Relation between initial conditions and entanglement sudden death for two-qubit extended Werner-like states*

Yang Bai-Yuan(杨百元), Fang Mao-Fa(方卯发)†, and Huang Jiang(黄 江)

Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control, Ministry of Education, and Department of Physics, Hunan Normal University, Changsha 410081, *China*

(Received 30 November 2012; revised manuscript received 26 February 2013)

In this paper, the dynamical behavior of entanglement of an uncoupled two-qubit system, which interacts with independent identical amplitude damping environments and is initially prepared in the extended Werner-like (EWL) states, is investigated. The results show that whether entanglement sudden death (ESD) of an EWL state will occur or not depends on initial purity and concurrence. The boundaries between ESD states and ESD-free states for two kinds of EWL states are found to be different. Furthermore, some regions are shown where ESD states can be transformed into ESD-free states by local unitary operations.

Keywords: entanglement sudden death, extended Werner-like states, ESD states, ESD-free states

PACS: 03.65.Ud, 03.67.–a **DOI:** 10.1088/1674-1056/22/8/080303

1. Introduction

Entanglement has been generally recognized as a crucial resource in quantum information and quantum computation.[1] It plays an essential role in various kinds of quantum information processing, including quantum state teleportation,[2] quantum cryptography,[3] quantum error correction,[4] quantum computational speed-up,[5] etc. Much attention has been paid to the study of entanglement, and the relation between entanglement and other measures of quantumness[6–9] is concerned lately. For an actual physical system, the fact that entanglement is fragile due to inevitable interaction between the system and its surrounding environment becomes the main limitation of achievement of quantum protocols. Recently, an extraordinary phenomenon that entanglement disappears completely at a finite time,[10–12] known as ESD,[13] is widely concerned. On the one hand, existence of ESD has been extensively demonstrated. ESD of a system, which consists of interacting[14–16] or non-interacting[17–19] entangled partners and is disturbed by common[20–22] or independent[23–26] amplitude damping noise,[10,27] phase damping noise,[28,29] or others,[30] has been found. On the other hand, it is important to avoid ESD in practice so that quantum information processing and quantum computation tasks based on entanglement can be successfully accomplished. Many methods have been proposed to delay or defeat ESD.[31,32] From a new perspective, Qian and Eberly[33] recently investigated the connections between ESD and initial conditions of a generic ESD-susceptible Bell superposition state, and obtained the phase boundaries between ESD states and ESD-free states in terms of initial parameters.

The extended Werner-like (EWL) states[34] are important physical states, and they can reduce to Bell-like states and Werner states,[35] which can be determined nondestructively[36] and are important in quantum information processing and quantum computation.[37] It is necessary to discuss the question of whether ESD of a two-qubit EWL state under amplitude noise will occur or not via initial parameters. In this paper, the relation between ESD and initial conditions for two kinds of EWL states is investigated. The phase boundaries between ESD states and ESD-free states are considered in terms of initial purity and concurrence. Furthermore, as indicated by Anna Jamróz, disentanglement time of some pure initial states and Werner states can be changed by local unitary operations,[38] and we find the regions formed by some EWL states where ESD state can be transformed into ESD-free state by local unitary operations, which generalizes Anna Jamróz's results.

This paper is organized as follows. In Section 2, initial states and concurrence are introduced. The entanglement evolution of a two-qubit system under amplitude noise is discussed in Section 3. The phase boundaries between ESD states and ESD-free states are shown in Section 4. In Section 5, regions where ESD can be avoided by local unitary operations are given. Finally, a short conclusion is presented in Section 6.

2. Initial states and concurrence

A two-qubit system is initially prepared in two kinds of EWL states,[34]

$$\rho_\phi(0) = r|\phi\rangle\langle\phi| + \frac{1-r}{4}\boldsymbol{I},$$
$$\rho_\psi(0) = r|\psi\rangle\langle\psi| + \frac{1-r}{4}\boldsymbol{I}, \quad (1)$$

*Project supported by the National Natural Science Foundation of China (Grant No. 11074072).

†Corresponding author. E-mail: mffang@hunnu.edu.cn

© 2013 Chinese Physical Society and IOP Publishing Ltd

where r indicates the purity of initial states and $\boldsymbol{I}$ is a 4×4 identity matrix. The states

$$|\phi\rangle = \sqrt{1-x^2}|01\rangle + x|10\rangle,$$
$$|\psi\rangle = \sqrt{1-x^2}|00\rangle + x|11\rangle \tag{2}$$

are the Bell-like states with $0 \le x \le 1$, and x measures their degree of entanglement. It is obvious that the states defined by Eq. (1) reduce to Werner states for $x = 1/\sqrt{2}$ and to Bell-like states for $r = 1$.

The concurrence[39] is used to quantify the amount of entanglement of a two-qubit system. Its expression is

$$C = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{3}$$

where $\lambda_i, i = 1,2,3,4$ are the eigenvalues of the matrix $\tilde{\rho} = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ in decreasing order. Here ρ is a two-qubit state, ρ^* is the complex conjugation of ρ, and σ_y is the well-known Pauli matrix. It is found that the concurrence of "X" states, i.e.

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{4}$$

can be simply given by their density matrix elements,[40]

$$C_X = 2\max\{0, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\}. \tag{5}$$

Obviously, the two kinds of EWL states given by Eq. (1) are "X" states. Then, the initial concurrence of the system can be easily obtained from Eq. (5),

$$C_\phi(0) = C_\psi(0) = 2\max\left\{0, \left(\frac{1}{4} + x\sqrt{1-x^2}\right)r - \frac{1}{4}\right\}. \tag{6}$$

From Eq. (6), one can find that entanglement exists for initial states only if $r > 1/3$. Therefore only initial states with $1/3 < r \le 1$ are considered in the following discussion.

3. The dynamics of entanglement

For a system consisting of two uncoupled qubits and interacting with two independent identical amplitude damping environments, given an initial state $\rho(0)$, its evolution can be written as

$$\rho(t) = \sum_{\alpha,\beta} \boldsymbol{E}_{\alpha,\beta}\rho(0)\boldsymbol{E}_{\alpha,\beta}^\dagger, \tag{7}$$

where $\boldsymbol{E}_{\alpha,\beta} = \boldsymbol{E}_\alpha \otimes \boldsymbol{E}_\beta$ and $\alpha,\beta = 1,2$[41] satisfy $\sum_{\alpha,\beta}\boldsymbol{E}_{\alpha,\beta}^\dagger\boldsymbol{E}_{\alpha,\beta} = \boldsymbol{I}$ for all t. The Kraus operators $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are given by

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad \boldsymbol{E}_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \tag{8}$$

with $\gamma = 1 - e^{-\Gamma t}$.

Let $p(t) = 1-\gamma$, substituting p for $p(t)$, in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, two kinds of EWL states given by Eq. (1) evolve to

$$\rho_\phi(t) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & E & 0 \\ 0 & E & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \tag{9}$$

and

$$\rho_\psi(t) = \begin{pmatrix} F & 0 & 0 & E \\ 0 & G & 0 & 0 \\ 0 & 0 & G & 0 \\ E & 0 & 0 & H \end{pmatrix}, \tag{10}$$

with

$$\begin{aligned} A &= \frac{1}{4}(-rp^2 + p^2 - 4p + 4), \\ B &= \frac{1}{4}p(-4rx^2 + rp - p + 2r + 2), \\ C &= \frac{1}{4}p(4rx^2 + rp - p - 2r + 2), \\ D &= \frac{1}{4}(1-r)p^2, \\ E &= rpx\sqrt{1-x^2}, \\ F &= rx^2p^2 - 2rx^2p - \frac{1}{4}rp^2 + rp + \frac{1}{4}p^2 - p + 1, \\ G &= -rx^2p^2 + rx^2p + \frac{1}{4}rp^2 - \frac{1}{2}rp - \frac{1}{4}p^2 + \frac{1}{2}p, \\ H &= rx^2p^2 - \frac{1}{4}rp^2 + \frac{1}{4}p^2. \end{aligned} \tag{11}$$

From Eq. (5), the expression of concurrence for these states at t can be obtained

$$\begin{aligned} C_\phi(t) &= 2\max\left\{0, rpx\sqrt{1-x^2}\right. \\ &\quad \left. - \frac{1}{4}p\sqrt{(-rp^2 + p^2 - 4p + 4)(1-r)}\right\}, \end{aligned} \tag{12}$$

$$\begin{aligned} C_\psi(t) &= 2\max\left\{0, rpx\sqrt{1-x^2}\right. \\ &\quad \left. - \frac{1}{4}p|2 - p - 2r + pr + 4(1-p)rx^2|\right\}. \end{aligned} \tag{13}$$

Figure 1 shows the entanglement evolution for two kinds of EWL-states with fixed r. Both ESD states and ESD-free states are found. Obviously, the boundaries which separate ESD states from ESD-free states in Figs. 1(a) and 1(b) are different for fixed r. Do ESD states and ESD-free states always exist for initial entangled EWL states with arbitrary purity? What difference is there for boundaries of two kinds of EWL states in terms of initial purity and concurrence? We will give answers in the next section.

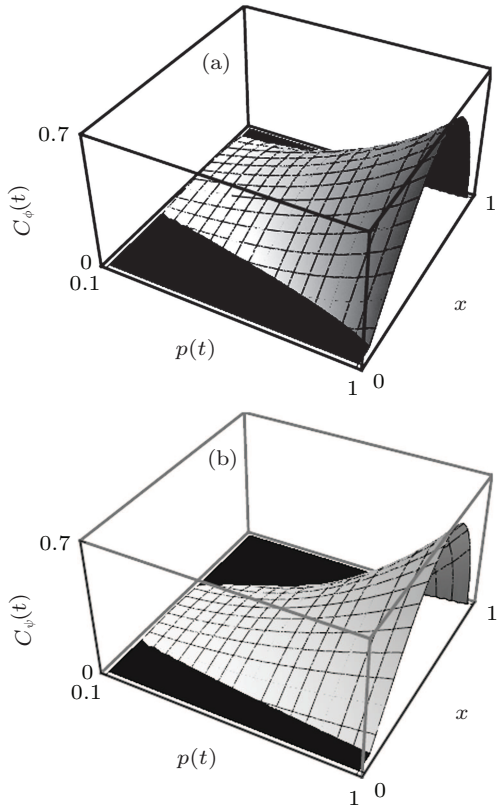


Fig. 1. The dynamics of entanglement for two kinds of EWL states. $C_\phi(t)$ (a) and $C_\psi(t)$ (b) as a function of the quantities $p(t)$ and x with $r = 0.75$. The black base-plane shows the region where ESD has already occurred.

4. Relation between initial conditions and ESD

In this section, ESD of two kinds of EWL-states is investigated in terms of initial purity and concurrence, and the boundaries which separate ESD states from ESD-free states are discussed.

4.1. System in initial states $\rho_\phi(0)$

We characterize the phenomenon of ESD for entire initial entangled states $\rho_\phi(0)$ in terms of r and $C_\phi(0)$. C_0 will be used to replace $C_\phi(0)$ in this subsection. From Eq. (6) one can find

$$x^2 = \frac{1}{2} - \frac{\sqrt{3r^2+4C_0r+2r-(2C_0+1)^2}}{4r}, \quad 0 \le x \le \frac{1}{\sqrt{2}}, \tag{14}$$

$$x^2 = \frac{1}{2} + \frac{\sqrt{3r^2+4C_0r+2r-(2C_0+1)^2}}{4r}, \quad \frac{1}{\sqrt{2}} < x \le 1. \tag{15}$$

That is to say, corresponding to given C_0 and r, there are two different EWL states $\rho_\phi(0)$. On the other hand, substituting Eqs. (14) and (15) into Eq. (12), one can easily obtain

$$C_\phi(t) = 2\max\left\{0, \frac{p}{4}(2C_0 - r + 1) - \frac{1}{4}p\sqrt{(-rp^2+p^2-4p+4)(1-r)}\right\}, \tag{16}$$

which means that the dynamical behaviors of entanglement of two different initial states with fixed C_0 and r are the same.

The condition when $C_\phi(t)$ becomes zero requests

$$p = \frac{2-\sqrt{4C_0^2+4C_0(1-r)+(1+r)^2}}{1-r}, \tag{17}$$

and ESD occurs only if $0 < p < 1$. Thus, one immediately obtains the region where ESD will occur

$$\frac{1}{3} < r < -1+2C_0+2\sqrt{1-2C_0}. \tag{18}$$

In addition, from Eq. (6), it is easy to obtain $C_0 \le 3r/2 - 1/2$ due to $0 \le x \le 1$, thus

$$r \ge \frac{2}{3}C_0 + \frac{1}{3}. \tag{19}$$

Figure 2 is obtained by using Eqs. (18) and (19), and it is the phase diagram of entire initial entangled EWL states $\rho_\phi(0)$ in terms of initial concurrence C_0 and purity r. In the figure, the dashed line $r = -1+2C_0+2\sqrt{1-2C_0}$ separates ESD states from ESD-free states; the thick solid line $r = 2C_0/3 + 1/3$ represents states with maximal concurrence when purity r is fixed, in fact, Werner states with $x = 1/\sqrt{2}$; the dot–dashed line represents Bell-like states $|\phi\rangle$ with $r = 1$. It is obvious that from the figure, there is no ESD for Bell-like states, and for Werner states, ESD occurs only if $1/3 < r < (-1+\sqrt{5})/2$. Moreover, ESD always exists for part of EWL states with initial purity less than $(-1+\sqrt{5})/2$. It is necessary to mention that a point in the region above the thick solid line represents two different states $\rho_\phi(0)$ via previous discussions.

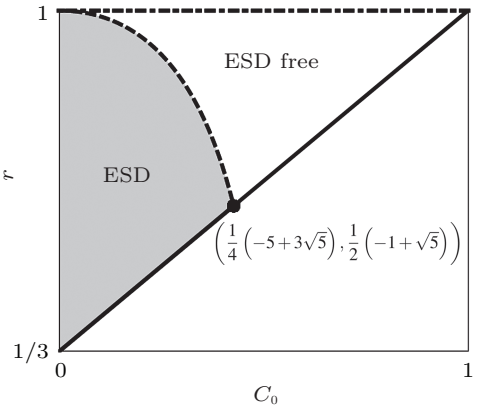


Fig. 2. Entanglement phase diagrams of entire initial entangled states $\rho_\phi(0)$ in terms of initial concurrence C_0 and purity r. The ESD states and ESD-free states are separated by the dashed line $r = -1+2C_0+2\sqrt{1-2C_0}$. The dot–dashed line represents Bell-like states, and the thick solid line Werner states.

4.2. System in initial states $\rho_\psi(0)$

The relation between ESD and initial conditions for entire initial entangled states $\rho_\psi(0)$ in terms of r and $C_\psi(0)$ is discussed in this subsection. C_0 replaces $C_\psi(0)$ here. Substituting Eqs. (14) and (15) into Eq. (13) yields

$$C_\psi(t) = 2\max\left\{0, \frac{p[2C_0-(1-p)(1+r-\sqrt{4r^2-(1+2C_0-r)^2})]}{4}\right\}, \quad 0 \le x \le \frac{1}{\sqrt{2}}, \tag{20}$$

$$C_\psi(t) = 2\max\left\{0, \frac{p[2C_0-(1-p)(1+r+\sqrt{4r^2-(1+2C_0-r)^2})]}{4}\right\}, \quad \frac{1}{\sqrt{2}} < x \le 1. \tag{21}$$

Two different EWL states $\rho_\psi(0)$ also exist for given C_0 and r due to $C_\phi(0) = C_\psi(0)$. However, equations (20) and (21) show that the entanglement evolution of different initial states $\rho_\psi(0)$ with fixed C_0 and r is different.

The initial entangled states request r to satisfy the condition shown in Eq. (19) also. However, the time when $C_\psi(t)$ becomes zero is quite different from the one when $C_\phi(t)$ equals zero. From Eqs. (20) and (21), when $C_\psi(t)$ is zero, p can be given by

$$p = 1 - \frac{2C_0}{1+r-\sqrt{4r^2-(1+2C_0-r)^2}}, \quad 0 \le x \le \frac{1}{\sqrt{2}}, \tag{22}$$

$$p = 1 - \frac{2C_0}{1+r+\sqrt{4r^2-(1+2C_0-r)^2}}, \quad \frac{1}{\sqrt{2}} < x \le 1. \tag{23}$$

One can find that the values of p are always between zero and one for initial states in the region $1/\sqrt{2} < x \le 1$, which means there must be ESD, while the existence of ESD for states bounded in another region $0 \le x \le 1/\sqrt{2}$ needs

$$\frac{1}{3} < r < -2C_0 + \sqrt{1+8C_0^2}. \tag{24}$$

Equations (19) and (24) lead to the phase diagram in Fig. 3 in terms of initial concurrence C_0 and purity r for initial states $\rho_\psi(0)$ in the region $0 \le x \le 1/\sqrt{2}$. The dashed line $r = -2C_0 + \sqrt{1+8C_0^2}$ separates ESD states from ESD-free states. The thick solid line again represents Werner states with $x = 1/\sqrt{2}$. The dot–dashed line represents Bell-like states $|\psi\rangle$ with $r = 1$. It is clear that ESD does not exist for Bell-like states with $0 \le x \le 1/\sqrt{2}$. However, ESD always occur for Werner states with $1/3 < r < 1$.

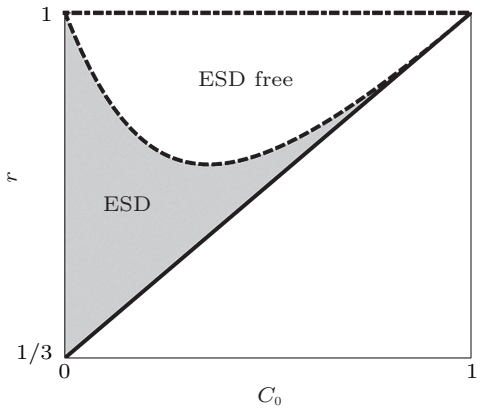


Fig. 3. Entanglement phase diagrams of part of initial entangled states $\rho_\psi(0)$ ($0 \le x \le 1/\sqrt{2}$) in terms of initial concurrence C_0 and purity r. The ESD states and ESD-free states are separated by the dashed line $r = -2C_0 + \sqrt{1+8C_0^2}$. The dot–dashed line represents Bell-like states, and the thick solid line Werner states.

5. Transformation induced by local unitary operations

For some pure initial states and Werner states, Anna Jamróz showed a possibility of changing disentanglement time by local unitary operations.[38] In this section, we demonstrate that regions formed by some EWL states exist where the original ESD states can be transformed into ESD-free states by local unitary operations.

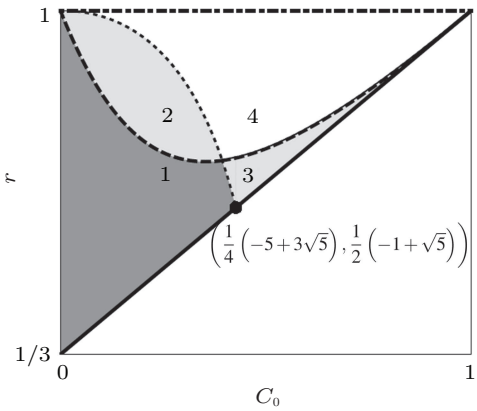


Fig. 4. Overlapped entanglement phase diagrams of entire initial entangled states $\rho_\phi(0)$ and part of initial entangled states $\rho_\psi(0)$ ($0 \le x \le 1/\sqrt{2}$) in terms of initial concurrence C_0 and purity r. The dotted line is described by $r = -1 + 2\sqrt{1-2C_0} + 2C_0$ and the dashed line by $r = -2C_0 + \sqrt{1+8C_0^2}$. The dot–dashed line represents Bell-like states, and the thick solid line Werner states.

Figure 4 has overlapped Figs. 2 and 3. According to discussions in the previous section, one can easily find that there are parts of Bell-like states $|\psi\rangle$ (i.e. $1/\sqrt{2} < x \le 1$) and Werner states ρ_ψ with $x = 1/\sqrt{2}$ (i.e. $(-1+\sqrt{5})/2 \le r \le 1$), which are ESD states and can be transformed into ESD-free states by a local unitary operation ($\boldsymbol{U} = \boldsymbol{\sigma}_1^x \otimes \boldsymbol{I}_2$). This is consistent with Jamróz's results. Furthermore, the conclusion can also be applied to states situated in the areas marked as 2 and 3 in Fig. 4. For example, we choose an initial state $\rho_1 = \rho_\phi(0)$ with $r = 4/5$, $x = \sqrt{15}/4$, which is in area 2. From Eq. (17), it is easy to obtain $p \approx 0.253$ when $C_{\rho_1}(t)$ is zero, which means ESD occurs. After the local unitary operation $\boldsymbol{U}$ on ρ_1, it turns to $\rho_2 = \rho_\psi(0)$ with $r = 4/5$, $x = 1/4$, and

$$C_{\rho_2}(t) = \frac{1}{5}\mathrm{e}^{-2\Gamma t} + \frac{\sqrt{15}-3}{10}\mathrm{e}^{-\Gamma t}$$

is always positive. In addition, since local unitary operations do not change purity and concurrence of a system, ρ_1 and ρ_2 are described by the same point in Fig. 4, i.e., ρ_2 is in area 2, thus, the local unitary operation has transformed an ESD state into an ESD-free state.

6. Conclusion

In conclusion, we have studied the dependence of ESD on initial purity and concurrence for two kinds of EWL states subject to an amplitude damping environment. Both ESD states and ESD-free states have been found for these two kinds of states. The boundaries which separate ESD states from ESD-free states have been explicitly described by equations with initial purity and concurrence. This result can provide guidelines for choosing a proper initial state for certain quantum information processing and quantum computation based on a quantum entangled system under noise. It is demonstrated that boundaries for two kinds of EWL states are different. The reason is as follows. Two states $\rho_\phi(0)$ and $\rho_\psi(0)$ with fixed r and C_0 possess different non-zero density matrix elements, and evolution of different density matrix elements under amplitude damping is also different. Thus, entanglement evolution of these two states is different. In fact, the two kinds of states are locally unitary-equivalent, therefore, the differences in boundaries allow one to transform an ESD state into an ESD-free one by a local unitary operation. This result may be useful to avoid ESD in quantum information processes and quantum computation.

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[3] Deutsch D, Ekert A, Jozsa R, Macchiavello C, Popescu S and Sanpera A 1996 *Phys. Rev. Lett.* **77** 2818
[4] Shor P W 1995 *Phys. Rev. A* **52** 2493
[5] Jozsa R and Linden N 2003 *Proc. R. Soc. Lond. A* **459** 2011
[6] Li M, Fei S M and Li-Jost X Q 2011 *Chin. Sci. Bull.* **56** 945
[7] Qian Y, Zhang Y Q and Xu J B 2012 *Chin. Sci. Bull.* **57** 1637
[8] Yue H D, Zhang Y and Gong J 2012 *Sci. China: Phys. Mech. Astron.* **55** 1641
[9] Zhu Y Y and Zhang Y 2012 *Sci. China: Phys. Mech. Astron.* **55** 2081
[10] Yu T and Eberly J H 2004 *Phys. Rev. Lett.* **93** 140404
[11] Yu T and Eberly J H 2006 *Opt. Commun.* **264** 393
[12] Yu T and Eberly J H 2007 *Science* **316** 555
[13] Al-Qasimi A and James D F V 2008 *Phys. Rev. A* **77** 012117
[14] Chou C H, Yu T and Hu B L 2008 *Phys. Rev. E* **77** 011112
[15] Scala M, Migliore R and Messina A 2008 *J. Phys. A: Math. Theor.* **41** 435304
[16] Xu J W and Chen Q H 2012 *Chin. Phys. B* **21** 040302
[17] Yönaç M, Yu T and Eberly J H 2006 *J. Phys. B* **39** S621
[18] Ikram M, Li F L and Zubairy M S 2007 *Phys. Rev. A* **75** 062336
[19] Fan H 2010 *Chin. Sci. Bull.* **55** 1758
[20] Ficek Z and Tanás R 2006 *Phys. Rev. A* **74** 024304
[21] Paz J P and Roncaglia A J 2008 *Phys. Rev. Lett.* **100** 220401
[22] Huang J, Fang M F, Yang B Y and Liu X 2012 *Chin. Phys. B* **21** 084205
[23] Sun Z, Wang X G and Sun C P 2007 *Phys. Rev. A* **75** 062312
[24] Ann K and Jaeger G 2007 *Phys. Rev. B* **76** 044101
[25] Xu J Z, Guo J B, Wen W, Bai Y K and Yan F L 2012 *Chin. Phys. B* **21** 080305
[26] Hu Y H and Wang J Q 2012 *Chin. Phys. B* **21** 014203
[27] Huang J H and Zhu S Y 2007 *Phys. Rev. A* **76** 062322
[28] Roszak K and Machnikowski P 2006 *Phys. Rev. A* **73** 022313
[29] Ann K and Jaeger G 2007 *Phys. Rev. B* **75** 115307
[30] Bellomo B, Lo Franco R and Compagno G 2007 *Phys. Rev. Lett.* **99** 160502
[31] Rau A R P, Ali M and Alber G 2008 *Europhys. Lett.* **82** 40002
[32] Siomau M and Kamli A A 2012 *Phys. Rev. A* **86** 032304
[33] Qian X F and Eberly J H 2012 *Phys. Lett. A* **45** 376
[34] Bellomo B, Lo Franco R and Compagno G 2008 *Phys. Rev. A* **77** 032342
[35] Werner R 1989 *Phys. Rev. A* **40** 4277
[36] Tang Y C, Li Y S, Hao L, Hou S Y, and Long G L 2012 *Phys. Rev. A* **85** 022329
[37] Popescu S 1994 *Phys. Rev. Lett.* **72** 797
[38] Jamróz A 2006 *J. Phys. A: Math. Gen.* **39** 7727
[39] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
[40] Yu T and Eberly J H 2007 *Quantum Inf. Comput.* **7** 459
[41] Werlang T, Souza S, Fanchini F F and Villas B C J 2009 *Phys. Rev. A* **80** 024103