Excitation spectrum and structure factor of a two-component Bose–Einstein condensate in different hyperfine states*

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The elementary excitation spectrum of a two-component Bose–Einstein condensate in different hyperfine states is obtained by Green's function method. It is found to have two branches. In the long wave-length limit, the two branches of the excitation spectrum are reduced to one phonon excitation and one single-particle excitation. The single-particle one has an energy gap. When the energy gap exists, we study the Landau critical velocity and the depletion of the condensate. With the obtained Green's functions, we calculate the structure factor of a two-component condensate. It is found that the static structure factor comprises only the branch of the phonon excitation and the single-particle excitation makes no contribution to the structure factor.

Keywords: elementary excitation, Bose–Einstein condensate, structure factor

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1. Introduction

The realization of Bose–Einstein condensation (BEC)[1,2] has aroused much interest in the past years, because it provides the unique opportunities for exploring quantum coherent phenomena and many-body physics.[3–7] The Bose–Einstein condensation for non-interacting particles is characterized as the macroscopic occupation number for one of the single-particle energy levels. For an interacting system, the criterion for BEC is generalized by Penrose and Onsager and by Yang as off-diagonal long range order.[8,9] The experimental realization of BEC is in dilute atomic gases, in which mean-field theory is well applied at nearly zero temperature.[3,4] The condition for diluteness is $na_s^3 \ll 1$, where n is the density of the gas and a_s is the s-wave scattering length. The interaction between atoms is characterized by the s-wave scattering length a_s, which can be manipulated by the use of lasers and magnetic fields. The transition between repulsive and attractive interaction can be controlled by a Feshbach resonance. When it is in the regime of strong interaction, a recent experiment shows a clear deviation from mean-field theory.[10] The ground state properties of a gas in a trap are usually studied by the Thomas–Fermi approximation. When the length scale of the trap is much greater than the coherence length $\xi = 1/\sqrt{8\pi n a_s}$, the gas is assumed to be homogeneous.

Many experimental and theoretical studies have been devoted to the subject of two-component Bose–Einstein condensate.[11–27] Most work has been done for the case of double-well trapping,[16–20] which is similar to the Josephson junctions of superconductors. The recent experiments have directly observed the phenomena of plasmon oscillation and macroscopic quantum self-trapping (MQST),[19,20] which have been theoretically predicted earlier.[16–18] The dynamics of a soliton in a two-component BEC has been studied in Refs. [21] and [22]. A two-component BEC can also be used to construct entanglement states, which are useful in quantum information and computation.[23,24] Another possible case is the mixture of the same isotope, but in different internal spin states, such as ^{87}Rb.[11,15] In this case, atoms can undergo transition between hyperfine states by an external field, which corresponds to the tunneling effect in the double-well case. The two-body interaction between the atoms of the same state may be different from the interaction of the different hyperfine states. The dynamics of this case is similar to the double-well condensate.[18] The thermal effects can act as the damping term, and the system under damping will evolve into a stationary state of two equivalent components.[17,27] The π phase difference between the two components corresponds to the energy minimum in the mean-field theory at zero temperature.[25]

In the present paper we will study the ground state properties of a two-component BEC in different hyperfine states. The concept of elementary excitation is important for the ground state BEC, and it can be studied in several ways. The excitation spectrum can be achieved by linearizing the hydrodynamic equations derived from the Gross–Pitaevskii equation.[4,26] However for a quantum Bose gas, the excitation spectrum was first obtained by Bogoliubov through a special transformation,[28] which has been well extended for

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many quantum theories. As is well known, the method of Green's function can be used to find the elementary excitations in many fields of condensed matter physics.[29] In the present paper, we extend the method developed by Beliaev[30] to the Bose–Einstein condensate of two equivalent components. In the mean-field approximation, the elementary excitation spectrum is found to have two branches. In the long-wave limit, the two branches of the excitation spectrum are reduced to one phonon excitation and one single-particle excitation. By use of the obtained excitation spectrum and the Green's functions, we calculate the depletion and the ground state energy of the condensate. In this paper, we also calculate the structure factor and find that only the phonon branch of excitation appears in the static structure factor.

2. Excitation spectrum and Landau critical velocity of a two-component BEC in different hyperfine states

The Hamiltonian for a homogeneous two-component Bose gas of different hyperfine states in a volume V can be written in the form[25]

$$\begin{aligned} K = & \sum_{i,p}\left(\frac{p^2}{2m}-\mu\right)a_{ip}^{\dagger}a_{ip} \\ & +\sum_{p}\frac{\eta}{2}(a_{1p}^{\dagger}a_{2p}+a_{2p}^{\dagger}a_{1p}) \\ & +\frac{1}{2V}\sum_{i,k_1k_2q}U_{\mathrm{s}}a_{ik_1+q}^{\dagger}a_{ik_2-q}^{+}a_{ik_1}a_{ik_2} \\ & +\frac{1}{V}\sum_{k_1k_2q}U_x a_{1k_1+q}^{\dagger}a_{2k_2-q}^{\dagger}a_{1k_1}a_{2k_2}, \end{aligned} \tag{1}$$

where η is a coupling parameter which shows the transition between the two kinds of particles, U_{s} and U_x characterize the two-body interactions between the same kinds of particles and between two different kinds of particles respectively, and μ is the chemical potential used to keep the conservation of the total number of particles.

Bose–Einstein condensation occurs in a state of zero momentum and the condensate wave function is

$$\Psi_i = \frac{1}{\sqrt{V}}\langle a_{i0}\rangle = \sqrt{n_{i0}}\,\mathrm{e}^{\mathrm{i}\theta_i},$$

where θ_i is the phase of the condensate and the angle brackets denote averaging with respect to the ground state. When the depletion is small at nearly zero temperature, condensate density n_{i0} can be replaced by density n_i. Because of the thermal effect and the transition term η, the system will evolve into a stationary state of equivalent components $n_1 = n_2 = n/2$.[17] As in Ref. [25], the phase difference of π corresponds to the minimum of the ground state energy of the system. In this paper, we consider the case of the phase difference being π. At zero temperature, the ground state energy density for the homogeneous two-component condensate with volume V is given as

$$\frac{E_0}{V} = \frac{1}{4}(U_{\mathrm{s}}+U_x)n^2 - \frac{1}{2}\eta n. \tag{2}$$

In accordance with Ref. [20], the chemical potential can be found to be

$$\mu = -\frac{\eta}{2} + \frac{n}{2}(U_{\mathrm{s}}+U_x). \tag{3}$$

In our earlier work,[27] we have studied this problem by Green's function method. There we have introduced four single-particle Green's functions at zero temperature, and derived their dynamical equations. The method we used is similar to the work of Beliaev,[30] which is based on the mean-field approximation. In Appendix A of this work, we rewrite the coupled equations in the new forms, and also describe them by Feynman diagrams. In the next section we will make use of the obtained Green's functions to calculate the structure factor.

For the convenience of the following discussion, we give the normal Green's functions in a symmetric form as

$$\begin{aligned} G(p,\omega) = & \frac{\omega+(p^2/2m)+n(U_{\mathrm{s}}+U_x)/2}{2(\omega^2-\varepsilon_1^2)+\mathrm{i}0} \\ & +\frac{\omega+(p^2/2m)+\eta+n(U_{\mathrm{s}}-U_x)/2}{2(\omega^2-\varepsilon_2^2)+\mathrm{i}0}, \end{aligned} \tag{4}$$

where

$$\varepsilon_1(p) = \sqrt{\frac{p^2}{2m}\left[\frac{p^2}{2m}+n(U_{\mathrm{s}}+U_x)\right]}, \tag{5}$$

$$\varepsilon_2(p) = \sqrt{\left(\frac{p^2}{2m}+\eta\right)\left[\frac{p^2}{2m}+\eta+n(U_{\mathrm{s}}-U_x)\right]}. \tag{6}$$

From the form of the denominator of the Green's functions, it is clear that the energy spectrum of the elementary excitations has two branches $\varepsilon_1(p)$ and $\varepsilon_2(p)$. It is possible for $\varepsilon_2(p)$ to be imaginary, and this situation corresponds to the instabilities as indicated in Refs. [25] and [28]. The second branch $\varepsilon_2(p)$ comprises the transition effect and the coupling of the two components. In this section, we continue to study the properties related to the excitations.

In the long-wavelength limit and $0 \ll \eta,\ n(U_{\mathrm{s}}-U_x)$, equations (5) and (6) can be reduced to

$$\varepsilon_1(p) = \sqrt{\frac{n(U_{\mathrm{s}}+U_x)}{2m}}p, \tag{7}$$

$$\varepsilon_2(p) = \sqrt{\eta[\eta+n(U_{\mathrm{s}}-U_x)]}+\frac{p^2}{2m^*}, \tag{8}$$

where the effective mass m^* is given by

$$m^* = \frac{2\sqrt{\eta[\eta+n(U_{\mathrm{s}}-U_x)]}}{2\eta+n(U_{\mathrm{s}}-U_x)}m. \tag{9}$$

It is clear that $\varepsilon_1(p)$ denotes the phonon excitations with the sound speed $\sqrt{n(U_{\mathrm{s}}+U_x)/2m}$ and $\varepsilon_2(p)$ refers to the single-particle excitations with an effective mass and an energy gap.

In this section, we focus our attention on the properties resulting from the energy gap.

In the experiments of the cold atomic gas ^{87}Rb,[3] the scattering length is often in a range $85a_0 < a_{\rm Rb} < 140a_0$, where $a_0 = 0.5292$ Å (1 Å = 0.1 nm) is the Bohr radius. The density is $n \approx 10^{12}$ atoms/cm^3–10^{14} atoms/cm^3, which can meet the condition of diluteness $na^3 \ll 0$, and the sound speed $s = \sqrt{nU_0/m}$ is always of the order of 1 mm/s. In Fig. 1 we display the two branches of the excitation spectra (5) and (6) with the parameters: the density of the ^{87}Rb gas $n = 4 \times 10^{13}$ atoms/cm^3, the two scattering lengths $a_{\rm s} = 120a_0$ and $a_x = 80a_0$ (a_0 is the Bohr radius), and $\eta = n(U_{\rm s} - U_x)/2$. From the selected parameters, we can obtain the sound speed of the first branch $\varepsilon_1(p)$ to be 1.6 mm/s. It is clear that the two branches of excitation spectrum are of a single-particle excitation and a phonon one.

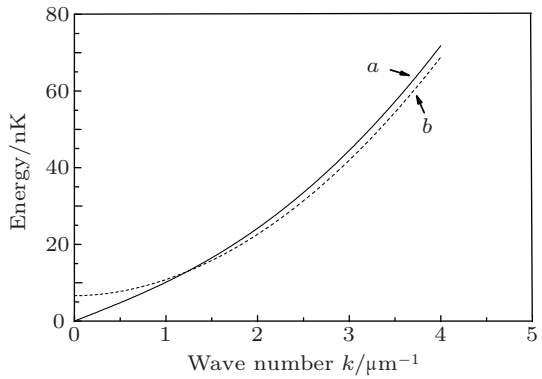


Fig. 1. Plots of excitation energy (in units of nK) versus wave number (in units of μm^{-1}), where the solid line (curve *a*) shows the excitation energy ε_1, and the dashed line (curve *b*) represents the branch ε_2 with the parameters: $n = 4 \times 10^{13}$ atoms/cm^3, $a_{\rm s} = 120a_0$, $a_x = 80a_0$, and $\eta = n(U_{\rm s} - U_x)/2$.

From Fig. 1, we can see that the second branch of excitation has an energy gap:

$$\Delta = \min \varepsilon_2(p) = \sqrt{\eta[\eta + n(U_{\rm s} - U_x)]}. \tag{10}$$

Normally the transition temperature $T_{\rm c}$ of BEC for a cold atomic gas is about μK. When the value η is about of the order of $n(U_{\rm s} - U_x)$, the energy gap characterized by $k_{\rm B}T$ is around the order of nK, which is much lower than the transition temperature $T_{\rm c}$. If the value η is much larger than the interaction term, the energy gap is dominated by η. When the temperature is much lower than the energy gap $k_{\rm B}T \ll \Delta$, the gas nearly cannot thermally excite atoms into the second excitation spectrum.

According to Landau criterion, a system cannot give rise to energy dissipation if its velocity, with respect to a container at rest, is smaller than the Landau critical velocity.[4,6] With the obtained excitation spectrum, we can study the Landau critical velocity, which is defined as

$$s_{\rm c} = \min_p(\varepsilon_p/p). \tag{11}$$

For the second branch, the minimum of the value ε_2/p can be obtained as

$$\begin{aligned} s_2 &= \min_p\left(\frac{\varepsilon_2}{p}\right) = \min_p\left(\frac{\Delta}{p} + \frac{p}{2m^*}\right) \\ &= 2\sqrt{\frac{\Delta}{2m^*}} = \sqrt{\frac{2\eta + n(U_{\rm s} - U_x)}{m}}. \end{aligned} \tag{12}$$

The first branch of excitation spectrum $\varepsilon_1(p)$ is in the phonon form with the sound speed $s_1 = \sqrt{n(U_{\rm s} + U_x)/2m}$. Then the Landau critical velocity can be obtained as

$$s_{\rm c} = \min(s_1, s_2). \tag{13}$$

When the transition effect $\eta = 0$, the second excitation is also reduced to a phonon one $s_2 = \sqrt{n(U_{\rm s} - U_x)/2m}$, which can be regarded as the Landau critical velocity. When $\eta < 2nU_x$, we can obtain $s_2 < s_1$ and the Landau critical velocity is s_2.

Formula (4) is in symmetry, from which we can easily obtain its relation with Bogoliubov transformation,[26]

$$\begin{aligned} G(p,\omega) &= \left(\frac{u_1^2}{\omega - \varepsilon_1 + {\rm i}0} - \frac{v_1^2}{\omega + \varepsilon_1 + {\rm i}0}\right) \\ &\quad + \left(\frac{u_2^2}{\omega - \varepsilon_2 + {\rm i}0} - \frac{v_2^2}{\omega + \varepsilon_2 + {\rm i}0}\right). \end{aligned} \tag{14}$$

The parameters u_i and v_i in Eq. (14) are provided in Appendix B. Compared with the one-component BEC, the Green's function (14) comprises two symmetric parts. Correspondingly, the relationship of the Bogoliubov parameters is changed into $u_i^2 - v_i^2 = 1/2$. The first pair parameters u_1 and v_1 are similar to the one-component case.[4] In the long wave-length limit, the excitation ε_1 is a phonon one and the parameters u_1^2 and v_1^2 diverge when $p \to 0$. When the momentum is large, the excitations behave as free particles and the parameter $v_1^2 \to 0$. Because of the energy gap, the case for the second branch of excitation is changed. Figure 2 shows the plots of parameter v_2^2 versus wave number for different values of the transition parameter η. For $\eta = 0$, the excitation ε_2 is also a phonon one and the parameter v_2^2 diverges when $p \to 0$. But for $\eta \neq 0$, the parameter v_2^2 becomes finite at $p = 0$ and decreases rapidly as η increases, which shows its single-particle properties.

From the result (14), we can obtain the single-particle spectral density of a two-component BEC as[31]

$$\begin{aligned} A_{11}(p,\omega) &= -2\,{\rm Im}\,G(p, \omega + {\rm i}0^+) \\ &= 2\pi \sum_i [u_i^2 \delta(\omega - \varepsilon_i) - v_i^2 \delta(\omega + \varepsilon_i)]. \end{aligned} \tag{15}$$

The spectral density function given by Eq. (15) is shown in Fig. 3, which consists of two positive-weight resonances at $\omega = \varepsilon_1$ and $\omega = \varepsilon_2$ and two negative-weight resonances at $\omega = -\varepsilon_1$ and $\omega = -\varepsilon_2$. Allowing for the damping effects of the excitations, the peaks in Fig. 2 each should have a finite width.[31]

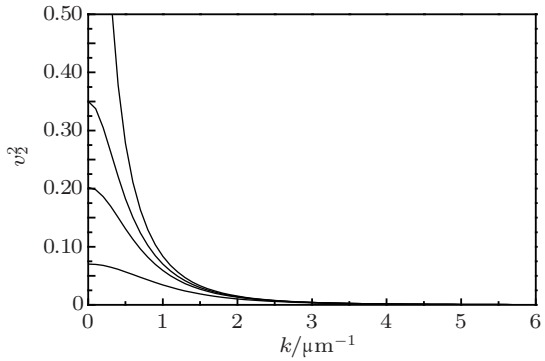


Fig. 2. Plots of Bogoliubov parameter v_2^2 of the second branch of the excitation spectrum versus wave number, where the lines from the top to the lowest correspond to the transition parameters $\eta = (0, 0.1, 0.2, 0.6)n(U_s - U_x)/2$ respectively. The other parameters are selected as the same as those in Fig. 1.

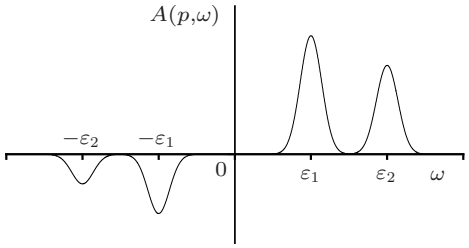


Fig. 3. Sketch of spectral density function $A(p,\omega)$ with a given momentum p for a two-component BEC in different hyperfine states at zero temperature.

The density of non-condensate atoms in a uniform gas can be found using the single-particle spectral density to be[32]

$$\begin{aligned}\tilde{n} &= \int \frac{\mathrm{d}p}{(2\pi\hbar)^3}\int \frac{\mathrm{d}\omega}{2\pi} f(\omega)A_{11}(p,\omega)\\ &= \int \frac{\mathrm{d}p}{(2\pi\hbar)^3}\sum_i [v_i^2 + (u_i^2 + v_i^2)f(\varepsilon_i)],\end{aligned} \tag{16}$$

where the function $f(\varepsilon)$ is the Bose distribution function at finite temperature. Even at zero temperature, there also exists quantum depletion from the condensate, which results from the interactions between the atoms.

In our earlier work,[27] we have studied the quantum depletion of the condensate at zero temperature. The first branch of excitation is a phonon one and the depletion from this branch can be obtained exactly as $n_1' = (1/2)((ms_1)^3/3\pi^2\hbar^3)$. In the case of $U_s = U_x$, the parameter $v_2 = 0$ and the second branch of excitation makes no contribution to the depletion of the condensate. In another case $\eta = 0$, the second excitation is reduced to a phonon one and the depletion from the second branch spectrum can be obtained similarly as $n_0' = (1/2)((ms_2)^3/3\pi^2\hbar^3)$.[27] For a normal case with an energy gap, the depletion n' from the second excitation branch can be calculated numerically, which is shown in Fig. 4 (curve a). It is clear that the depletion decreases rapidly when the energy gap first appears, and then decreases slowly with the large parameter η. The line of Fig. 4 (curve a) is not strictly monotonic and fluctuates with small amplitudes. Figure 4 (curve b) is obtained by approximation (B12), which integrates the excitation energy from the energy gap. The two lines coincide well in the region of small values of η, where the energy gap is small. In the region of large η, they both become small enough to be ignored.

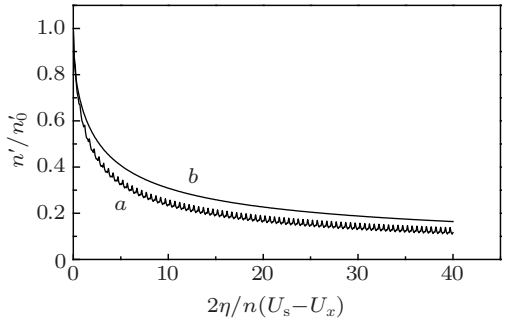


Fig. 4. Depletion from the second excitation in terms of the transition effect. The value n_0' is the depletion of phonon case $\eta = 0$. The line a represents the numerical result, and line b refers to the approximation result.

3. Static structure factor of a two-component BEC in different hyperfine states

The static structure factor of atomic gas is defined as[33,34]

$$S(p) = \langle 0|\hat{\rho}(p)\hat{\rho}^+(p)|0\rangle/N, \tag{17}$$

where $\hat{\rho}^+(p) = \sum_{i,k} a_{i,k+p}^\dagger a_{i,k}$ is the Fourier transform of the atomic density operator at momentum p, N is the total number of the gas, and $|0\rangle$ is the ground state at zero temperature.

In our earlier work,[27] we have used Green's function method to study the excitation and ground state properties of the same system. Now in this section we make use of the obtained Green's functions to calculate the static structure factor (17). Because the two components are equivalent and we have made the assumption that the phase of the first component is zero and the second is π, we can obtain the following relations:

$$\langle a_{10}a_{20}\rangle = -\frac{n}{2}, \quad \langle a_{10}^+ a_{20}\rangle = -\frac{n}{2}. \tag{18}$$

First, we should reduce the static structure factor (17) into pairs by the mean-field approximation:

$$\begin{aligned}S(p) &= \langle 0|(a_{1p}a_{1p}^\dagger + a_{1-p}^\dagger a_{1-p} + a_{1-p}^\dagger a_{1p}^\dagger + a_{1p}a_{1-p})\\ &\quad -(a_{2p}a_{1p}^\dagger + a_{2-p}^\dagger a_{1-p} + a_{2-p}^\dagger a_{1p}^\dagger + a_{2p}a_{1-p})|0\rangle.\end{aligned} \tag{19}$$

In our earlier work,[27] we have introduced four Green's functions at zero temperature:

$$G(p,t) = -\mathrm{i}\langle Ta_{1p}(t)a_{1p}^\dagger(0)\rangle, \tag{20}$$

$$F(p,t) = -\mathrm{i}\langle Ta_{1-p}^\dagger(t)a_{1p}^\dagger(0)\rangle, \tag{21}$$

$$H(p,t) = -\mathrm{i}\langle Ta_{2p}(t)a_{1p}^\dagger(0)\rangle, \tag{22}$$

$$L(p,t) = -\mathrm{i}\langle Ta_{2-p}^\dagger(t)a_{1p}^\dagger(0)\rangle. \tag{23}$$

Similar to result (4), we can rewrite the other three Green's functions in more symmetric forms:

$$F(p,\omega) = -\frac{n(U_s+U_x)/2}{2(\omega^2-\varepsilon_1^2)+\mathrm{i}0} - \frac{n(U_s-U_x)/2}{2(\omega^2-\varepsilon_2^2)+\mathrm{i}0}, \quad (24)$$

$$H(p,\omega) = -\frac{\omega+(p^2/2m)+n(U_s+U_x)/2}{2(\omega^2-\varepsilon_1^2)+\mathrm{i}0} + \frac{\omega+(p^2/2m)+\eta+n(U_s-U_x)/2}{2(\omega^2-\varepsilon_2^2)+\mathrm{i}0}, \quad (25)$$

$$L(p,\omega) = \frac{n(U_s+U_x)/2}{2(\omega^2-\varepsilon_1^2)+\mathrm{i}0} - \frac{n(U_s-U_x)/2}{2(\omega^2-\varepsilon_2^2)+\mathrm{i}0}. \quad (26)$$

To study the static structure factor (19), we can make use of the above Green's functions. For example,

$$\langle 0|a_{1p}a_{1p}^{\dagger}|0\rangle = \lim_{t\to 0^+}\mathrm{i}(-\mathrm{i}\langle Ta_{1p}(t)a_{1p}^{\dagger}(0)\rangle). \quad (27)$$

To study the static structure factor (19), we should know the other two Green's functions, i.e.,

$$G'(p,t) = -\mathrm{i}\langle Ta_{1p}^{\dagger}(t)a_{1p}(0)\rangle, \quad (28)$$

$$H'(p,t) = -\mathrm{i}\langle Ta_{2p}^{\dagger}(t)a_{1p}(0)\rangle. \quad (29)$$

The Green's functions (28) and (29) in energy representation can be obtained by the same procedure in our earlier work,[27] and they are given as

$$G'(p,\omega) = \frac{-\omega+(p^2/2m)+n(U_s+U_x)/2}{2(\omega^2-\varepsilon_1^2)+\mathrm{i}0} + \frac{-\omega+(p^2/2m)+\eta+n(U_s-U_x)/2}{2(\omega^2-\varepsilon_2^2)+\mathrm{i}0}, \quad (30)$$

$$H'(p,\omega) = -\frac{-\omega+(p^2/2m)+n(U_s+U_x)/2}{2(\omega^2-\varepsilon_1^2)+\mathrm{i}0} + \frac{-\omega+(p^2/2m)+\eta+n(U_s-U_x)/2}{2(\omega^2-\varepsilon_2^2)+\mathrm{i}0}. \quad (31)$$

The solutions of the Green's functions (30) and (31) can be testified by the following commutation relations

$$\langle 0|a_{1p}a_{1p}^{\dagger}-a_{1p}^{\dagger}a_{1p}|0\rangle = \lim_{t\to 0^+}\mathrm{i}\int[G(p,\omega)-G'(p,\omega)]\,\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi} = 1, \quad (32)$$

$$\langle 0|a_{2p}a_{1p}^{\dagger}-a_{1p}^{\dagger}a_{2p}|0\rangle = \lim_{t\to 0^+}\mathrm{i}\int[H(p,\omega)-H'(p,\omega)]\,\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi} = 0. \quad (33)$$

It is easily seen that $F(p,\omega) = F^{+}(p,\omega)$ and $L(p,\omega) = L^{+}(p,\omega)$. By making use of the obtained Green's functions, we can obtain the static structure factor (19) as

$$S(p) = \frac{\varepsilon^{(0)}(p)}{\varepsilon_1(p)}, \quad (34)$$

where $\varepsilon^{(0)}(p) = p^2/2m$ is the energy of free particles. The result (34) is the well-known Feynman–Bijl formula.[32,33] The dynamical factor can be obtained as[31,32]

$$S(p,\omega) = S(p)\delta(\omega-\varepsilon_1(p)). \quad (35)$$

In their recent work, Sun and Pindzola[36] have studied the static structure factor of a two-component BEC in the case of big number ratio, in which the two components are greatly separated from each other and they should only calculate the structure factor for a certain component of dominating number. However our work is done for a two-component BEC in different hyperfine states, in which the two components are equivalent. So the structure factor must incorporate the contributions from both kinds of atoms.

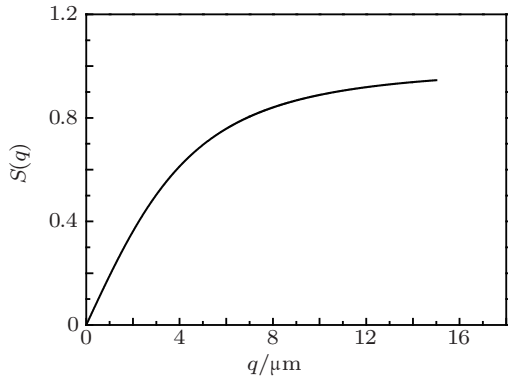


Fig. 5. Static structure factor $S(q)$ as a function of the wave number $q = p/\hbar$ (in unit of $\mu\mathrm{m}^{-1}$). The parameters are selected as follows: the density of ^{87}Rb gas $n = 1.0\times 10^{13}$ atoms/cm^3, the scattering lengths $a_s = 120a_0$, and $a_x = 80a_0$ (a_0 is the Bohr radius).

In Section 2, we have obtained two branches of excitations of a two-component BEC in different hyperfine states. The excitation $\varepsilon_1(p)$ is of the phonon form and the effective interaction is the average of the two different interactions $(U_s+U_x)/2$. The single-particle excitation $\varepsilon_2(p)$ comprises the transition effects and the coupling between the two components of BEC. However, there is only one excitation $\varepsilon_1(p)$ appearing in structure factor (34). This question can be understood by rewriting structure factor (17) into the coordinate picture as follows:[32,33]

$$S(p) = 1 + N_0\sum_R \mathrm{e}^{\mathrm{i}p\cdot R}[g(R)-1], \quad (36)$$

where $g(R)$ is the ground-state pair-correlation function. In this case, correlation function $g(R)$ gives the probability to find two atoms separated by a displacement R no matter what kind of atoms they are. In the low-energy limit, the effective interaction between atoms in hard-sphere potential can be characterized by $U_i = 4\pi\hbar^2 a_i/m$, in which a_i is the scattering length and it can be regarded as the diameter of the hard atom.[3,37] In the phonon limit the pair-correlation function $g(R)$ will be greatly dependent on the inter-atom interaction. Then for a two-component BEC, the pair-correlation function $g(R)$ is dependent on average interaction $(U_s+U_x)/2$, and the transition effect η makes no contribution to the pair-correlation function

$g(R)$. So only the phonon excitation $\varepsilon_1(p)$ appears in the structure factor (34) and the second excitation $\varepsilon_2(p)$ exerts no effect on the structure factor. It deserves to be mentioned that for the atomic gas the correlation function $g(R)$ can be simulated by the Monte Carlo method,[32,38] from which the information about the static structure factor $S(p)$ can be obtained.

Spectroscopic studies have been well used in condensed matter physics, which is often related to the dynamic structure factor.[38] The dynamic structure factor is measureable through inelastic scattering, in which the probe particle is weakly coupled to the many-body system. Neutron and light scattering have long been used to probe the microscopic excitations of liquid helium, which has played a crucial role in understanding the physics of the superfluid ^{4}He.[32] In the case of dilute atomic gases, excitations in the phonon regime have been well studied by the technique of two-photon Bragg transitions,[34] and the excitations in the free-particle regime have been accessed by near-resonant light scattering.[39] The same experimental research can be conducted for a two-component BEC, and the speed of Bogoliubov sound is changed into $s = \sqrt{[n(U_s + U_x)]/2m}$, in which the effective interaction is the average of the two different interactions. The results (34) and (35) can be extended to a trapped two-component BEC by a local density approximation.[33,35]

4. Conclusions

In this paper, we apply the Green's function method and Bogoliubov transformation to a two-component Bose–Einstein condensate in different hyperfine states. The elementary excitation spectrum is found to have two branches. Under the condition of strong coupling and the long wave-length limit, one branch of the excitation spectrum $\varepsilon_1(p)$ is reduced to the phonon form, and the other $\varepsilon_2(p)$ is reduced to the single-particle excitation with an effective mass and a shift in energy. This shift can be seen as an energy gap. Landau critical velocity is studied in the case of an energy gap. The energy gap can also exert an influence on the depletion of the condensate. When the two different kinds of interactions are equal, i.e., $U_s = U_x$, there is no excitation occupying the branch of the single-particle excitation spectrum $\varepsilon_2(p)$. When the transition between the two different particles is forbidden, i.e., $\eta = 0$, the excitation spectrum is reduced to two phonon forms in the long wave-length limit. In this case the depletion of the condensate is reduced for the two different kinds of interactions. The depletion from the single-particle spectrum decreases rapidly as the energy gap appears. In this paper, we also calculate the static structure factor $S(p)$ of a two-component BEC. It is found that the static structure factor $S(p)$ only comprises the branch of the phonon excitation $\varepsilon_1(p)$, and the single-particle excitation $\varepsilon_2(p)$ exerts no effect. In a way similar to the one-component case, the technique of the two-photon Bragg transitions can be used to study the phonon excitation of a two-component BEC and the effective interaction is changed to the average of the two different interactions.

Appendix A: The Feynman diagrams for a two-component Bose–Einstein condensate

In addition to the normal Green's function (20), we introduce another Green's function as follows:

$$G(-p,t) = -\mathrm{i}\langle T a^{\dagger}_{1-p}(t) a_{1-p}(0)\rangle. \tag{A1}$$

The zero-order solutions for the Green's functions (20) and (A1) can be easily obtained as[29]

$$\begin{aligned} G^{(0)}(p,\omega) &= \frac{1}{\omega - (p^2/2m) + \mu + \mathrm{i}0}, \\ G^{(0)}(-p,\omega) &= \frac{-1}{\omega + (p^2/2m) - \mu - \mathrm{i}0}. \end{aligned} \tag{A2}$$

The dynamical equations of the Green's functions have been obtained in our earlier work.[27] By making use of Eq. (A2), we can rewrite them into the new forms as follows:

$$\begin{aligned} G(p,\omega) = G^{(0)}(p,\omega)\Big[1 &+ \Big(nU_s + \frac{n}{2}U_x\Big)G(p,\omega) \\ &+ \frac{n}{2}U_s F(p,\omega) + \Big(\frac{\eta}{2} - \frac{n}{2}U_x\Big)H(p,\omega) \\ &- \frac{n}{2}U_x L(p,\omega)\Big], \end{aligned} \tag{A3}$$

$$\begin{aligned} F(p,\omega) = G^{(0)}(-p,\omega)\Big[&\Big(nU_s + \frac{n}{2}U_x\Big)F(p,\omega) \\ &+ \frac{n}{2}U_s G(p,\omega) - \frac{n}{2}U_x H(p,\omega) \\ &+ \Big(\frac{\eta}{2} - \frac{n}{2}U_x\Big)L(p,\omega)\Big], \end{aligned} \tag{A4}$$

$$\begin{aligned} H(p,\omega) = G^{(0)}(p,\omega)\Big[&\Big(nU_s + \frac{n}{2}U_x\Big)H(p,\omega) \\ &+ \frac{n}{2}U_s L(p,\omega) - \frac{n}{2}U_x F(p,\omega) \\ &+ \Big(\frac{\eta}{2} - \frac{n}{2}U_x\Big)G(p,\omega)\Big], \end{aligned} \tag{A5}$$

$$\begin{aligned} L(p,\omega) = G^{(0)}(-p,\omega)\Big[&\Big(nU_s + \frac{n}{2}U_x\Big)L(p,\omega) \\ &+ \frac{n}{2}U_s H(p,\omega) - \frac{n}{2}U_x G(p,\omega) \\ &+ \Big(\frac{\eta}{2} - \frac{n}{2}U_x\Big)F(p,\omega)]. \end{aligned} \tag{A6}$$

The diagrams of the four Green's functions are introduced in Fig. A1.

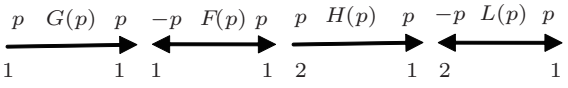


Fig. A1. Diagrams of the four Green's functions. The numbers 1 and 2 indicate the two kinds of particles, and the direction of the line represents the creation operator.

As done in Refs. [29] and [40], the four dynamical equations can be written as graphical equations consisting of skeleton diagrams, which are shown in Fig. A2. The circles represent the self-energy functions.

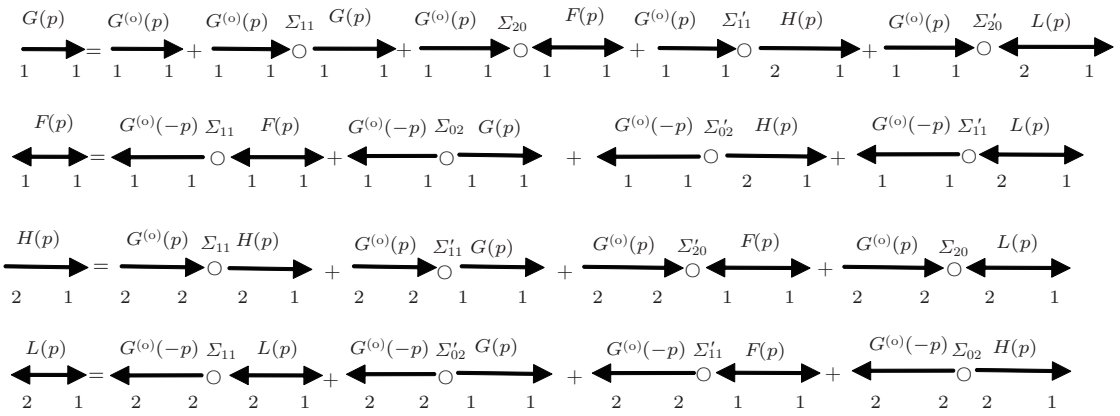


Fig. A2. Graphical equations of the four Green's functions consisting of skeleton diagrams. The circles represent the self-energy functions.

In a way similar to the one-component case,[29,40] we can obtain two relations as

$\Sigma_{02}(p) = \Sigma_{20}(p)$ and $\Sigma_{02}(p) = \Sigma_{02}(-p)$. In Fig. A3, we display all the non-zero diagrams of the self-energy functions $\Sigma_{11}(p)$, $\Sigma'_{11}(p)$, $\Sigma_{02}(p)$, and $\Sigma'_{02}(p)$ in the first order of perturbation theory.

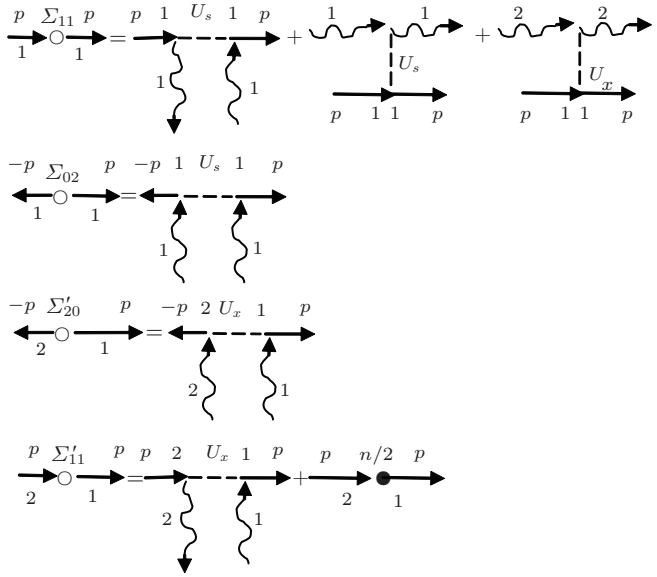


Fig. A3. Diagrams of the four self-energy functions in the first order of perturbation theory. The wavy lines represent the condensates.

As done in Refs. [29] and [40], the interaction value $U(p)$ can be replaced by U_0 at $p = 0$. From Fig. A1, we can easily obtain

$$\Sigma_{11} = nU_s + \frac{n}{2}U_x, \quad \Sigma_{02} = \frac{n}{2}U_s,$$
$$\Sigma'_{02} = -\frac{n}{2}U_x, \quad \Sigma'_{11} = \frac{\eta}{2} - \frac{n}{2}U_x. \tag{A7}$$

To deduce Eq. (A7), we have taken into account the π phase difference between the two components. From Eqs. (A3), (A4), (A5), and (A6), we can also obtain the relation (A7), which clearly shows that the Green's function method developed in this paper is equivalent to the first order approximation in the perturbation theory.

Appendix B: The Bogoliubov transformation of a two-component BEC in different hyperfine states

For a two-component BEC, we can construct Bogoliubov excitations as

$$B_{1p} = (u_1 a_{1p} + v_1 a^\dagger_{1-p}) - (u_1 a_{2p} + v_1 a^\dagger_{2-p}), \tag{B1}$$
$$B_{2p} = (u_2 a_{1p} + v_2 a^\dagger_{1-p}) + (u_2 a_{2p} + v_2 a^\dagger_{2-p}). \tag{B2}$$

Here, B_{1p} and B_{2p} are new Bose operators, which satisfy the following commutation relation:

$$[B_{ip}, B^\dagger_{jp'}] = \delta_{ij}\delta(p - p'). \tag{B3}$$

The corresponding parameters are given as

$$u_i^2 = \frac{1}{4}\left(\frac{\tilde{\varepsilon}_i}{\varepsilon_i} + 1\right), \quad v_i^2 = \frac{1}{4}\left(\frac{\tilde{\varepsilon}_i}{\varepsilon_i} - 1\right), \tag{B4}$$

where

$$\tilde{\varepsilon}_1 = \frac{p^2}{2m} + \frac{n}{2}(U_s + U_x),$$
$$\tilde{\varepsilon}_2 = \frac{p^2}{2m} + \eta + \frac{n}{2}(U_s - U_x). \tag{B5}$$

From Eq. (B4), we can easily obtain the following relations

$$u_i^2 - v_i^2 = \frac{1}{2}, \tag{B6}$$
$$u_1 v_1 = \frac{n(U_s + U_x)}{8\varepsilon_1}, \quad u_2 v_2 = \frac{n(U_s - U_x)}{8\varepsilon_2}. \tag{B7}$$

In Ref. [25], the canonical Hamiltonian (1) has been simplified into the form of pair operators. By use of the new Bogoliubov

operators (B1) and (B2), we can construct the Hamiltonian (1) in the diagonal form.

$$K = K_0 + \sum_p \varepsilon_1 B_{1p}^\dagger B_{1p} + \sum_p \varepsilon_2 B_{2p}^\dagger B_{2p}. \tag{B8}$$

In Eq. (B8), K_0 is the ground state energy, with the depletion effects ignored.

From the Bogoliubov transformations (B1) and (B2), we can obtain the atomic operators as follows:

$$a_{1p} = (u_1 B_{1p} - v_1 B_{1-p}^\dagger) + (u_2 B_{2p} - v_2 B_{2-p}^\dagger), \tag{B9}$$

$$a_{2p} = (u_2 B_{2p} - v_1 B_{2-p}^\dagger) - (u_1 B_{1p} - v_1 B_{1-p}^\dagger). \tag{B10}$$

Though the two operators are not the same in form, they both obey the commutation rules strictly.

The depletion from the second branch of excitation can be written as

$$n' = \frac{\sqrt{2}}{8}\frac{(ms_2)^3}{\pi^2\hbar^3} \times \int_0^\infty \mathrm{d}\varepsilon' \sqrt{\varepsilon'} \left(\frac{\varepsilon' + \eta' + 1}{\sqrt{(\varepsilon' + \eta')(\varepsilon' + \eta' + 2)}} - 1 \right), \tag{B11}$$

where the variables are replaced into $\varepsilon' = p^2/mn(U_s - U_x)$ and $\eta' = 2\eta/n(U_s - U_x)$. This can be calculated numerically. If we emphasize the role of the energy gap, we can make the following approximation:

$$\begin{aligned} n' &\approx \frac{\sqrt{2}}{8}\frac{(ms_2)^3}{\pi^2\hbar^3} \int_{\Delta'}^\infty \mathrm{d}\varepsilon' \sqrt{\varepsilon'} \left(\frac{\varepsilon' + 1}{\sqrt{\varepsilon'(\varepsilon' + 2)}} - 1 \right) \\ &= \frac{\sqrt{2}}{8}\frac{(ms_2)^3}{\pi^2\hbar^3} \left[2\left(\sqrt{\eta'^2 + \eta'} + 2 \right)^{1/2} + \frac{2}{3}\left(\sqrt{\eta'^2 + \eta'} \right)^{3/2} \right. \\ &\quad \left. - \frac{2}{3}\left(\sqrt{\eta'^2 + \eta'} + 2 \right)^{3/2} \right]. \end{aligned} \tag{B12}$$

In the above approximation, the excitation energy is selected as the one of $\eta = 0$ and it is integrated from the energy gap.

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