Optical field's quadrature excitation studied by new Hermite-polynomial operator identity*

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We study the optical field's quadrature excitation state $X^m|0\rangle$, where $X=(a+a^\dagger)/\sqrt{2}$ is the quadrature operator. We find it is ascribed to the Hermite-polynomial excitation state. For the first time, we determine this state's normalization constant which turns out to be a Laguerre polynomial. This is due to the integration method within the ordered product of operators (IWOP). The normalization for the two-mode quadrature excitation state is also completed by virtue of the entangled state representation.

Keywords: quadrature excitation, Hermite-polynomial excitation state, operator identities about Laguerre polynomials, IWOP method, entangled state representation

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1. Introduction

In recent years, quantum control has been a major topic in quantum information and communication. One of the controlling schemes is using photon adding or photon subtracting operation[1,2] on some well-known optical field states, where $a^\dagger$ and a are creation and annihilation operator, respectively, $[a,a^\dagger]=1$. In fact, the photon subtraction[3] and the addition[4] are now practically realized in the laboratory. Lee and Nha considered a coherent superposition of photon addition and subtraction, $fa+ga^\dagger, (|f|^2+|g|^2=1)$, acting on a coherent state and a thermal state to form non-Gaussian states,[5] and investigated the emerging non-classical properties via the Wigner function; Wang *et al.* studied the case when the coherent operation acts on a squeezed thermal state by examining its quadrature squeezing, sub-Poissonian statistics, and negativity of the Wigner function.[6] The coherent operation, when $f=g=1/\sqrt{2}$, leads to $(a+a^\dagger)^m/\sqrt{2^m}=X^m$, $X=(a+a^\dagger)/\sqrt{2}$ is the quadrature operator of the optical field. In this paper we shall construct the state $CX^m|0\rangle$, where C is the normalization constant to be determined. As will be shown in Section 2, $X^m|0\rangle$ actually is a Hermite-polynomial excitation. Thus in Section 3 we shall derive some useful identities about Hermite-polynomials of creation and annihilation operators. To further determine C, in Section 4 we derive the expansion of X^{2m} as a normally ordered associated Laguerre polynomial; its reciprocal relation is shown in Section 5. In Section 6 the normalization for two-mode quadrature excitation state is also completed by virtue of the entangled state representation. We shall use the technique of integration within an ordered product (IWOP) of operators[7,8] to fulfill our task — determining the normalization constant of the quadrature excitation state.

2. Expansion of X^n as a normally ordered Hermite-polynomial operator

Our departure point is considering the generating function formula of X^n, using the Baker–Hausdorff formula we have

$$\begin{aligned} \mathrm{e}^{\lambda X} &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}X^n \\ &= \mathrm{e}^{\lambda\left(a+a^\dagger\right)/\sqrt{2}} \\ &= :\mathrm{e}^{(-\mathrm{i}\lambda)\mathrm{i}X-(-\mathrm{i}\lambda/2)^2}: \\ &= :\sum_{n=0}^{\infty}\frac{(-\mathrm{i}\lambda/2)^n}{n!}\mathrm{H}_n\left(\mathrm{i}X\right):, \end{aligned} \tag{1}$$

where : : denotes normal ordering, and in the last step we have used the generating function formula of Hermite-polynomials

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_n\left(x\right) &= \mathrm{e}^{2tx-t^2}, \\ \mathrm{H}_n\left(x\right) &= \frac{\partial^n}{\partial t^n}\mathrm{e}^{2tx-t^2}|_{t=0}. \end{aligned} \tag{2}$$

Comparing the same power of λ of the two sides of Eq. (1) we have the normal ordering expansion of X^n

$$X^n=(2\mathrm{i})^{-n}:\mathrm{H}_n\left(\mathrm{i}X\right):. \tag{3}$$

Thus due to $a\left|0\right\rangle=0$, we obtain

$$X^m\left|0\right\rangle=(2\mathrm{i})^{-m}:\mathrm{H}_m\left(\mathrm{i}X\right):\left|0\right\rangle=(2\mathrm{i})^{-m}\mathrm{H}_m\left(\frac{\mathrm{i}a^\dagger}{\sqrt{2}}\right)\left|0\right\rangle, \tag{4}$$

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which is a Hermite-polynomial excitation on the vacuum state. In order to know the normalization constant we should calculate

$$\langle 0|X^m X^m|0\rangle = (4)^{-m}\langle 0|\mathrm{H}_m\left(\frac{-\mathrm{i}a}{\sqrt{2}}\right)\mathrm{H}_m\left(\frac{\mathrm{i}a^\dagger}{\sqrt{2}}\right)|0\rangle, \quad (5)$$

so we need to know how to transform

$$\mathrm{H}_m(-\mathrm{i}a/\sqrt{2})\mathrm{H}_m(\mathrm{i}a^\dagger/\sqrt{2})$$

into its normal ordering form.

3. Normal ordering for $\mathrm{H}_n(fa)\mathrm{H}_m(ga^\dagger)$

In order to put $\mathrm{H}_n(fa)\mathrm{H}_m(ga^\dagger)$ into its normal ordering, we consider its generating function

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n s^m}{n!m!}\mathrm{H}_n(fa)\mathrm{H}_m(ga^\dagger)\\
&= \mathrm{e}^{-t^2+2tfa}\mathrm{e}^{-s^2+2sga^\dagger}\\
&= \mathrm{e}^{4ftsg}:\mathrm{e}^{-t^2+2tfa-s^2+2sga^\dagger}:\\
&= \sum_{k=0}\frac{(4ftsg)^k}{k!}:\mathrm{e}^{-t^2+2tfa-s^2+2sga^\dagger}:. \qquad (6)
\end{aligned}$$

Because $a^\dagger$ and a are permutable within : :, and can be treated as c-number within : :, so

$$\begin{aligned}
G &= \sum_{k=0}\frac{1}{k!}:\left(\frac{\partial}{\partial a^\dagger}\right)^k\left(\frac{\partial}{\partial a}\right)^k\mathrm{e}^{-t^2+2tfa}\mathrm{e}^{-s^2+2sga^\dagger}:\\
&= \sum_{k=0}\frac{(fg)^k}{k!}:\left(\frac{\partial}{\partial(ga^\dagger)}\right)^k\left(\frac{\partial}{\partial(fa)}\right)^k\\
&\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n s^m}{n!m!}\mathrm{H}_m(ga^\dagger)\mathrm{H}_n(fa):. \qquad (7)
\end{aligned}$$

Then using the recurrence relation of Hermite polynomial

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l}\mathrm{H}_n(x) = \frac{2^l n!}{(n-l)!}\mathrm{H}_{n-l}(x), \quad (8)$$

equation (7) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n s^m}{n!m!}\mathrm{H}_n(fa)\mathrm{H}_m(ga^\dagger)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n s^m}{n!m!}\sum_{k=0}\frac{(4fg)^k}{k!}:\frac{n!m!}{(m-k)!(n-k)!}\\
&\quad\times\mathrm{H}_{m-k}(ga^\dagger)\mathrm{H}_{n-k}(fa):, \qquad (9)
\end{aligned}$$

so normal ordering for $\mathrm{H}_n(fa)\mathrm{H}_m(ga^\dagger)$ is

$$\begin{aligned}
&\mathrm{H}_n(fa)\mathrm{H}_m(ga^\dagger)\\
&= \sum_{k=0}^{\min(n,m)}\frac{(4fg)^k}{k!}\frac{n!m!}{(n-k)!(m-k)!}\\
&\quad\times\mathrm{H}_{m-k}(ga^\dagger)\mathrm{H}_{n-k}(fa), \qquad (10)
\end{aligned}$$

which is a new operator identity. Using Eq. (10) we know that equation (5) leads to

$$\begin{aligned}
&\langle 0|X^m X^m|0\rangle\\
&= (4)^{-m}\sum_{k=0}^{m}\frac{2^k}{k!}\frac{m!m!}{(m-k)!(m-k)!}\mathrm{H}_{m-k}^2(0). \qquad (11)
\end{aligned}$$

Due to $\mathrm{H}_{2n}(0) = (-1)^n 2^n(2n-1)!!$, $\mathrm{H}_{2n+1}(0) = 0$, we need to make further summation over k in Eq. (11).

4. Expansion of X^{2m} as a normally ordered associated Laguerre-polynomial operator

In order to complete the summation in Eq. (11). By noticing $\langle 0|X^m X^m|0\rangle = \langle 0|X^{2m}|0\rangle$, we consider its generating function $\mathrm{e}^{\lambda X^2}$, using the normally ordered Gaussian form of the coordinate representation's completeness,

$$\int_{-\infty}^{\infty}\mathrm{d}x|x\rangle\langle x| =: \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}\mathrm{e}^{-(x-X)^2}:=1 \quad (12)$$

and the IWOP method we have the normally ordered expansion formula

$$\begin{aligned}
\mathrm{e}^{\lambda X^2} &= \int_{-\infty}^{\infty}\mathrm{d}x\mathrm{e}^{\lambda x^2}|x\rangle\langle x|\\
&= :\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}\mathrm{e}^{\lambda x^2}\mathrm{e}^{-(x-X)^2}:\\
&= (1-\lambda)^{-1/2}:\exp\left[\frac{-\lambda X^2}{\lambda-1}\right]:. \qquad (13)
\end{aligned}$$

Comparing with the generating function of the associated Laguerre polynomials

$$(1-t)^{-\alpha-1}\exp\left[\frac{xt}{t-1}\right] = \sum_{n=0}^{\infty}\mathrm{L}_n^{\alpha}(x)t^n, \quad (14)$$

we see when $\alpha = -1/2$,

$$\begin{aligned}
:\sum_{n=0}^{\infty}\mathrm{L}_n^{-1/2}(-X^2)\lambda^n: &= (1-\lambda)^{-1/2}:\exp\left[\frac{-\lambda X^2}{\lambda-1}\right]:\\
&= \mathrm{e}^{\lambda X^2}, \qquad (15)
\end{aligned}$$

which is a new operator identity. Then comparing with

$$\mathrm{e}^{\lambda X^2} = \sum_{m=0}^{\infty}\frac{\lambda^m}{m!}X^{2m},$$

we see

$$X^{2m} = m!:\mathrm{L}_m^{-1/2}(-X^2):, \quad (16)$$

and

$$\langle 0|X^{2m}|0\rangle = m!\langle 0|:\mathrm{L}_m^{-1/2}(-X^2):|0\rangle = m!\mathrm{L}_m^{-1/2}(0). \quad (17)$$

Comparing with Eq. (11) we obtain a new formula

$$(4)^{-m}\sum_{k=0}^{m}\frac{2^k}{k!}\frac{m!}{(m-k)!(m-k)!}\mathrm{H}_{m-k}^2(0) = \mathrm{L}_m^{-1/2}(0), \quad (18)$$

so the normalized quadrature excitation state is

$$\frac{1}{\sqrt{m!\mathrm{L}_m^{-1/2}(0)}}X^m|0\rangle, \tag{19}$$

where

$$\mathrm{L}_m^{\alpha}(0)=\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)}, \tag{20}$$

Γ is the Gamma function.

5. The reciprocal relation of Eq. (16)

We can also derive the reciprocal relation of Eq. (16). In reference to $X^{2n}=(2\mathrm{i})^{-2n}:\mathrm{H}_{2n}(\mathrm{i}X):$, we obtain the relation connecting Hermite polynomial and Laguerre polynomial

$$\mathrm{H}_{2n}(\mathrm{i}X)=(-1)^n 2^{2n}n!\mathrm{L}_n^{-1/2}(-X^2), \tag{21}$$

this still holds for $\mathrm{i}X\to x$, i.e.,

$$\mathrm{H}_{2n}(x)=(-1)^n 2^{2n}n!\mathrm{L}_n^{-1/2}(x^2). \tag{22}$$

Since $\mathrm{H}_{2n}(X)=2^{2n}:X^{2n}:$, we also have

$$:X^{2n}:=(-1)^n n!\mathrm{L}_n^{-1/2}(X^2), \tag{23}$$

and equation (23) is the reciprocal relation of Eq. (16). Further, from Eqs. (13) and (14) we see

$$\begin{aligned}&(1-t)^{-\alpha-1}\exp\left[\frac{X^2t}{t-1}\right]\\ &=(1-t)^{-\alpha-1/2}:\exp[-tX^2]:\\ &=\sum_{n=0}^{\infty}\mathrm{L}_n^{\alpha}(X^2)t^n.\end{aligned} \tag{24}$$

It then follows that

$$\begin{aligned}&:(-X^2)^n:\\ &=\frac{\partial^n}{\partial t^n}:\exp[-tX^2]:|_{t=0}\\ &=:\sum_{m=0}^{\infty}\mathrm{L}_m^{\alpha}(X^2)\frac{\partial^n}{\partial t^n}\left[t^m(1-t)^{\alpha+1/2}\right]|_{t=0}:\\ &=:\sum_{m=0}^{\infty}\mathrm{L}_m^{\alpha}(X^2)\sum_l\binom{n}{l}\left(\frac{\partial^{n-l}}{\partial t^{n-l}}t^m\right)\frac{\partial^l}{\partial t^l}(1-t)^{\alpha+1/2}|_{t=0}:\\ &=\sum_{m=0}^{n}\mathrm{L}_m^{\alpha}(X^2)\binom{n}{n-m}(-1)^{n-m}\binom{\alpha+1/2}{n-m}(n-m)!.\end{aligned} \tag{25}$$

In particular, taking $\alpha=1/2$ in Eq. (24) or in Eq. (25), we have

$$\begin{aligned}:\exp[-tX^2]:&=:\sum_{n=0}^{\infty}\frac{t^n}{n!}(-X^2)^n:\\ &=\sum_{n=0}^{\infty}\mathrm{L}_n^{1/2}(X^2)t^n(1-t).\end{aligned} \tag{26}$$

Comparing the term of t^n on both sides of Eq. (26) we have

$$:(-X^2)^n:=n!\left[\mathrm{L}_n^{1/2}(X^2)-\mathrm{L}_{n-1}^{1/2}(X^2)\right]. \tag{27}$$

Combining Eq. (27) with Eq. (23) we derive

$$\mathrm{L}_n^{1/2}(X^2)-\mathrm{L}_{n-1}^{1/2}(X^2)=\mathrm{L}_n^{-1/2}(X^2), \tag{28}$$

which coincides with the corresponding formula in Ref. [9].

6. Two-mode case

In the two-mode case, the field quadrature operator is $(X+Y)/\sqrt{2}, Y=(b+b^{\dagger})/\sqrt{2}$, $[b,b^{\dagger}]=1$. Now we consider the two-mode quadrature excitation state

$$\left(\frac{X+Y}{\sqrt{2}}\right)^m|00\rangle. \tag{29}$$

Using the entangled state representation[10–13]

$$\exp\left(-|\xi|^2/2+a_1^{\dagger}\xi+a_2^{\dagger}\xi^*-a_1^{\dagger}a_2^{\dagger}\right)|00\rangle\equiv|\xi\rangle, \tag{30}$$

which satisfies

$$\left(a+b^{\dagger}\right)|\xi\rangle=\xi|\xi\rangle,\quad\left(a^{\dagger}+b\right)|\xi\rangle=\xi^*|\xi\rangle \tag{31}$$

and

$$\begin{aligned}&(X+Y)|\xi\rangle=\sqrt{2}\xi_1|\xi\rangle,\ (P_x-P_y)|\xi\rangle=\sqrt{2}\xi_2|\xi\rangle,\\ &\xi=\xi_1+\mathrm{i}\xi_2,\end{aligned} \tag{32}$$

and the IWOP method as well as

$$|00\rangle\langle 00|=:\exp\{-a_1^{\dagger}a_1-a_2^{\dagger}a_2\}:, \tag{33}$$

we have

$$\begin{aligned}&\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi|=\int\frac{\mathrm{d}^2\xi}{\pi}:\mathrm{e}^{-|\xi-a_1-a_2^{\dagger}|^2}:=1,\\ &\mathrm{d}^2\xi\equiv\mathrm{d}\xi_1\mathrm{d}\xi_2,\end{aligned} \tag{34}$$

or

$$\begin{aligned}&\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi|\\ &=\int\frac{\mathrm{d}^2\xi}{\pi}:\mathrm{e}^{-\left(\xi_1-(X+Y)/\sqrt{2}\right)^2-\left(\xi_2-(P_x-P_y)/\sqrt{2}\right)^2}:\\ &=1,\end{aligned} \tag{35}$$

where $P_x=(a-a^{\dagger})/(\sqrt{2}\mathrm{i})$, $P_y=(b-b^{\dagger})/(\sqrt{2}\mathrm{i})$, so

$$\begin{aligned}&\mathrm{e}^{\lambda(X+Y)^2}\\ &=\int\frac{\mathrm{d}^2\xi}{\pi}\mathrm{e}^{2\lambda\xi_1^2}|\xi\rangle\langle\xi|\\ &=\int\frac{\mathrm{d}^2\xi}{\pi}\mathrm{e}^{2\lambda\xi_1^2}:\mathrm{e}^{-\left(\xi_1-(X+Y)/\sqrt{2}\right)^2-\left(\xi_2-(P_x-P_y)/\sqrt{2}\right)^2}:\\ &=\int\frac{\mathrm{d}\xi_1}{\sqrt{\pi}}\mathrm{e}^{2\lambda\xi_1^2}:\mathrm{e}^{-\left(\xi_1-(X+Y)/\sqrt{2}\right)^2}:\\ &=(1-2\lambda)^{-1/2}:\exp\left[\frac{-\lambda(X+Y)^2}{2\lambda-1}\right]:.\end{aligned} \tag{36}$$

Using Eq. (14) we see when $\alpha = -1/2$,

$$:\sum_{n=0}^{\infty} \mathrm{L}_n^{-1/2}\left[-\left(\frac{X+Y}{\sqrt{2}}\right)^2\right](2\lambda)^n := \mathrm{e}^{\lambda(X+Y)^2}, \tag{37}$$

which is a new operator identity. Then comparing with

$$\mathrm{e}^{2\lambda\left((X+Y)/\sqrt{2}\right)^2} = \sum_{m=0}^{\infty} \frac{(2\lambda)^m}{m!}\left(\frac{X+Y}{\sqrt{2}}\right)^{2m},$$

we obtain

$$\left(\frac{X+Y}{\sqrt{2}}\right)^{2m} = m! :\mathrm{L}_m^{-1/2}\left[-\left(\frac{X+Y}{\sqrt{2}}\right)^2\right]:, \tag{38}$$

and

$$\begin{aligned}\langle 00|\left(\frac{X+Y}{\sqrt{2}}\right)^{2m}|00\rangle &= \langle 00|: m!\mathrm{L}_m^{-1/2}\left[-\left(\frac{X+Y}{\sqrt{2}}\right)^2\right]: |00\rangle \\ &= m!\mathrm{L}_m^{-1/2}(0),\end{aligned} \tag{39}$$

so the normalization for the state in Eq. (29) is still $1\Big/\sqrt{m!\mathrm{L}_m^{-1/2}(0)}$.

In summary, we have introduced the quadrature excitation state $X^m|0\rangle$ which turns out to be a Hermite-polynomial excitation state, and derived its normalization constant $1\Big/\sqrt{m!\mathrm{L}_m^{-1/2}(0)}$, which is concise. The extension to two-mode quadrature excitation is also obtained by virtue of the entangled state representation. As we have mentioned in the introduction section, the quadrature excitation of a vacuum state is the special case of the coherent operation proposed by Su-Yong Lee and Hyunchul Nha[5] who proposed an experimental realization conditioned on photodetections using a parametric down-converter with two beamsplitters to realize the coherent control. Finally, we mention that the result in this paper can also help to derive the normalization of the quadrature excitation of the coherent state.

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