Casson fluid flow and heat transfer over a nonlinearly stretching surface*

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A boundary layer analysis is presented for non-Newtonian fluid flow and heat transfer over a nonlinearly stretching surface. The Casson fluid model is used to characterize the non-Newtonian fluid behavior. By using suitable transformations, the governing partial differential equations corresponding to the momentum and energy equations are converted into non-linear ordinary differential equations. Numerical solutions of these equations are obtained with the shooting method. The effect of increasing Casson parameter is to suppress the velocity field. However the temperature is enhanced with the increasing Casson parameter.

Keywords: nonlinear stretching, Casson fluid, heat transfer, similarity transformations

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1. Introduction

The boundary layer flow over a stretching surface is often encountered in some engineering disciplines. In particular, the flow induced by a stretching boundary is important in the extrusion processes in plastic and metal industries.[1] During a manufacturing process, a stretched sheet interacts with the ambient fluid both thermally and mechanically.[2] Therefore, the flow behavior over a stretching surface, which determines the rate of cooling, is important.[3,4] Moreover, flow and heat transfer phenomena over a stretching surface have promising applications in a number of technological processes including the production of polymer films or thin sheets.[5,6] Crane[7] investigated the steady flow of an incompressible viscous fluid over an elastic sheet, where the flow was caused solely by the stretching of the sheet in its own plane with a velocity varying linearly with the distance from a fixed point. Heat transfer in the flow over a stretching surface maintained at constant as well as at variable temperatures was investigated by Carragher and Crane,[8] and Gupta and Gupta.[9] The work of Crane was subsequently extended by some authors to Newtonian/non-Newtonian boundary layer flows with various velocities and thermal boundary conditions.[10] The analysis of heat transfer due to a continuously stretching surface through an ambient fluid is one of the thrust areas of current research.[11] Ali[12] obtained the similarity solutions of the laminar boundary layer equations describing heat and flow in a quiescent fluid driven by a stretched surface subject to suction or injection. Mukhopadhyay and Andersson[13] analyzed the slip effects on the forced convective flow due to an unsteady stretching surface. Mukhopadhyay[14] reported the effects of partial slip on the mixed convective flow passing an unsteady stretching surface. Some important aspects of the magnetohydrodynamic (MHD) flow over a stretching/shrinking sheet can be found in Refs. [15]–[19]. Moreover, the heat transfer analysis plays a vital role in handling and processing of non-Newtonian fluids. An assessment of technological applications requires the knowledge of the rheological characteristics of the non-Newtonian fluids.[20]

Recently, due to increasing industrial and technological applications, the flows of non-Newtonian fluids attract much attention of researchers. For example, if one uses a non-Newtonian fluid as the coolant or heat exchangers, the required pumping power may be greatly reduced. Therefore, the fundamental analysis of the flow field of non-Newtonian fluids in a boundary layer adjacent to a stretching sheet or an extended surface is very important, and is an essential part in the study of fluid dynamics and heat transfer.[21–23] Some materials, eg., melts, muds, condensed milk, glues, printing ink, emulsions, soaps, sugar solution, paints, shampoos, tomato paste, show the non-Newtonian properties of fluids. The features of non-Newtonian fluids are different from those of viscous fluids. The governing equations of non-Newtonian fluids are highly non-linear and much more complicated compared to those of Newtonian fluids. Due to complexity, no single constitutive equation exhibiting all properties of such fluids is available.[24,25] So, several models of non-Newtonian fluids have been suggested in the literature through the differential, integral, and rate type categories. In the literature, the vast majority of non-Newtonian fluid models are concerned with simple models like the power law and grade two or three.[26–30] The simplest model for the rheological effects of viscoelastic fluids is the Maxwell model, where the dimensionless relaxation time is small, and it can predict the stress relaxation.

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Some recent investigations on the Maxwell fluids are given in Refs. [31] and [32]. There is another non-Newtonian fluid model known as the Casson fluid model. In the literature, it is sometimes claimed that for many materials, the Casson model is better than the general viscoplastic model in fitting the rheological data. So it becomes the preferred rheological model for blood and chocolate. A Casson fluid exhibits a yield stress. If a shear stress less than the yield stress is applied to the fluid, it behaves like a solid, whereas if a shear stress greater than the yield stress is applied, it starts to move. Representative studies dealing with this model can be found in Refs. [33]– [36].

In most of the above mentioned papers, the physical situations were related to a linearly stretching sheet. But the stretching may not necessarily be linear, it may be quadratic, power-law, exponential, and so on. Some physical situations arise in which the need of a nonlinearly stretching sheet should be faced.[37,38] Hence, the flow and heat transfer caused by a nonlinearly stretching sheet is of paramount interest. Bataller[39] obtained the similarity solutions for flow and heat transfer over a nonlinearly stretching sheet by considering both uniform and variable surface temperatures. Being motivated by the possible industrial applications and the previous studies regarding the flow and heat transfer of non-Newtonian fluids over the stretching sheet, we present a numerical solution for flow and heat transfer of a laminar boundary layer flow of the Casson fluid. Using suitable transformations, we derive a third-order ordinary differential equation corresponding to the momentum equation and a second-order differential equation corresponding to the energy equation. These equations are then solved numerically with the help of the shooting method. The exact solution of the transformed momentum for the linearly stretching sheet is obtained. Numerical calculations are carried out for different values of the dimensionless parameters used in this problem for illustrating the results graphically. The analysis of the results shows that the flow field is appreciably influenced by the governing parameters. The estimation of the skin friction, which is very important for industrial applications, is also presented in this analysis. It is expected that the results obtained will not only provide useful information for applications, but also serve as a complement to the previous studies.

2. Problem formulation

Consider the flow of an incompressible viscous fluid passing a flat sheet coinciding with plane $y = 0$. The flow is confined to $y > 0$. Two equal and opposite forces are applied along the x axis so that the wall is stretched with the origin fixed. The rheological equation of state for an isotropic and incompressible flow of the Casson fluid is

$$\tau_{ij} = \begin{cases} 2(\mu_{\rm B} + p_{\rm y}/\sqrt{2\pi})e_{ij}, & \pi > \pi_{\rm c}, \\ 2(\mu_{\rm B} + p_{\rm y}/\sqrt{2\pi_{\rm c}})e_{ij}, & \pi < \pi_{\rm c}, \end{cases}$$

where π is the product of the component of deformation rate with itself, $\pi = e_{ij}e_{ij}$, e_{ij} is the (i, j)-th component of the deformation rate, $\pi_{\rm c}$ is a critical value of this product based on the non-Newtonian model, $\mu_{\rm B}$ is the plastic dynamic viscosity of the non-Newtonian fluid, and $p_{\rm y}$ is the yield stress of the fluid.

The continuity, momentum, and energy equations governing such type of flow are given as

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = \nu\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x} + \upsilon\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where u and υ are the velocity components in the x and y directions, respectively, ν is the kinematic viscosity, ρ is the fluid density, $\beta = \mu_{\rm B}\sqrt{2\pi_{\rm c}}/p_{\rm y}$ is the parameter of the Casson fluid, and κ is the thermal diffusivity of the fluid.

The appropriate boundary conditions for the problem are given by

$$u = U = cx^n, \quad \upsilon = 0, \quad T = T_{\rm w}, \ \text{at} \ y = 0, \tag{4}$$

$$u \to 0, \quad T \to T_\infty, \ \text{as} \ y \to \infty. \tag{5}$$

Here, c ($c > 0$) is a parameter related to the surface stretching speed, $T_{\rm w}$ is the uniform temperature at the sheet, T_∞ is the free stream temperature, and n is the power index related to the surface stretching speed.

Introducing the following transformations:

$$\eta = y\sqrt{\frac{c(n+1)}{2\nu}}x^{\frac{n-1}{2}}, \quad u = cx^n f'(\eta),$$

$$\upsilon = -\sqrt{c\nu\left(\frac{n+1}{2}\right)}x^{\frac{n-1}{2}}\left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right],$$

$$\frac{T - T_\infty}{T_{\rm w} - T_\infty} = \theta(\eta), \tag{6}$$

and substituting Eq. (6) in Eqs. (2) and (3), we obtain the reduced governing equations

$$\left(1 + \frac{1}{\beta}\right)f''' + ff'' - \frac{2n}{n+1}f'^2 = 0, \tag{7}$$

$$\theta'' + Prf\theta' = 0. \tag{8}$$

The boundary conditions take the following form:

$$f' = 1, \quad f = 0, \quad \theta = 1, \ \text{at} \ \eta = 0, \tag{9}$$

$$f' \to 0, \quad \theta \to 0, \ \text{as} \ \eta \to \infty, \tag{10}$$

where the prime denotes the differentiation with respect to η, and $Pr = \nu/\kappa$ is the Prandtl number.

The exact solution of Eq. (7) for the linearly stretching sheet, i.e., for $n = 1$, is given by

$$f(\eta) = \sqrt{1+\frac{1}{\beta}}\left(1-\exp\left(-\frac{\eta}{\sqrt{1+1/\beta}}\right)\right).$$

As there is no exact solution for the nonlinearly stretching sheet, we have to proceed with numerical techniques.

3. Numerical technique

Equations (7) and (8) are the third- and the second-order non-linear boundary value problems, respectively. So we develop the most effective numerical shooting technique with the fourth-order Runge–Kutta integration algorithm.[40–43] In order to integrate Eqs. (7) and (8) as the initial value problem, one requires values for $f''(0)$ and $\theta'(0)$, but no such values are given at the boundary. Suitable guess values for $f''(0)$ and $\theta'(0)$ are chosen, and then the integration is carried out. The most important factor in the shooting method is to choose an appropriate finite value of η_∞. In order to determine η_∞ for the boundary value problem, we start with some initial guess values for some particular sets of physical parameters to obtain $f''(0)$ and $\theta'(0)$. The solving procedure is repeated with another larger value of η_∞ until two successive values of $f''(0)$ and $\theta'(0)$ differ only by the specified significant digit. The last value of η_∞ is chosen to be the most appropriate value of η_∞ for that particular set of parameters. The value of η_∞ may be different for another set of physical parameters. Once the finite value of η_∞ is determined, the integration is carried out. We compare the calculated values for $f'$and θ at $\eta = \eta_\infty(= 10)$ (say) with the given boundary conditions $f'(10) = 0$ and $\theta(10) = 0$ and adjust the estimated values $f''(0)$ and $\theta'(0)$ to give a better approximation to the solution. We take the series of values for $f''(0)$ and $\theta'(0)$ and apply the fourth-order classical Runge–Kutta method with step-size $h = 0.01$. The above procedure is repeated until we get the converged results within a tolerance limit of 10^{-5}.

4. Discussion

We first verify the accuracy of the applied numerical scheme. Comparisons with Vajravelu's results[37] for skin friction coefficient $f''(0)$ of the viscous Newtonian fluid and heat transfer coefficient $\theta'(0)$ are presented in Table 1 for various values of nonlinear stretching parameter n and Prandtl number Pr. An excellent agreement is found. The results of $f''(0)$ corresponding to the Newtonian fluid also agree with Cortell's results.[44] A comparison of the results of stream function profiles $f(\eta)$ and velocity profiles $f'(\eta)$ with the exact results for the linearly stretching sheet ($n = 1$) is presented in Fig. 1, and an excellent agreement is found.

Table 1. Comparison of $f''(0)$ and $\theta'(0)$ for viscous Newtonian fluid.

n	Vajravelu[36]			Cortell[44]	Present study		
	$f''(0)$	$\theta'(0)$		$f''(0)$	$f''(0)$	$\theta'(0)$	
		$Pr = 0.72$	$Pr = 7.0$			$Pr = 0.72$	$Pr = 7.0$
1	–1.0000	–0.4590	–1.8953	–1.000000	–1.0000	–0.4591	–1.8953
5	–1.1945	–0.4394	–1.8610	–	–1.1944	–0.4394	–1.8611
10	–1.2348	–0.4357	–1.8541	–1.234875	–1.2348	–0.4358	–1.8541

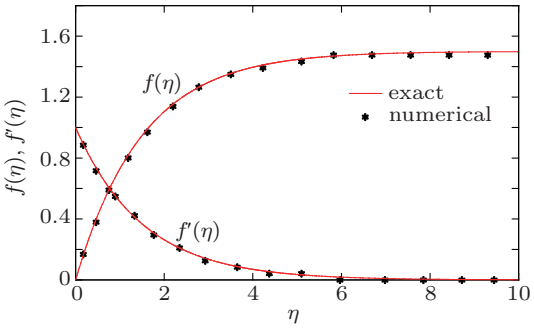


Fig. 1. (color online) Stream function $f(\eta)$ and velocity $f'(\eta)$ for linearly stretching sheet.

In order to analyze the results, numerical computations have been carried out using the method described in the previous section for various values of Casson parameter β, nonlinear stretching parameter n, and Prandtl number Pr. For illustrations of the results, the numerical results are plotted in Figs. 2–6.

Figures 2(a) and 2(b) depict the velocity and the temperature profiles for different nonlinearly stretching parameter n. The fluid velocity is found to decrease with increasing n, while the temperature increases in this case. The effect of nonlinear stretching parameter n is significant when n is low. But the decrease of the velocity profile is negligible for a large n since the coefficient $2n/(n+1)$ in Eq. (7) approaches 2 as $n \to \infty$.[37]

Let us now concentrate on the effects of Casson parameter β on the velocity and the temperature profiles. Figures 3(a) and 3(b) display the effects of Casson parameter β on the velocity and the temperature profiles for a nonlinearly stretching sheet ($n = 10$). The velocity is found to decrease with increasing Casson parameter β, whereas the temperature increases in this case. The decreasing nature of the momentum boundary layer thickness with increasing β is noted. However the thermal boundary layer thickness increases with increasing β.

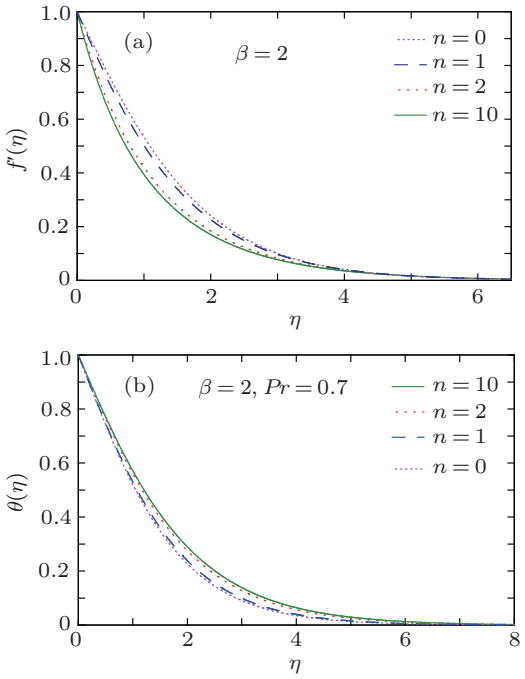


Fig. 2. (color online) Variations of (a) velocity $f'(\eta)$ and (b) temperature $\theta(\eta)$ with η for several values of power index n.

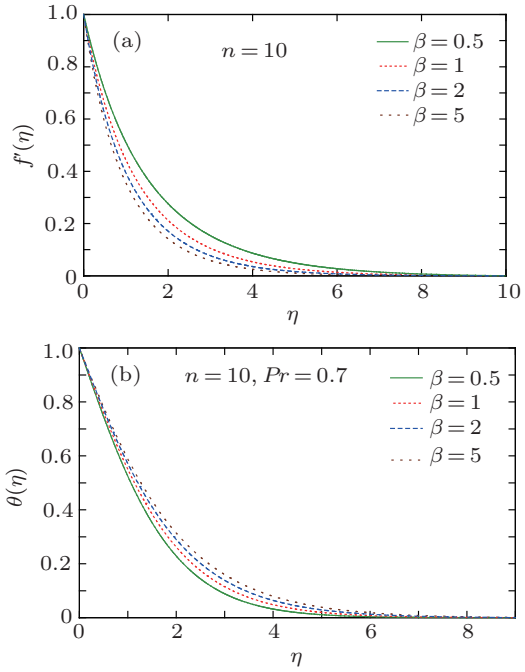


Fig. 3. (color online) Variations of (a) velocity $f'(\eta)$ and (b) temperature $\theta(\eta)$ with η for several values of Casson parameter β.

Figures 4(a) and 4(b) exhibit the velocity and the temperature profiles for variable values of Casson parameter β for linearly/nonlinearly stretching sheets. The fluid velocity is found to decrease with increasing Casson parameter β for both cases, whereas the temperature at a fixed point increases with increasing β. The fluid velocity in the case of nonlinearly stretching sheet ($n = 10$) is much more suppressed than that of linearly stretching sheet ($n = 1$) (Fig. 4(a)). On the other hand, the increase of the temperature field in the case of nonlinearly stretching sheet ($n = 10$) is much more pronounced than that of linearly stretching sheet ($n = 1$) (Fig. 4(b)), i.e., at a fixed point of the sheet, the temperature in the nonlinearly stretching sheet is higher than that in the linearly stretching sheet.

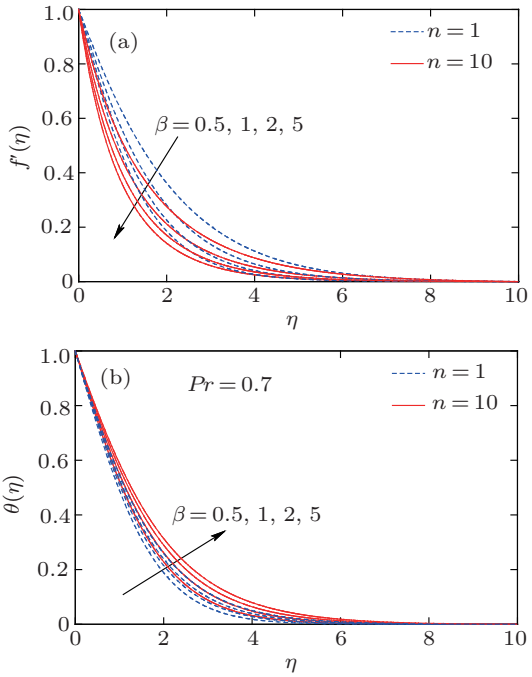


Fig. 4. (color online) Variations of (a) velocity $f'(\eta)$ and (b) temperature $\theta(\eta)$ with η for several values of Casson parameter β for linearly/nonlinearly stretching sheets.

The effects of Prandtl number can be found from Fig. 5 for linearly/nonlinearly stretching sheets. The temperature decreases when the Prandtl number increases for both linearly and nonlinearly stretching sheets. An increase in the Prandtl number reduces the thermal boundary layer thickness. The Prandtl number signifies the ratio of momentum diffusivity to thermal diffusivity. In heat transfer problems, the Prandtl number controls the relative thickening of the momentum and the thermal boundary layers. When the Prandtl number is small, heat diffuses quickly compared to the velocity (momentum), which means that for liquid metals, the thermal boundary layer is much thicker than the momentum boundary layer. Fluids with lower Prandtl numbers have higher thermal conductivities (and thicker thermal boundary layers) so that heat can diffuse from the sheet faster than that for higher Pr fluids (thinner boundary layers). Hence the Prandtl number can be used to increase the rate of cooling. Moreover, the thermal boundary layer thickness decreases with increasing Prandtl number, as shown in Fig. 5.

Figure 6(a) exhibits the variation of the skin-friction coefficient with Casson parameter β for two values of n. It is found that the skin-friction coefficient increases with increasing β and the magnitude of it is higher for nonlinearly stretching sheet than that for linearly stretching sheet. From this figure, it is very clear that the shear stress at the wall is negative. Physically, a negative sign implies that the surface exerts a dragging force on the fluid, and a positive sign implies the

opposite. Figure 6(b) displays the variation of heat transfer coefficient $\theta'(0)$ with Casson parameter β. The increase of Casson parameter β leads to the increase of the heat transfer coefficient. Wall temperature gradient $\theta'(0)$ increases with increasing n (Fig. 6(b)), i.e., more heat is carried out from the sheet, resulting in a decrease of the thermal boundary layer thickness, hence the heat transfer rate increases. Negative values of $(\theta'(0))$ indicate that heat flows from the sheet to the ambient fluid.

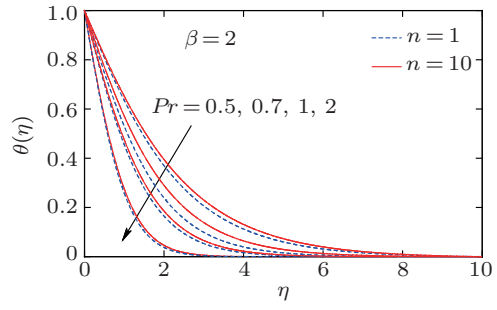


Fig. 5. (color online) Variation of temperature $\theta(\eta)$ with η for several values of Prandtl number Pr for linearly/nonlinearly stretching sheets.

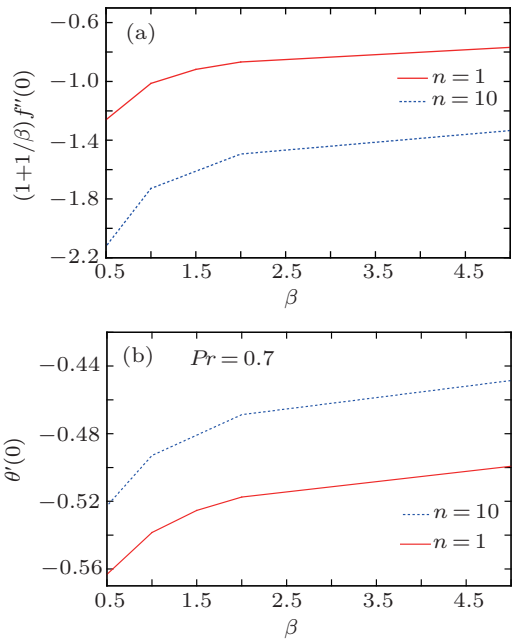


Fig. 6. (color online) (a) Skin-friction coefficient and (b) wall temperature gradient $\theta'(0)$ versus Casson parameter β for linearly/nonlinearly stretching sheets.

5. Conclusions

Numerical solutions for the steady boundary layer Casson fluid flow and heat transfer passing a nonlinearly stretching surface are obtained. The following conclusions can be drawn from the results based on the numerical computations. (i) The effect of nonlinear stretching parameter is to suppress the velocity field, which in turn causes the enhancement of the skin-friction coefficient. (ii) The temperature increases with increasing nonlinearly stretching parameter. (iii) The momentum boundary layer thickness decreases with increasing Casson parameter, while the thermal boundary layer thickness increases in this case. (iv) The surface shear stress increases as the Casson parameter increases. (v) For all values of nonlinearly stretching parameter, heat flows from the sheet to the ambient fluid. (vi) With increasing Pr, the temperature decreases, and an increase in the Prandtl number reduces the thermal boundary layer thickness. The results may be helpful for possible technological applications in liquid-based systems involving stretchable materials.

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