Relativistic symmetries in the Hulthén scalar–vector–tensor interactions

Majid Hamzavi[a)†] and Ali Akbar Rajabi[b)]

[a)] Department of Science and Engineering, Abhar Branch, Islamic Azad University, Abhar, Iran
[b)] Physics Department, Shahrood University of Technology, Shahrood, Iran

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In the presence of spin and pseudospin (p-spin) symmetries, the approximate analytical bound states of the Dirac equation for scalar–vector–tensor Hulthén potentials are obtained with any arbitrary spin–orbit coupling number κ using the Pekeris approximation. The Hulthén tensor interaction is studied instead of the commonly used Coulomb or linear terms. The generalized parametric Nikiforov–Uvarov (NU) method is used to obtain energy eigenvalues and corresponding wave functions in their closed forms. It is shown that tensor interaction removes degeneracy between spin and p-spin doublets. Some numerical results are also given.

Keywords: Dirac equation, Hulthén scalar–vector–tensor potential, spin and p-spin symmetry, NU method

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1. Introduction

One of the most important concepts for understanding the traditional magic numbers in stable nuclei is the spin symmetry breaking.[1–4] On the other hand, the pseudospin (p-spin) symmetry was observed originally almost 40 years ago as a mechanism to explain different aspects of the nuclear structure, which is one of the most interesting phenomena in the relativistic quantum mechanics. Within the framework of the Dirac equation, the p-spin symmetry is used to feature deformed nuclei and superdeformation and also to establish an effective shell-model.[4–8] However, spin symmetry is relevant to mesons.[9] The spin symmetry occurs when the difference between the scalar $S(r)$ and vector $V(r)$ potentials is constant, i.e., $\Delta(r) = C_{\rm s}$ and the p-spin symmetry occurs when the sum of the scalar and vector potentials is constant, i.e., $\Sigma(r) = C_{\rm ps}$.[10,11] The p-spin symmetry refers to a quasi-degeneracy of single nucleon doublets with non-relativistic quantum number $(n, l, j = l+1/2)$ and $(n-1, l+2, j = l+3/2)$, where n, l, and j are single nucleon radial, orbital, and total angular quantum numbers, respectively.[12,13] The total angular momentum $j = \tilde{l} + \tilde{s}$ with $\tilde{l} = l+1$ is a pseudo-angular momentum and $\tilde{s}$ is p-spin angular momentum.[14–18] For more reviews of the p-spin symmetry, readers can refer to the very recent work[9] and the references therein. In Ref. [19], authors explored the origin of the p-spin symmetry and its breaking mechanism by combining the supersymmetry quantum mechanics, perturbation theory and similarity renormalization group method. Tensor potentials were introduced into the Dirac equation with the substitution $\boldsymbol{p} \to \boldsymbol{p} - {\rm i}m\omega\beta \cdot \hat{r}U(r)$ and a spin–orbit coupling was added to the Dirac Hamiltonian.[20,21] For more reviews of tensor interaction, readers can refer to Refs. [22]–[29] in which authors used different potentials and different kinds of tensor potential like Coulomb and linear ones, but here we study a tensor potential in the Hulthén form.

The Hulthén potential has the form

$$v(r) = -v_0 \frac{{\rm e}^{-\delta r}}{1-{\rm e}^{-\delta r}}, \tag{1}$$

where δ is the screening parameter which is used for determining the range of the Hulthén potential; the parameter v_0 represents δZe^2, with Ze being the charge of the nucleon.[30] The intensity of the Hulthén potential is denoted by v_0 under the condition of $\delta > 0$. This potential has been used in several branches of physics and its discrete and continuum states have been studied by a variety of techniques such as the algebraic perturbation calculations which are based on the dynamical group structure $SO(2,1)$,[31] the formalism of supersymmetric quantum mechanics within the framework of the variational method,[32] the supersymmetry and shape invariance property,[33] the asymptotic iteration method,[34,35] the Nikiforov–Uvarov (NU) method,[36] and the approach proposed by Biedenharn for the Dirac–Coulomb problem.[37,38]

Ikhdair and Sever[14] obtained the energy eigenvalues and wave functions of the Dirac particle in the field of the scalar and vector Hulthén potentials including a Coulomb-like tensor interaction. The motivation of this work is to obtain the energy eigenvalues and wave functions of the Dirac particle in the fields of the scalar and vector Hulthén potentials including a Hulthén tensor interaction in view of spin and p-spin symmetries.

The rest of this paper is organized as follows. In Section 2, we briefly introduce the Dirac equation with scalar, vector, and tensor potentials with arbitrary spin–orbit coupling

†Corresponding author. E-mail: majid.hamzavi@gmail.com

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number κ including tensor interaction under spin and p-spin symmetric limits. The generalized parametric NU method is presented in Section 3. The energy eigenvalue equations and corresponding wave functions in terms of the Jacobi polynomials are obtained in Section 4. We also give some remarks and numerical results in this section. We end with a conclusion in Section 5.

2. Dirac equation including tensor coupling

The Dirac equation for fermionic massive spin-1/2 particles moving in attractive scalar $S(r)$, repulsive vector $V(r)$, and tensor $U(r)$ potentials is (in units $\hbar = c = 1$)

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\boldsymbol{\beta}(M+S(r))-i\boldsymbol{\beta}\boldsymbol{\alpha}\cdot\hat{r}U(r)]\psi(\boldsymbol{r})=[E-V(r)]\psi(\boldsymbol{r}), \quad (2)$$

where E is the relativistic energy of the system, $\boldsymbol{p} = -\mathrm{i}\boldsymbol{\nabla}$ is the three-dimensional momentum operator, and M is the mass of the fermionic particle. Further, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the 4×4 Dirac matrices given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix}, \quad (3)$$

where $\boldsymbol{I}$ is 2×2 unitary matrix and $\boldsymbol{\sigma}$ are three-vector spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)$$

The total angular momentum operator $\boldsymbol{J}$, and the spin–orbit $K = (\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, where $\boldsymbol{L}$ is orbital angular momentum of the spherical nucleons, commutes with the Dirac Hamiltonian. The eigenvalues of the spin–orbit coupling operator are $\kappa = (j+1/2) > 0$ and $\kappa = -(j+1/2) < 0$ for unaligned spin $j = l-1/2$ and the aligned spin $j = l+1/2$, respectively. (H^2, K, J^2, J_z) can be taken as a complete set of the conservative quantities. Thus, the spinor wave functions can be classified according to their angular momentum j, spin–orbit quantum number κ, and the radial quantum number n written as follows:

$$\psi_{n\kappa}(\boldsymbol{r}) = \begin{pmatrix} f_{n\kappa}(\boldsymbol{r}) \\ g_{n\kappa}(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \dfrac{F_{n\kappa}(r)}{r}\mathrm{Y}^{l}_{jm}(\theta,\varphi) \\ \mathrm{i}\dfrac{G_{n\kappa}(r)}{r}\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi) \end{pmatrix}, \quad (5)$$

where $f_{n\kappa}(\boldsymbol{r})$ is the upper (large) component and $g_{n\kappa}(\boldsymbol{r})$ is the lower (small) component of the Dirac spinors. $\mathrm{Y}^{l}_{jm}(\theta,\varphi)$ and $\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi)$ are spin and p-spin spherical harmonics, respectively, and m is the projection of the angular momentum on the z axis. Substituting Eq. (5) into Eq. (2) and using the following relations

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}+\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}), \quad (6)$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{P}) = \boldsymbol{\sigma}\cdot\hat{r}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{P}+\mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}\right), \quad (7)$$

together with the following properties

$$\begin{aligned}
&(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi) = (\kappa-1)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi),\\
&(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{l}_{jm}(\theta,\phi) = -(\kappa-1)\mathrm{Y}^{l}_{jm}(\theta,\phi),\\
&(\boldsymbol{\sigma}\cdot\hat{r})\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi) = -\mathrm{Y}^{l}_{jm}(\theta,\phi),\\
&(\boldsymbol{\sigma}\cdot\hat{r})\mathrm{Y}^{l}_{jm}(\theta,\phi) = -\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi), \quad (8)
\end{aligned}$$

one obtains two coupled differential equations for upper and lower radial wave functions $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r) = (M+E_{n\kappa}-\Delta(r))G_{n\kappa}(r), \quad (9a)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r) = (M-E_{n\kappa}+\Sigma(r))F_{n\kappa}(r), \quad (9b)$$

where

$$\Delta(r) = V(r)-S(r), \quad (10a)$$

$$\Sigma(r) = V(r)+S(r), \quad (10b)$$

are the difference and the sum potentials, respectively. Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ from Eqs. (9a) and (9b), we finally obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components, respectively:

$$\begin{aligned}
&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\right]F_{n\kappa}(r)\\
&+\frac{\mathrm{d}\Delta(r)/\mathrm{d}r}{M+E_{n\kappa}-\Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)\\
&=\left[(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))\right]F_{n\kappa}(r), \quad (11)\\
&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)+\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\right]G_{n\kappa}(r)\\
&+\frac{\mathrm{d}\Sigma(r)/\mathrm{d}r}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r)\\
&=\left[(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))\right]G_{n\kappa}(r), \quad (12)
\end{aligned}$$

where $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1) = l(l+1)$. The quantum number κ is related to the quantum numbers for spin symmetry l and p-spin symmetry $\tilde{l}$ as

$$\kappa = \begin{cases} -(l+1) = -\left(j+\dfrac{1}{2}\right) \ (\mathrm{s}_{1/2}, \mathrm{p}_{3/2}, \text{etc.}), & j = l+\dfrac{1}{2}, \ \text{aligned spin} \ (\kappa<0), \\ +l = +\left(j+\dfrac{1}{2}\right) \ (\mathrm{p}_{1/2}, \mathrm{d}_{3/2}, \text{etc.}), & j = l-\dfrac{1}{2}, \ \text{unaligned spin} \ (\kappa>0), \end{cases}$$

and the quasidegenerate doublet structure can be expressed in terms of a p-spin angular momentum $\tilde{s} = 1/2$

$$\kappa = \begin{cases} -\tilde{l} = -\left(j+\dfrac{1}{2}\right) \ (\mathrm{s}_{1/2}, \mathrm{p}_{3/2}, \text{etc.}), & j = \tilde{l}-\dfrac{1}{2}, \ \text{aligned p-spin} \ (\kappa<0), \\ +(\tilde{l}+1) = +\left(j+\dfrac{1}{2}\right) \ (\mathrm{d}_{3/2}, f_{5/2}, \text{etc.}), & j = \tilde{l}+\dfrac{1}{2}, \ \text{unaligned p-spin} \ (\kappa>0), \end{cases}$$

where $\kappa = \pm 1, \pm 2, \ldots$. For example, $(1s_{1/2}, 0d_{3/2})$ and $(1p_{3/2}, 0f_{5/2})$ can be considered as p-spin doublets.

2.1. The p-spin symmetric limit

Ginocchio[11,39,40] showed that there is a connection between p-spin symmetry and the time component of a vector potential that is nearly equal to the scalar potential, i.e., $S(r) \approx -V(r)$. Also, Meng *et al.*[41,42] showed that if $\mathrm{d}[V(r)+S(r)]/\mathrm{d}r = \mathrm{d}\Sigma(r)/\mathrm{d}r = 0$ or $\Sigma(r) = C_{\mathrm{ps}} =$ constant, p-spin symmetry is exact in the Dirac equation. In this part, we take attractive scalar $S(r)$, repulsive vector $V(r)$, and tensor $U(r)$ potentials as

$$V(r) = V_0 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}, \quad S(r) = -S_0 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}, \tag{13}$$

$$U(r) = -U_0 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}, \tag{14}$$

and the difference between scalar and vector potentials becomes

$$\Delta(r) = \Delta_0 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}, \tag{15}$$

where $\Delta_0 = V_0 + S_0$. Under this symmetry, from Eq. (12), one obtains

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} - 2\kappa U_0 \frac{\mathrm{e}^{-\delta r}}{r(1-\mathrm{e}^{-\delta r})} + U_0\delta\Big(\frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2}\Big) - U_0^2 \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} - \tilde{\gamma}\Delta_0 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} - \tilde{\beta}^2\Big] G_{n\kappa}(r) = 0, \tag{16}$$

where $\tilde{\gamma} = E_{n\kappa} - M - C_{\mathrm{ps}}$, and $\tilde{\beta}^2 = (M + E_{n\kappa})(M - E_{n\kappa} + C_{\mathrm{ps}})$. Also, $\kappa = -\tilde{l}$ and $\kappa = \tilde{l}+1$ for $\kappa < 0$ and $\kappa > 0$, respectively.

2.2. Spin symmetric limit

In the spin symmetry we require $\mathrm{d}\Delta(r)/\mathrm{d}r = 0$ or $\Delta(r) = C_{\mathrm{s}} =$ constant[43–46] and $\Sigma(r)$ is expressed as

$$\Sigma(r) = \Sigma_0 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}, \tag{17}$$

where $\Sigma_0 = -(S_0 - V_0)$. Then equation (11) becomes

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} - 2\kappa U_0 \frac{\mathrm{e}^{-\delta r}}{r(1-\mathrm{e}^{-\delta r})} - U_0\delta\Big(\frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2}\Big) - U_0^2 \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} + \gamma\Sigma_0 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} - \beta^2\Big] G_{n\kappa}(r) = 0, \tag{18}$$

where $\gamma = M + E_{n\kappa} - C_{\mathrm{s}}$ and $\beta^2 = (M - E_{n\kappa})(M + E_{n\kappa} - C_{\mathrm{s}})$. Also, $\kappa = l$ and $\kappa = -l-1$ for $\kappa < 0$ and $\kappa > 0$, respectively.

Because of the spin–orbit coupling term, equations (15) and (16) cannot be solved analytically, so we shall use the shifted approximation[36] in order to deal with the spin–orbit coupling term as

$$\frac{1}{r^2} \approx \delta^2 \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2}, \tag{19a}$$

or equivalently

$$\frac{1}{r} \approx \delta \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}. \tag{19b}$$

The exponential numerator in Eq. (19) is expanded for small values of r and higher order terms are ignored up to the first-order term.[36] In Fig. 1, we plot the centrifugal term and its approximation with two different values of δ, to show the accuracy of the approximation.

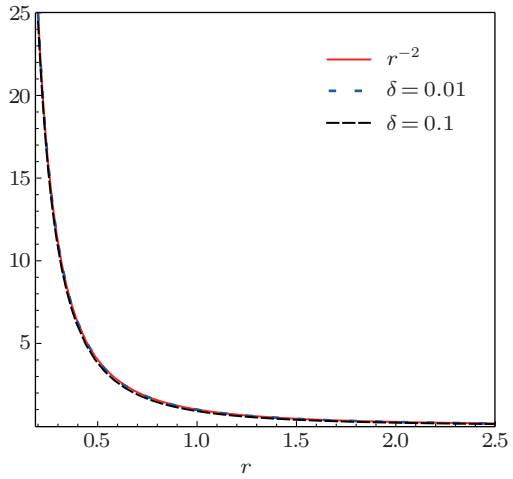


Fig. 1. (color online) Centrifugal term r^{-2} (red solid curve) and its approximation in Eq. (19a) with $\delta = 0.01$ fm^{-1} (blue dashed curve) and $\delta = 0.1$ fm^{-1} (black dashed curve). The unit 1 fm $= 10^{-15}$ m.

3. Parametric NU method

This powerful mathematical tool is usually used to solve the second-order differential equations. Now we consider the following differential equation[47]

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{20}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of the second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without the need to check the validity of solution, we present a shortcut for the method. Hence, firstly we write the general form of the Schrödinger-like equation (20) in a more general form as[47,48]

$$\psi_n''(s) + \Big(\frac{c_1 - c_2 s}{s(1-c_3 s)}\Big)\psi_n'(s) + \Big(\frac{-p_2 s^2 + p_1 s - p_0}{s^2(1-c_3 s)^2}\Big)\psi_n(s) = 0, \tag{21}$$

satisfying the wave functions

$$\psi_n(s) = \varphi(s) y_n(s). \tag{22}$$

Secondly, we compare Eq. (21) with its counterpart (20) to obtain the following parameter values,

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1-c_3 s),$$

$$\tilde{\sigma}(s) = -p_2 s^2 + p_1 s - p_0. \tag{23}$$

Now, following the NU method,[47] we obtain the energy equation[48]

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} - c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3c_8 - 2\sqrt{c_8 c_9} = 0, \tag{24}$$

and the corresponding wave functions

$$\begin{aligned}
&\rho(s) = s^{c_{10}}(1-c_3 s)^{c_{11}}, \quad \varphi(s) = s^{c_{12}}(1-c_3 s)^{c_{13}}, \\
&\qquad c_{12} > 0, \ c_{13} > 0, \\
&y_n(s) = \mathrm{P}_n^{(c_{10},c_{11})}(1-2c_3 s), \ c_{10} > -1, \ c_{11} > -1, \\
&\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}}(1-c_3 s)^{c_{13}} \mathrm{P}_n^{(c_{10},\, c_{11})}(1-2c_3 s),
\end{aligned} \tag{25}$$

where $\mathrm{P}_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$, and $x \in [-1,1]$ are Jacobi polynomials with the following constants:

$$\begin{aligned}
&c_4 = \frac{1}{2}(1-c_1), \ c_5 = \frac{1}{2}(c_2 - 2c_3), \\
&c_6 = c_5^2 + p_2, \ c_7 = 2c_4c_5 - p_1, \\
&c_8 = c_4^2 + p_0, \ c_9 = c_3(c_7 + c_3c_8) + c_6, \\
&c_{10} = c_1 + 2c_4 - 2\sqrt{c_8} - 1 > -1, \\
&c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, \ c_3 \neq 0, \\
&c_{12} = c_4 - \sqrt{c_8} > 0, \ c_{13} = -c_4 + \frac{1}{c_3}(\sqrt{c_9} - c_5) > 0, \\
&c_3 \neq 0,
\end{aligned} \tag{26}$$

where $c_{12} > 0$, $c_{13} > 0$, and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

In the special case where $c_3 = 0$, the wave function (22) becomes

$$\begin{aligned}
&\lim_{c_3 \to 0} \mathrm{P}_n^{(c_{10},c_{11})}(1-2c_3 s) = L_n^{c_{10}}(c_{11} s), \\
&\lim_{c_3 \to 0} (1-c_3 s)^{c_{13}} = \mathrm{e}^{c_{13} s}, \\
&\psi(s) = N s^{c_{12}} \mathrm{e}^{c_{13} s} L_n^{c_{10}}(c_{11} s).
\end{aligned} \tag{27}$$

4. Relativistic bound states for the Hulthén-like and tensor potentials

4.1. The p-spin symmetric solution

Substituting Eq. (19) into Eq. (16) gives

$$\begin{aligned}
&\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \kappa(\kappa-1)\delta^2 \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} - 2\kappa U_0 \delta \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} \\
&+ U_0 \delta \Big(\frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} \Big) \\
&- U_0^2 \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} - \tilde{\gamma}\Delta_0 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} - \tilde{\beta}^2 \Big] G_{n\kappa}(r) = 0
\end{aligned} \tag{28}$$

and further making the change of variables $s = \mathrm{e}^{-\delta r}$, one obtains

$$\begin{aligned}
&\frac{\mathrm{d}^2 G_{n\kappa}(s)}{\mathrm{d}s^2} + \frac{1-s}{s(1-s)} \frac{\mathrm{d}G_{n\kappa}(s)}{\mathrm{d}s} \\
&+ \frac{1}{\delta^2 s^2 (1-s)^2} \left[-\tilde{A} s^2 + \tilde{B} s(1-s) - \tilde{\beta}^2 (1-s)^2 \right] G_{n\kappa}(s) = 0,
\end{aligned} \tag{29}$$

where

$$\tilde{A} = \kappa(\kappa-1)\delta^2 + U_0(U_0 + 2\kappa\delta - \delta), \tag{30a}$$

$$\tilde{B} = -\tilde{\gamma}\Delta_0 + U_0\delta. \tag{30b}$$

Comparing Eq. (28) and Eq. (21), we can easily obtain the coefficients c_i $(i = 1,2,3)$ and analytical expressions p_j $(j = 0,1,2)$ as follows:

$$\begin{aligned}
&c_1 = 1, \ \ p_0 = \frac{\tilde{\beta}^2}{\delta^2}, \quad c_2 = 1, \ \ p_1 = \frac{\tilde{B} + 2\tilde{\beta}^2}{\delta^2}, \\
&c_3 = 1, \ \ p_2 = \frac{\tilde{A} + \tilde{B} + \tilde{\beta}^2}{\delta^2}.
\end{aligned} \tag{30c}$$

The values of coefficients c_i $(i = 4,5,\ldots,13)$ are found from Eq. (26) as follows:

$$\begin{aligned}
&c_4 = 0, \quad c_5 = -\frac{1}{2}, \quad c_6 = \frac{1}{4} + \frac{\tilde{A} + \tilde{B} + \tilde{\beta}^2}{\delta^2}, \\
&c_7 = -\frac{\tilde{B} + 2\tilde{\beta}^2}{\delta^2}, \ \ c_8 = \frac{\tilde{\beta}^2}{\delta^2}, \\
&c_9 = p_2 - p_1 + p_0 + \frac{1}{4} = \Big(\kappa - \frac{1}{2}\Big)^2 + \frac{U_0}{\delta}\Big(\frac{U_0}{\delta} + 2\kappa - 1\Big), \\
&c_{10} = -2\sqrt{\frac{\tilde{\beta}^2}{\delta^2}}, \ \ c_{11} = 2\tilde{\eta}_\kappa, \ \ c_{12} = -\sqrt{\frac{\tilde{\beta}^2}{\delta^2}}, \ \ c_{13} = \tilde{\eta}_\kappa + \frac{1}{2},
\end{aligned} \tag{31}$$

where

$$\tilde{\eta}_\kappa = \sqrt{\Big(\kappa - \frac{1}{2}\Big)^2 + \frac{U_0}{\delta}\Big(\frac{U_0}{\delta} + 2\kappa - 1\Big)}. \tag{32}$$

By using Eq. (24), we can obtain the energy eigenvalues of the Hulthén potential with p-spin symmetric limit as

$$\begin{aligned}
&\left(n + \frac{1}{2} + \sqrt{\Big(\kappa - \frac{1}{2}\Big)^2 + \frac{U_0}{\delta}\Big(\frac{U_0}{\delta} + 2\kappa - 1\Big)} - \sqrt{\frac{\tilde{\beta}^2}{\delta^2}} \right)^2 \\
&= \frac{-\tilde{\gamma}\Delta_0 + \tilde{\beta}^2}{\delta^2} + \Big(\kappa - \frac{1}{2}\Big)^2 + \frac{U_0}{\delta}\Big(\frac{U_0}{\delta} + 2\kappa\Big) - \frac{1}{4}.
\end{aligned} \tag{33}$$

Some numerical results of the above equation are given in Table 1. In Table 1, we obtain numerical results in the p-spin symmetry in the presence and absence of tensor Hulthén form potential. We take a set of parameter values, $M = 5.0$ fm^{-1} (1 fm $= 10^{-15}$ m), $C_{\mathrm{ps}} = 0$, $V_0 = 2.5$ fm^{-1}, $S_0 = 2.9$ fm^{-1}, and $\delta = 0.1$ fm^{-1}. It is observed that the degeneracy exists in the following doublets $(1\mathrm{s}_{1/2}, 0\mathrm{d}_{3/2})$, $(1\mathrm{p}_{3/2}, 0\mathrm{f}_{5/2})$, $(1\mathrm{d}_{5/2}, 0\mathrm{g}_{7/2})$, $(1\mathrm{f}_{7/2}, 0\mathrm{h}_{9/2})$, and so on, and in the absence of the tensor potential $(U_0 = 0)$ and they are considered as p-spin doublets.

In Fig. 2, we investigate the effect of the tensor potential on the p-spin doublet splitting by considering the following pairs of orbitals: $(1\mathrm{d}_{5/2}, 0\mathrm{g}_{7/2})$, $(2\mathrm{f}_{7/2}, 1\mathrm{h}_{9/2})$. From Fig. 2, we observe that in the case of $U_0 = 0$ (no tensor interaction), members of p-spin doublets have the same energy. However, in the presence of the tensor potential $U_0 \neq 0$, these degeneracies are removed.

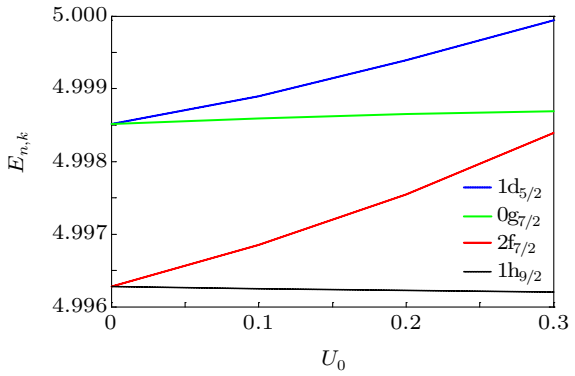


Fig. 2. (color online) Contributions of the tensor potential parameter to the energy level in the case of p-spin symmetry.

To find the corresponding wave functions, referring to Table 1 and relations in Eq. (23), we find the functions

$$\rho(s) = s^{-2\sqrt{\tilde{\beta}^2/\delta^2}}(1-s)^{2\tilde{\eta}_\kappa},$$
$$\phi(s) = s^{-\sqrt{\tilde{\beta}^2/\delta^2}}(1-s)^{\tilde{\eta}_\kappa+1/2},$$

$$y_n(s) = \mathrm{P}_n^{(-2\sqrt{\tilde{\beta}^2/\delta^2},2\tilde{\eta}_\kappa)}(1-2s). \tag{34}$$

By using $G_{n\kappa}(s) = \phi(s)y_n(s)$, we obtain the lower component of the Dirac spinor from relation (25) as

$$G_{n\kappa}(s) = \tilde{N}_{n\kappa}s^{-\sqrt{\tilde{\beta}^2/\delta^2}}(1-s)^{\tilde{\eta}_\kappa+1/2} \times \mathrm{P}_n^{(-2\sqrt{\tilde{\beta}^2/\delta^2},2\tilde{\eta}_\kappa)}(1-2s), \tag{35}$$

or equivalently

$$G_{n\kappa}(r) = \tilde{N}_{n\kappa}\mathrm{e}^{-\sqrt{\tilde{\beta}^2}r}(1-\mathrm{e}^{-\delta r})^{\tilde{\eta}_\kappa+1/2} \times \mathrm{P}_n^{(-2\sqrt{\tilde{\beta}^2/\delta^2},2\tilde{\eta}_\kappa)}(1-2\mathrm{e}^{-\delta r}), \tag{36}$$

where $\tilde{N}_{n\kappa}$ is the normalization constant. The upper component of the Dirac spinor from Eq. (9b) can be calculated from

$$F_{n\kappa}(r) = \frac{1}{M-E_{n\kappa}+C_{\mathrm{ps}}}\Big(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\Big)G_{n\kappa}(r), \tag{37}$$

where $E \neq M + C_{\mathrm{ps}}$.

Table 1. Bound state energy eigenvalues in units of fm^{-1} of the p-spin symmetry Hulthén potential for several values of n and κ with $U_0 = 0.1$.

$\tilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}, U_0 \neq 0$	$E_{n,\kappa<0}, U_0 = 0$	$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}, U_0 \neq 0$	$E_{n-1,\kappa>0}, U_0 = 0$
1	1, −1	$1s_{1/2}$	4.999528842	4.998866406	0, 2	$0d_{3/2}$	4.998896324	4.998866406
2	1, −2	$1p_{3/2}$	4.999147492	4.998655904	0, 3	$0f_{5/2}$	4.998720844	4.998655904
3	1, −3	$1d_{5/2}$	4.998896324	4.998512210	0, 4	$0g_{7/2}$	4.998591717	4.998512210
4	1, −4	$1f_{7/2}$	4.998720844	4.998407889	0, 5	$0h_{9/2}$	4.998492820	4.998407889
1	2, −1	$2s_{1/2}$	4.998552452	4.997499195	1, 2	$1d_{3/2}$	4.997268501	4.997499195
2	2, −2	$2p_{3/2}$	4.997816234	4.996981851	1, 3	$1f_{5/2}$	4.996848860	4.996981851
3	2, −3	$2d_{5/2}$	4.997268501	4.996589219	1, 4	$1g_{7/2}$	4.996517976	4.996589219
4	2, −4	$2f_{7/2}$	4.996848860	4.996281439	1, 5	$1h_{9/2}$	4.996250655	4.996281439

4.2. Spin symmetric solution

In this subsection we will obtain the energy eigenvalues and the corresponding wave functions for the spin symmetric limit of the Hulthén potential, i.e. solutions of Eq. (18). Substituting Eq. (19) into Eq. (18) gives

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \kappa(\kappa+1)\delta^2\frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} - 2\kappa U_0\delta\frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} - U_0\delta\Big(\frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2}\Big) - U_0^2\frac{\mathrm{e}^{-2\delta r}}{(1-\mathrm{e}^{-\delta r})^2} + \gamma\Sigma_0\frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} - \beta^2\Big]F_{n\kappa}(r) = 0, \tag{38}$$

and further making the change of variables $s = \mathrm{e}^{-\delta r}$, one obtains

$$\frac{\mathrm{d}^2G_{n\kappa}(s)}{\mathrm{d}s^2} + \frac{1-s}{s(1-s)}\frac{\mathrm{d}G_{n\kappa}(s)}{\mathrm{d}s} + \frac{1}{\delta^2s^2(1-s)^2}[-As^2 + Bs(1-s) - \tilde{\beta}^2(1-s)^2]F_{n\kappa}(s) = 0, \tag{39}$$

where

$$A = \kappa(\kappa+1)\delta^2 + U_0(U_0 + 2\kappa\delta + \delta), \tag{40a}$$

$$B = \gamma\Sigma_0 - U_0\delta. \tag{40b}$$

On the other hand, to avoid repetition in our solution, the energy states of Eq. (38) in the spin symmetric case are as follows:

$$\Big(n + \frac{1}{2} + \sqrt{\Big(\kappa+\frac{1}{2}\Big)^2 + \frac{U_0}{\delta}\Big(\frac{U_0}{\delta} + 2\kappa + 1\Big)} - \sqrt{\frac{\beta^2}{\delta^2}}\Big)^2 = \frac{-\gamma\Sigma_0 + \beta^2}{\delta^2} + \Big(\kappa+\frac{1}{2}\Big)^2 + \frac{U_0}{\delta}\Big(\frac{U_0}{\delta} + 2\kappa\Big) - \frac{1}{4}. \tag{41}$$

Also, we obtain the upper component of the Dirac spinor as

$$F_{n\kappa}(r) = N_{n\kappa}\mathrm{e}^{-\sqrt{\beta^2}r}(1-\mathrm{e}^{-\delta r})^{\eta_\kappa+1/2} \times \mathrm{P}_n^{(-2\sqrt{\beta^2/\delta^2},2\eta_\kappa)}(1-2\mathrm{e}^{-\delta r}), \tag{42}$$

where $N_{n\kappa}$ is normalization constant and

$$\eta_\kappa = \sqrt{\Big(\kappa+\frac{1}{2}\Big)^2 + \frac{U_0}{\delta}\Big(\frac{U_0}{\delta} + 2\kappa + 1\Big)}. \tag{43}$$

The lower component of the Dirac spinor from Eq. (9a) can be calculated from

$$G_{n\kappa}(r) = \frac{1}{M+E_{n\kappa}-C_{\mathrm{s}}}\Big(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\Big)F_{n\kappa}(r), \tag{44}$$

where $E \neq -M + C_{\mathrm{s}}$.

In Table 2, we use the same parameters as those in the above subsection. We can observe that each pair of orbitals ($n\mathrm{p}_{1/2}$, $n\mathrm{p}_{3/2}$), ($n\mathrm{d}_{3/2}$, $n\mathrm{d}_{5/2}$), and ($n\mathrm{f}_{5/2}$, $n\mathrm{f}_{7/2}$) has the same energy in the absence of the tensor potential ($U_0 = 0$). Thus, they can be viewed as the spin doublets, i.e., the state $1\mathrm{p}_{1/2}$ with $n = 1$ and $\kappa = 1$ forms a spin doublet with the $1\mathrm{p}_{3/2}$ state with $n = 1$ and $\kappa = -2$. On the other hand, in the presence of the tensor potential ($U_0 \neq 0$), one can notice that the degeneracy between every pair of spin doublets is removed.

In Fig. 3, we investigate the effect of the tensor potential on the spin doublet splitting by considering the following pairs of orbitals: ($1\mathrm{p}_{3/2}$, $1\mathrm{p}_{1/2}$), ($1\mathrm{f}_{7/2}$, $1\mathrm{f}_{5/2}$) and one can observe that the results obtained in the spin symmetric limit resemble the ones observed in the p-spin symmetric limit.

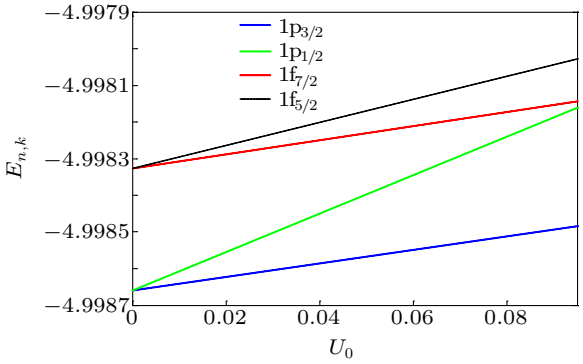


Fig. 3. (color online) Contributions of the tensor potential parameter to the energy level in the case of spin symmetry.

Table 2. Bound state energy eigenvalues in units of fm^{-1} of the spin symmetry Hulthén potential for several values of n and κ with $U_0 = 0.1$.

l	$n, \kappa < 0$	$(l, j = l + 1/2)$	$E_{n,\kappa<0}, U_0 \neq 0$	$E_{n,\kappa<0}, U_0 = 0$	$n, \kappa > 0$	$(l, j = l - 1/2)$	$E_{n,\kappa>0}, U_0 \neq 0$	$E_{n,\kappa>0}, U_0 = 0$
1	0, −2	$0\mathrm{p}_{3/2}$	−4.999038017	−4.999752463	0, 1	$0\mathrm{p}_{1/2}$	−4.999559444	−4.999752463
2	0, −3	$0\mathrm{d}_{5/2}$	−4.999445752	−4.999751647	0, 2	$0\mathrm{d}_{3/2}$	−4.999611783	−4.999751647
3	0, −4	$0\mathrm{f}_{7/2}$	−4.999559444	−4.999751236	0, 3	$0\mathrm{f}_{5/2}$	−4.999641705	−4.999751236
4	0, −5	$0\mathrm{g}_{9/2}$	−4.999611783	−4.999750989	0, 4	$0\mathrm{g}_{7/2}$	−4.999661028	−4.999750989
1	1, −2	$1\mathrm{p}_{3/2}$	−4.998475156	−4.998659482	1, 1	$1\mathrm{p}_{1/2}$	−4.998134589	−4.998659482
2	1, −3	$1\mathrm{d}_{5/2}$	−4.998252842	−4.998456496	1, 2	$1\mathrm{d}_{3/2}$	−4.998061278	−4.998456496
3	1, −4	$1\mathrm{f}_{7/2}$	−4.998134589	−4.998327074	1, 3	$1\mathrm{f}_{5/2}$	−4.998011397	−4.998327074
4	1, −5	$1\mathrm{g}_{9/2}$	−4.998061278	−4.998237543	1, 4	$1\mathrm{g}_{7/2}$	−4.997975269	−4.998237543

5. Conclusion

By using the Pekeris approximation, we have obtained bound state solutions of the Hulthén scalar–vector–tensor potentials under p-spin and spin symmetries with arbitrary spin–orbit coupling number κ. This work has improved on the work of Ref. [10] in which the authors used a different kind of tensor potential as Coulomb-like potential. However, here we study the problem with Hulthén tensor potential. We use a powerful mathematical tool, i.e. generalized parametric NU method, to solve the second-order differential equations. Also, we have presented some results in Figs. 2 and 3.

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