frac{\mathrm{i}k^{2\alpha}}{4}\sqrt{16386\pi_1}\tan_\alpha^2[2\Psi_3(x,t)], \tag{44}$$

with

$$\Psi_3(x,t)=kx+k^3\pi_1\Big(\frac{2105}{k^\alpha+3}\Big)^{1/\alpha}\times\Big(\int_0^t f(\tau)\,\mathrm{d}\tau+\pi_2B(t)-\frac{\pi_2t^2}{2}\Big)+c_3.$$

5. Summary and discussion

The anomalous motion of two weakly nonlinear water wave modes with nearly coincident linear long-wave speeds can be described by Eq. (1). If the problem is considered in a random environment, we can get fractional random coupled KdV equations. In order to give the exact solutions of random coupled KdV equations, we only consider this problem in a white noise environment, i.e., we investigate the Wick-type stochastic fractional coupled KdV equation (2).

In Ref. [32] the author introduced a new method to look for travelling wave solutions of NFDEs. This method is called fractional sub-equation method, and it is based on the homogeneous balance principle[30] and Jumarie's Riemann–Liouville derivatives.[28] Also, we remark that He *et al.*[33] discussed the geometrical explanation of the fractional derivative and in Ref. [34] Ghany applied some results of complex transformation to obtain exact solutions for stochastic fractional Zakharov–Kuznetsov equations. Also, Ghany and Hyder[35] studied the Cauchy problem for stochastic Zakharov–Kuznetsov equation forced by a random term of additive white noise type. Moreover, Fan *et al.*[36–39] proposed and developed a sub-equation method which greatly exceeds the applicability of the existing tanh, extended tanh methods, and Jacobi elliptic function expansion method to construct

a uniform series of exact solutions for the NPDEs including polynomial, exponential, rational, trigonometric periodic wave, hyperbolic, solitary wave, Jacobi and Weierstrass doubly periodic wave types. More details about the sub-equation method and exp-function method can be found in Refs. [30] and [40]–[43].

The motivation of this paper is to use the modified fractional sub-equation method to seek new exact solutions for the variable coefficients fractional coupled KdV equations with the modified Riemann–Liouville derivatives. These solutions include the traveling wave-type solutions of generalized exponential, hyperbolic, and trigonometric types. Then, with the help of Hermite transform and white noise theory, we employ these solutions to find new white noise functional solutions for the stochastic fractional coupled KdV equations in a white noise environment, especially in its Wick version.

The planner which we have proposed in this paper can be used for solving several nonlinear fractional evolution equations, both deterministic and Wick-type stochastic. Note that we have only discussed the solutions of stochastic fractional coupled KdV equations driven by Gaussian white noise. There is a unitary mapping between the Gaussian white noise space and the Poisson white noise space. This connection was given by Benth and Gjerde.[44] Hence, with the help of this connection, we can derive some Poisson white noise functional solutions, if the coefficients $P(t)$, $Q(t)$, and $R(t)$ are Poisson white noise functions in Eq. (2). As $\alpha \to 1$ we have a new set of exact travelling wave solutions and white noise functional solutions for variable coefficients and Wick-type stochastic coupled KdV equations with integer derivatives. Since the fractional Riccati equation has other solutions if we select other values of λ, μ, and ν, there are many other exact solutions of the variable coefficients and Wick-type stochastic fractional coupled KdV equations.

Acknowledgements

The authors are very grateful to the referee for his valuable suggestions.

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