Tunneling dynamics of superfluid Fermi gas in a triple-well potential*

Gou Xue-Qiang(苟学强), Meng Hong-Juan(蒙红娟), Wang Wen-Yuan(王文元), and Duan Wen-Shan(段文山)†

College of Physics and Electronic Engineering, Northwest Normal University, Lanzhou 730070, China

(Received 17 November 2012; revised manuscript received 21 January 2013)

The tunneling dynamics of superfluid Fermi gas in a triple-well potential in the unitarity regime is investigated in the present paper. The fixed points of the $(0,0)$ mode and the (π,π) mode are given. We find that the interaction parameter U and the coupling strength k could have an extreme effect on the quantum tunneling dynamics. We also find that, in the zero mode, only Josophson oscillation appears. However, for the π mode, the trapping phenomena take place. An irregular oscillation of the particle number in each well could appear by adjusting the scanning period T. It is noted that if the scanning period is less than a critical point T^*, the particle number will come back to the fixed point with small oscillation, while if $T > T^*$ the particle number cannot come back to the fixed point, but with irregular oscillations. The dependence of the critical point T^* on the system parameter of coupling strength k is numerically given.

Keywords: Fermi superfluid gas, fixed points, unitarity regime, triple wells

PACS: 03.75.Kk, 03.75.Lm, 37.10.Jk **DOI:** 10.1088/1674-1056/22/8/080307

1. Introduction

In recent years, much attention has been paid to the dynamics of the ultracold atomic Fermi gas.[1–5] The main motivations for studying fermionic systems are in their many respects complementary to the bosonic case. Many relevant experiments have been carried out, focusing on the dynamic behavior of a Bardeen–Cooper–Schrieffer (BCS) to a Bose–Einstein condensation (BEC) crossover with magnetic-field-induced Feshbach resonance. In this technique, the s-wave scattering length a_{sc} can be controlled with large variation from negative to positive, or crossover from a BCS superfluid to a BEC of molecular pairs.[6,7] There have also been major theoretical achievements for this superfluid system, such as Josephson oscillation,[6–11] self-trapping,[12–17] Rosen–Zener transition,[18,19] Landau–Zener transition,[1,20] as well as interfere,[21–31] and so on.

In this paper, we study the tunneling dynamics of Fermi gas in a triple-well system. As quantum tunneling can occur between several wells simultaneously, the tunneling dynamics in the triple wells would show more interesting behavior; moreover, the study of the triple-well system would provide a bridge between the simple double-well and the multi-well system, helping us more in-depth study of the nature of the Fermi gas.

We established a triple-well model based on the macroscopic superfluid hydrodynamic equation. Compared with two-well system, more peculiar tunneling phenomena appear due to the interaction of the atoms and the coupling strengths among the three potential wells.

The present paper is organized as follows. In Section 2, we show the basic equation. In Section 3, we study tunneling phenomena from different phases. In Section 4, we give the summary and discussion.

2. Basic equation

At extreme low temperatures, the dynamical collective properties of a one-dimensional (1D) trapped superfluid Fermi gas are expected to be properly described by the following equation[8,32–36]

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}+V(x)+\mu-\mathrm{i}\frac{\partial}{\partial t}\right]\Psi(x,t)=0, \tag{1}$$

where $V(x)$ is the external potential acting on the fermions, μ is the chemical potential, and $\Psi(x,t)$ is the order parameter of fermions.

The chemical potential for this system has been studied by many researchers[10,11,37,38] which can be approximated by $\mu = C_0 n^{\gamma_0}$ where $C_0 = 4\pi\hbar^2 a_{sc}/m$, for the deep BEC regime $\gamma_0 = 1$, and the unitarity regime $\gamma_0 = 2/3$, n is the particle density and m is the mass of the dimer.

For the triple-well system under mean-field approximation, the order parameter of the system $\Psi(x,t)$ can be expressed as the superposition of individual wave functions in each well,[6,39,40] i.e., $\Psi(x,t) = \psi_1(t)\phi_1(x) + \psi_2(t)\phi_2(x) + \psi_3(t)\phi_3(x)$, where $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$ are the spatial mode functions, $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ are the probability amplitude in three wells respectively. In general, the functions $\psi_{1,2,3}(t)$ are complex and satisfy the condition: $|\psi_{1,2,3}(t)|^2 = N_{1,2,3}(t)$, the total particle number is given by

*Project supported by the National Fundamental Research Program of China (Grant Nos. 2007CB814800 and 2011CB921503), the National Natural Science Foundation of China (Grant Nos. 11275156, 91021021, and 10875098), and the Natural Science Foundation of Northwest Normal University (Grant No. NWNU-KJCXGC-03-48).

†Corresponding author. E-mail: duanws@nwnu.edu.cn

© 2013 Chinese Physical Society and IOP Publishing Ltd

$|\psi_1(t)|^2+|\psi_2(t)|^2+|\psi_3(t)|^3 = N_1(t)+N_2(t)+N_3(t) = N$. Then the following equations should be obtained

$$\mathrm{i}\dot{\psi}_1 = E_1\psi_1 + U_1 N_1^{\gamma_0}\psi_1 - k_{12}\psi_2 - k_{13}\psi_3, \quad (2)$$

$$\mathrm{i}\dot{\psi}_2 = E_2\psi_2 + U_2 N_2^{\gamma_0}\psi_2 - k_{12}\psi_1 - k_{23}\psi_3, \quad (3)$$

$$\mathrm{i}\dot{\psi}_3 = E_3\psi_3 + U_3 N_3^{\gamma_0}\psi_3 - k_{13}\psi_1 - k_{23}\psi_2, \quad (4)$$

where

$$E_i = \frac{1}{2}\int\left(\frac{\partial\phi_i}{\partial x}\right)^2 \mathrm{d}x + \int V(x)\phi_i(x)\phi_j(x)\,\mathrm{d}x$$

is the zero-point energies in each well, and

$$k_{i,j} = \frac{1}{2}\int\left(\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x}\right)\mathrm{d}x + \int V(x)\phi_i(x)\phi_j(x)\,\mathrm{d}x,$$

$(i=1,2,3, j=1,2,3)$ is the coupling strength between the i-th well and the j-th well, $U_i = C_0\int\phi_i^{2\gamma_0+2}(x)\,\mathrm{d}x$, $(i=1,2,3)$ is the nonlinear parameter depending on the atomic interaction in wells $1,2,3$ respectively.

By writing the wave $\psi_{1,2,3}(t)$ in terms of its amplitude $\sqrt{N_{1,2,3}}$ and phase $\theta_{1,2,3}$, $\psi_{1,2,3}(t) = \sqrt{N_{1,2,3}}\,\mathrm{e}^{\mathrm{i}\theta_{1,2,3}}$, we define $\varphi_1 = \theta_2 - \theta_1$, $\varphi_2 = \theta_3 - \theta_2$ and letting $k_{ij} = k, U_i = U, E_i = E$, we then obtain

$$\dot{N}_1 = -2k\sqrt{N_1N_2}\sin\varphi_1 - 2k\sqrt{N_1N_3}\sin(\varphi_1+\varphi_2), \quad (5)$$

$$\dot{\varphi}_1 = U(N_1^{\gamma_0} - N_2^{\gamma_0}) - k\left(\sqrt{\frac{N_2}{N_1}} - \sqrt{\frac{N_1}{N_2}}\right)\cos\varphi_1 + k\sqrt{\frac{N_3}{N_2}}\cos\varphi_2 - k\sqrt{\frac{N_3}{N_1}}\cos(\varphi_1+\varphi_2), \quad (6)$$

$$\dot{N}_2 = 2k\sqrt{N_1N_2}\sin\varphi_1 - 2k\sqrt{N_2N_3}\sin\varphi_2, \quad (7)$$

$$\dot{\varphi}_2 = U(N_2^{\gamma_0} - N_3^{\gamma_0}) - k\left(\sqrt{\frac{N_3}{N_2}} - \sqrt{\frac{N_2}{N_3}}\right)\cos\varphi_2 + k\sqrt{\frac{N_1}{N_3}}\cos(\varphi_1+\varphi_2) - k\sqrt{\frac{N_1}{N_2}}\cos\varphi_1, \quad (8)$$

$$N_3 = N - N_1 - N_2. \quad (9)$$

3. Fixed points analysis

We now focus our attention on the unitarity regime, in which the s-wave scattering length is large enough. According to the adiabatic theorem, the characters of the tunneling probabilities should be entirely determined by the eigenstates properties which correspond to the stability of the fixed points in the classical Hamiltonian. So we study the fixed points of Eqs. (5)–(9).

The fixed points can be obtained from Eqs. (5)–(9) by letting $\dot{N}_1 = 0, \dot{\varphi}_1 = 0, \dot{N}_2 = 0, \dot{\varphi}_2 = 0$. The fixed points are: $(\varphi_1^{\mathrm{f}}, \varphi_2^{\mathrm{f}}) = (0,0), (\pi,\pi), (0,\pi), (\pi,0)$. For the case of $(\varphi_1^{\mathrm{f}}, \varphi_2^{\mathrm{f}}) = (0,0)$, the population imbalance has a trivial solution of $(N_1^{\mathrm{f}}, N_2^{\mathrm{f}}) = (1/3, 1/3)$, but for the other cases, $N_1^{\mathrm{f}}, N_2^{\mathrm{f}}$ have a nontrivial solution. There are three cases of the nontrivial solution corresponding to the fixed points $(\varphi_1^{\mathrm{f}}, \varphi_2^{\mathrm{f}}) = (\pi,\pi), (0,\pi), (\pi,0)$. These three cases are similar, therefore, we choose a typical case of (π,π) mode as an example of nontrivial solution. So in the following we only study the case of both $(0,0)$ mode and (π,π) mode.

3.1. Zero $(0,0)$ mode

As is well known, the double wells or triple wells offer a powerful tool to study the quantum tunneling phenomena since almost every parameter, i.e., the inter-well tunneling strength, on-site interaction strength, and also the energy bias between the two wells, can be tuned experimentally. Due to this reason, we study how the on-site interaction strength affects the tunneling phenomena among the three wells. We now assume that the on-site interaction parameter changes with time linearly as follows: $U = \alpha t$, where α is the sweeping rate. We exploit 4-5th Runge–Kutta algorithm and numerically solve Eqs. (5)–(9). The results are shown in Fig. 1, where the initial conditions are $N_1 = 0.38$, $N_2 = 0.32$.

It is noted from Fig. 1 that the evolution of the populations in the three wells oscillates around the fixed point $(N_{1,2,3}^{\mathrm{f}} = 1/3)$. It indicates that the oscillation amplitude increases, while the frequency of the oscillation approximately remains unchanged as the difference between initial value and the fixed point $(N_i - N_i^{\mathrm{f}})$, $(i=1,2,3)$ increases. In other words, the mean population difference between wells is zero.

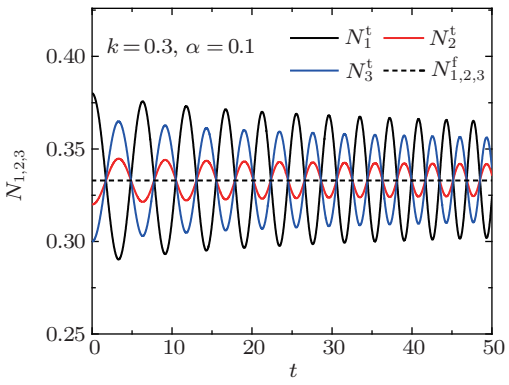


Fig. 1. (color online) The time evolution of the population and the corresponding fixed points.

The oscillation frequency can be approximately obtained by the following procedure. We introduce the infinitesimal variables of $\delta_1, \delta_2, \alpha_1, \alpha_2$ by $N_1 = N_1^{\mathrm{f}} + \delta_1$, $\varphi_1 = \varphi_1^{\mathrm{f}} + \alpha_1$, $N_2 = N_2^{\mathrm{f}} + \delta_2$, and $\varphi_2 = \varphi_2^{\mathrm{f}} + \alpha_2$, where $N_1^{\mathrm{f}}, \varphi_1^{\mathrm{f}}, N_2^{\mathrm{f}}, \varphi_2^{\mathrm{f}}$ are the fixed points. By substituting them into Eqs. (5)–(9), we obtain the following equations

$$\dot{\delta}_1 = -\frac{4}{3}k\alpha_1 - \frac{2}{3}k\alpha_2, \quad (10)$$

$$\dot{\alpha}_1 = \left(\frac{2}{3}3^{1/3}U + \frac{9}{2}k\right)\delta_1 + \left(-\frac{2}{3}3^{1/3}U - \frac{9}{2}k\right)\delta_2, \quad (11)$$

$$\dot{\delta}_2 = \frac{2}{3}k\alpha_1 - \frac{2}{3}k\alpha_2, \quad (12)$$

$$\dot{\alpha}_2 = \left(\frac{2}{3}3^{1/3}U + \frac{9}{2}k\right)\delta_1 + \left(\frac{4}{3}3^{1/3}U + 9k\right)\delta_2. \quad (13)$$

Based on the above equations and set $\alpha_1 = \alpha_2 = e^{\lambda t}$, we obtain

$$\lambda^2 = -\left(\frac{4}{3}3^{1/3}Uk + 9k^2\right). \tag{14}$$

We then obtain the THE oscillation frequency as follows:

$$\omega = \sqrt{\frac{4}{3}3^{1/3}Uk + 9k^2}. \tag{15}$$

It is easy to note that the oscillation frequency increases as U increases. We note from Fig. 2 that the analytical results are in good agreement with the numerical ones.

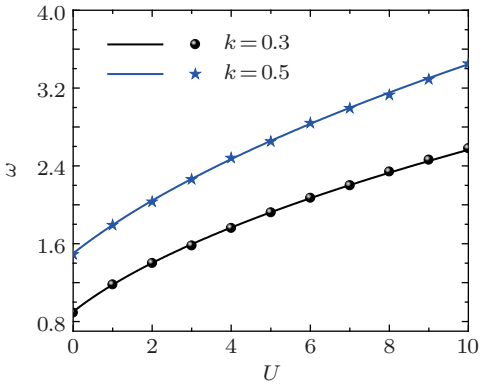


Fig. 2. (color online) The oscillation frequency change with the variation of U for the analytical (line) and the numerical solution (dot).

3.2. π mode

In this section, we study the tunneling phenomena of (π, π) mode. The dependence of the fixed point on the interaction parameter U is shown in Fig. 3. The red line refers to the second well and the blue lines refer to the first and the third wells.

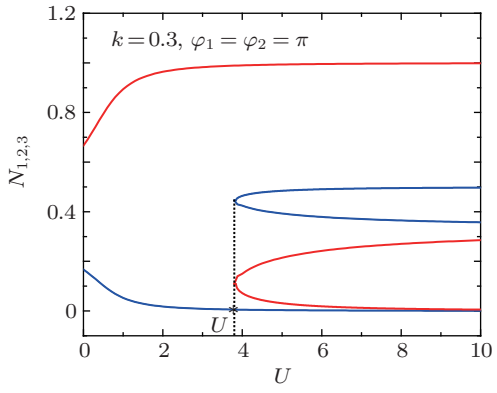


Fig. 3. (color online) The evolution of fixed point on U.

It is noted from Fig. 3 that the number of the fixed point depends on the interaction parameter U. If U is less than a critical value U^*, i.e., $U < U^*$, there is only one fixed point, however, if $U > U^*$, a new fixed point appears. It is also noted that the values of the fixed points depend on the interaction parameter U. The particle number in each well is not equal to the mean value of $1/3$. The tunneling phenomena in three wells take place. It is completely different from the $(0,0)$ mode in which case the mean number in each well equals to $1/3$.

We assume that U changes adiabatically with time in the form of $U = \alpha t$, where α is a small parameter. For the adiabatic limit, the value of α should satisfy $\alpha \ll 1$, there are many research studies in BECs.[20,35,40] In the present Fermi gas, we have taken many values of α in the range of $\alpha < 1$, and we found that U also changes adiabatically when $\alpha = 0.1$, so in the present paper, we take $\alpha = 0.1$ as an example. The evolution of the particle numbers (N_1, N_2, N_3) in three wells are shown in Fig. 4 and Fig. 5.

Figure 4 shows the results of the case in which the initial values of N_1, N_2, and N_3 are close to their corresponding fixed points, while figure 5 is the case in which the initial values of N_1, N_2, and N_3 are not close to their corresponding fixed points. It is noted that the less the difference between initial values of N_1, N_2, N_3 and the corresponding fixed points $N_1^{\rm f}$, $N_2^{\rm f}$, $N_3^{\rm f}$, the less the differences of $N_i(t) - N_i^{\rm f}(t)$, $i = 1, 2, 3.$ at the time t.

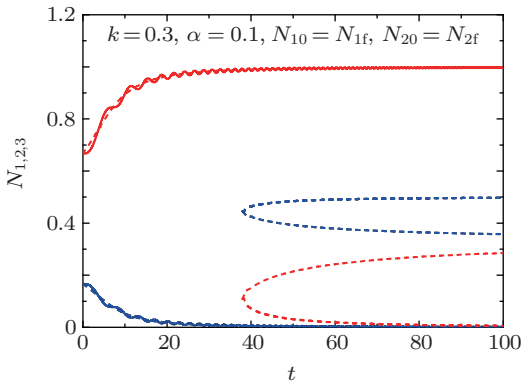


Fig. 4. (color online) The time evolution of probability of particles (solid line), and the corresponding fixed points $N_{1,2,3}^{\rm f}$ (dashed line).

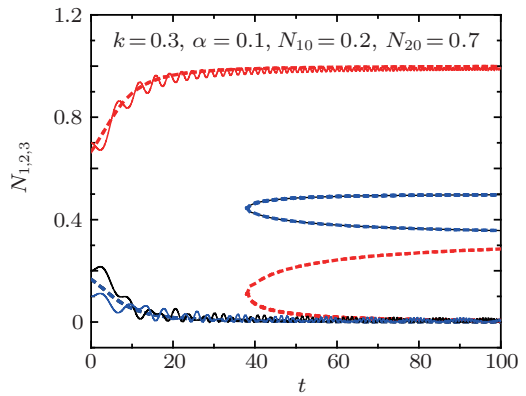


Fig. 5. (color online) The time evolution of probability of particles (solid line), and the corresponding fixed points $N_{1,2,3}^{\rm f}$ (dashed line).

In order to further understand these phenomena, we define a function as follows:

$$\Delta = \overline{\Sigma |N_i^t - N_i^{\rm f}|}, \tag{16}$$

where $N_i^{\rm f}$ refers to the value of a fixed point for the i-th well at time t, N_i^t refers to the value of a real particle number at time t in the i-th well. The dependence of Δ on the initial value of $\Delta|_{t=0} = N_i^0 - N_i^{\rm f}(0)$ is shown in Fig. 6. It is observed that as the initial value of $\Delta|_{t=0}$ increases, the mean difference between particle number and its corresponding fixed point at time t in each well increases.

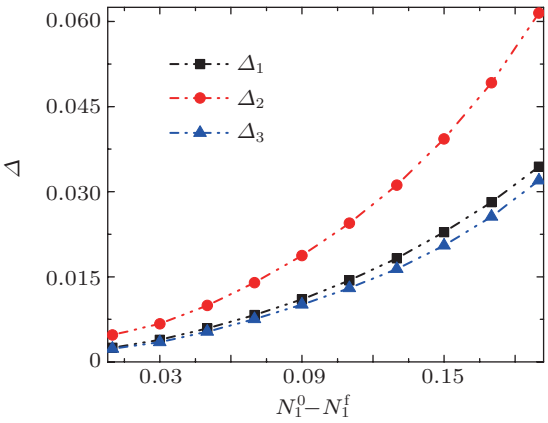


Fig. 6. (color online) The function Δ versus $N_1^0 - N_1^{\rm f}$, where $k = 0.3$. The black, red, and blue lines correspond to the first, second, and third wells, respectively.

In this section we also study the effect of variation of the interaction parameter U on the tunneling probability between wells, however, we assume that U changes with time t in different forms as follows:

$$U = a_0 + a_1 \sin(\pi t/T), \tag{17}$$

where, we set $a_0 = 9.0$, $a_1 = -8.5$. This ensures that the final value of the interaction parameter U returns to the initial value after the evolution of the period of time.

We take the fixed point as an initial value of the particle number in each well. Then we observe the dependence of the particle number in each well on time as U changes with time t in the form of Eq. (17). It is shown in Fig. 7 that, in the adiabatic limit, i.e., $T = 200 \gg 1$, the particle number is close to the fixed point in each well until it reaches a critical point ($t = t^*$) in which there is a discontinuity of the critical point. However, when $t > t^*$ the particle number in each well (N_1, N_2, N_3) oscillates regularly with large amplitude until it reaches another critical point $t = t^{**}$ at which the system comes back to its fixed point, but N_1, N_2, N_3 oscillate around their corresponding fixed points with much smaller amplitude.

Similar results are shown in Fig. 8 where $T = 1500$. However, an obvious difference is observed between Fig. 7 and Fig. 8. In Fig. 8 the oscillations when $t \gg t^*$ are irregular ($t = t^*$ corresponds to $U = U^*$), especially when $t \gg t^{**}$ ($t = t^{**}$ also corresponds to $U = U^*$) the oscillations may be more irregular. From our numerical results, we find that there is a critical point T^*. If $T < T^*$ the oscillation is regular, namely, if $t > t^{**}$ the particle numbers in three wells oscillate around the fixed points with small amplitude. If $T > T^*$ the oscillation is irregular, or the particle numbers in the three wells do not oscillate around the fixed points but with irregular pattern if $t > t^{**}$.

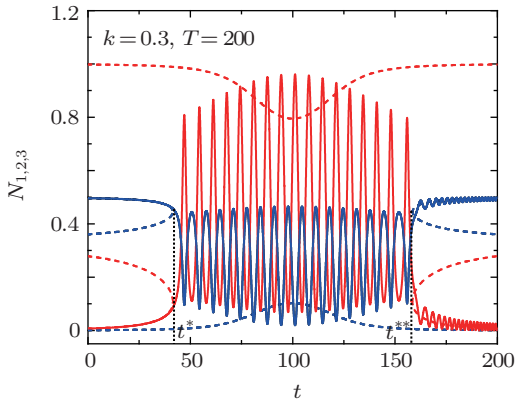


Fig. 7. (color online) The time evolution of probability of particles (solid line), and the corresponding fixed points $N_{1,2,3}^{\rm f}$ (dashed line).

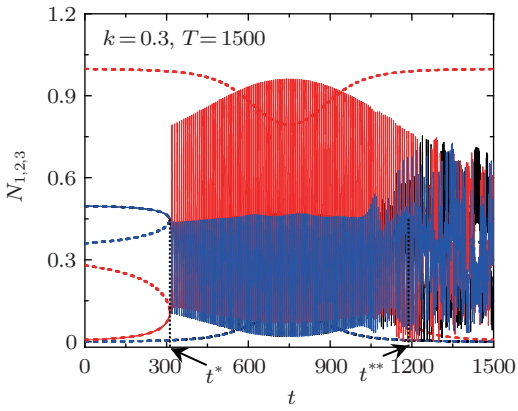


Fig. 8. (color online) The evolution of probability of particles with time (solid line), and the corresponding fixed points $N_{1,2,3}^{\rm f}$ (dashed line).

The dependence of the critical point T^* on k is shown in Fig. 9. We can see that the critical value T^* decreases as k increases.

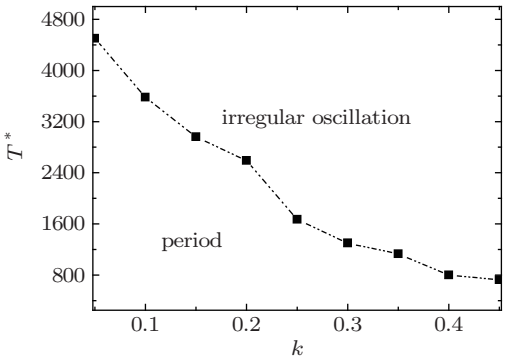


Fig. 9. The dependence of the critical value T^* on parameter k.

We have studied the tunneling dynamics of the superfluid Fermi gas in a triple-well system in the unitarity regime. Similar results can be obtained in the deep BEC regime.

In the deep BEC regime, the evolutions of fixed points with interaction parameter U are similar to the unitarity regime, the only difference is the critical interaction for the change of fixed points. In this case, the tunneling phenomenon is also similar to the unitarity regime, so we do not repeat the processes further.

4. Conclusion

In the present paper, we have investigated the tunneling dynamics of the Fermi gas in a triple-well in the BEC regime and unitarity regime. We find the interaction parameter U and the parameter of coupling strength k could affect the quantum tunneling. By analyzing the fixed points of the quantum system, the two corresponding phases, that is, the zero mode and π mode, are explored in depth. We found, in the zero mode, the probability of particles in each well oscillates around the fixed point, the dependence of these phenomena on the fixed points in three wells is also evident. For the π mode, we found that different initial values result in different tunneling behaviors, and irregular oscillations can be controlled by adjusting T. We also give the relationship between critical T^* and k for the tunneling and irregular oscillations. The present results would help us to give deep insight into the dynamics of the superfluid Fermi gas in triple wells. We hope our study will stimulate experiments in this direction.

References

[1] Wu B and Niu Q 2000 *Phys. Rev. A* **61** 023402
[2] Zhang J M, Liu W M and Zhou D L 2008 *Phys. Rev. A* **78** 043618
[3] Zhang P, Wu J H, Zhang X F and Liu W M 2010 *Phys. Rev. A* **82** 043628
[4] Ye D F, Fu L B and Liu J 2008 *Phys. Rev. A* **77** 013402
[5] Wu Y and Yang Y Y 2003 *Phys. Rev. A* **68** 013608
[6] Adhikari S K, Lu H and Pu H 2009 *Phys. Rev. A* **80** 063607
[7] Adhikari S K and Salasnich L 2008 *Phys. Rev. A* **77** 033618
[8] Wang W Y, Duan W S, Sun J A and Yang Y 2011 *Eur. Phys. J. B* **84** 283
[9] Li W D, Zhou X J, Wang Y Q, Liang J Q and Liu W M 2001 *Phys. Rev. A* **64** 015602
[10] Ancilotto F, Salasnich L and Toigo F 2009 *Phys. Rev. A* **79** 033627
[11] Salasnich L, Ancilotto F and Toigo F 2009 *Laser Phys.* **19** 636
[12] Morales-Molina L and Gong J B 2008 *Phys. Rev. A* **78** 041403
[13] Ostrovskaya E A, Kivshar Y S, Lisak M, Hall B, Cattani F and Anderson D 2000 *Phys. Rev. A* **61** 031601
[14] Fu L B, Liu J and Chen S G 2002 *Phys. Lett. A* **298** 388
[15] Choi S and Sundaram B 2008 *Phys. Rev. A* **77** 053613
[16] Chiara G D, Calarco T, Fishman S and Morigi G 2008 *Phys. Rev. A* **78** 043414
[17] Fu L B, Ye D F, Lee C, Zhang W and Liu J 2009 *Phys. Rev. A* **80** 013619
[18] Meng H J, Yang Y, Wang W Y, Qi P T, Ma Y Y, Ma Y, Wang S J and Duan W S 2012 *Acta Phys. Sin.* **61** 060303 (in Chinese)
[19] Li S C, Fu L B, Duan W S and Liu J 2008 *Phys. Rev. A* **78** 063621
[20] Liu J, Wu B and Niu Q 2003 *Phys. Rev. Lett.* **90** 170404
[21] Ramsey N F 1950 *Phys. Rev.* **78** 695
[22] Donley E A, Claussen N R, Thompson S T and Wieman C E 2002 *Nature* **417** 529
[23] Santarelli G, Laurent P, Lemonde P, Clairon A, Mann A G, Chang S, Luiten A N and Salomon C 1999 *Phys. Rev. Lett.* **82** 4619
[24] Fertig C and Gibble K 2000 *Phys. Rev. Lett.* **85** 1622
[25] Gustavson T L, Bouyer P and Kasevich M A 1997 *Phys. Rev. Lett.* **78** 2046
[26] Chen A X, Qiu W Y and Wang Z P 2008 *Chin. Phys. B* **17** 4204
[27] Weiss D S, Young B C and Chu S 1993 *Phys. Rev. Lett.* **70** 2706
[28] Weel M and Kumarakrishnan A 2003 *Phys. Rev. A* **67** 061602
[29] Widera A, Mandel O, Greiner M, Kreim S, Hänsch T W and Bloch I 2004 *Phys. Rev. Lett.* **92** 160406
[30] Li J H 2010 *Chin. Phys. B* **19** 040502
[31] Zhang X F, Liu X X and Zhang P 2010 *Chin. Phys. Lett.* **27** 070306
[32] Adhikari S K and Salasnich L 2008 *Phys. Rev. A* **78** 043616
[33] Wen W, Shen S Q and Huang G X 2010 *Phys. Rev. B* **81** 014528
[34] Wen W and Huang G X 2009 *Phys. Rev. A* **79** 023605
[35] Liu J, Fu L B, Ou B Y, Chen S G, Choi D I, Wu B and Niu Q 2002 *Phys. Rev. A* **66** 023404
[36] Tacla A B, Boixo S, Datta A, Shaji A and Caves C M 2010 *Phys. Rev. A* **82** 053636
[37] Peng P and Li G Q 2009 *Chin. Phys. B* **18** 3221
[38] Giorgini S, Pitaevskii L P and Stringeri S 2008 *Rev. Mod. Phys.* **80** 1215
[39] Wang G F, Ye D F, Fu L B, Chen X Z and Liu J 2006 *Phys. Rev. A* **74** 033414
[40] Liu J, Zhang C W, Raizen M G and Niu Q 2006 *Phys. Rev. A* **73** 013601