Uniqueness, reciprocity theorem, and plane waves in thermoelastic diffusion with a fractional order derivative*

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(Received 13 December 2012; revised manuscript received 18 January 2013)

In this work, a theory of thermoelasticity with diffusion is taken into consideration by using the methodology of fractional calculus. The governing equations for particle motion in a homogeneous anisotropic fractional order generalized thermoelastic diffusive medium are presented. Uniqueness and reciprocity theorems are proved. The plane wave propagation in the homogeneous transversely isotropic thermoelastic diffusive medium with fractional order derivative is studied. For the two-dimensional problem, there exist a quasi-longitudinal wave, a quasi-transverse wave, a quasi-mass diffusion wave, and a quasi-thermal wave. From the obtained results, the different characteristics of waves, like phase velocity, attenuation coefficient, specific loss, and penetration depth, are computed numerically and presented graphically. Some special cases are also discussed.

Keywords: thermoelastic diffusion, fractional calculus, uniqueness theorem, reciprocity theorem

PACS: 46.70.–p, 46.15.–x, 46.05.+b **DOI:** 10.1088/1674-1056/22/7/074601

1. Introduction

Diffusion is defined as the spontaneous movement of particles from a high concentration region to a low concentration one, and it occurs in response to a concentration gradient expressed as the change in the concentration due to the change in position. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish the isotope separation. Today, thermal diffusion remains a practical approach to separate isotopes of noble gases (e.g., xexon) and other light isotopes (e.g., carbon) for research purposes.

Nowacki[1–3] developed the theory of thermoelastic diffusion using the coupled thermoelastic model. Nowacki[4] derived the basic equations for the generalized thermoelastic diffusion. Sherief and Saleh[5] developed the generalized theory of thermoelastic diffusion with one relaxation time, which allows finite speeds of propagation of waves. With the diffusion effects considered, Kumar and Kansal[6] derived the basic equations for the generalized thermoelastic diffusion (G-L model) and discussed the Lamb waves. Kumar and Chawla[7] investigated the fundamental solution in an orthotropic thermoelastic diffusion material. However, the important Green's function for the two-dimensional problem with a steady point heat source in an orthotropic thermoelastic diffusion material has not been discussed so far.

During recent years, several interesting models have been developed by using fractional calculus to study the physical processes, particularly in the areas of heat conduction, diffusion, viscoelasticity, mechanics of solids, control theory, electricity, etc. It has been realized that the use of fractional order derivatives and integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There exists many materials and physical situations like amorphous media, colloids, glassy and porous materials, manmade and biological materials/polymers, transient loading, etc., for which the classical thermoelasticity based on the Fourier type heat conduction breaks down. In such cases, one needs to use a generalized thermoelasticity theory based on an anamolous heat conduction model involving time fractional (non integer order) derivatives.

Povstenko[8] proposed a quasi-static uncoupled theory of thermoelasticity based on the heat conduction equation with a time-fractional derivative of order α. Because the heat conduction equation in the case $1 \leq \alpha \leq 2$ interpolates the parabolic equation $\alpha = 1$ and the wave equation $\alpha = 2$, that theory interpolates a classical thermoelasticity and a thermoelasticity without energy dissipation. He also obtained the stresses corresponding to the fundamental solutions of a Cauchy problem for the fractional heat conduction equation in one-dimensional and two-dimensional cases.

Povstenko[9] investigated the nonlocal generalizations of the Fourier law and heat conduction by using time and space fractional derivatives. Youssef[10] proposed a new model of thermoelasticity theory based on a new consideration of heat conduction with fractional order and proved the uniqueness theorem. Jiang and Xu[11] obtained a fractional heat conduction equation with a time fractional derivative in the general orthogonal curvilinear coordinate and also in other orthogonal coordinate systems. Povstenko[12] investigated the fractional

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radial heat conduction in an infinite medium with a cylindrical cavity and the associated thermal stresses.

Ezzat[13] constructed a new model of the magneto-thermoelasticity theory based on a new consideration of the heat conduction equation by using the Taylor series expansion of time fractional order developed by Jumarie[14] as

$$\left[\boldsymbol{q}+\frac{\tau_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha}\boldsymbol{q}}{\partial t^{\alpha}}\right]=-K\boldsymbol{\nabla}T.$$

Ezzat[15] studied the state space approach to thermoelectric fluid with fractional order heat transfer. The Laplace transform and the state-space techniques were used to solve a one-dimensional application for a conducting half space of thermoelectric elastic material. Povstenko[16] investigated the generalized Cattaneo-type equations with time fractional derivatives and formulated the theory of thermal stresses. Biswas and Sen[17] proposed a scheme for the optimal control and a pseudo state space representation for a particular type of fractional dynamical equation. Jiang and Qi[18] investigated a thermal wave model of bioheat transfer with the modified Riemann–Liouville fractional derivative.

Uniqueness, reciprocity theorems, and variational principle in the theory of thermoelastic materials with voids were established by Lesan.[19] Uniqueness and reciprocity theorems for the equations of the generalized thermoleastic diffusion problem in isotropic media were proved by Sherief *et al.*[20] on the basis of the variational principle equation. Aouadi[21,22] proved the uniqueness and the reciprocity theorems for the equations of the generalized thermoleastic diffusion problem in both isotropic and anisotropic media by using the Laplace transformation method. Sherief *et al.*[23] introduced a new model of thermoelasticity using fractional calculus, proved a uniqueness theorem, and derived a reciprocity relation and a variational principle. El-Karamany and Ezzat[24] introduced two models in which the fractional derivatives and integrals were used to modify the Cattaneo heat conduction law,[25] and proved the uniqueness and reciprocity theorems in the context of thermoelasticity theory, the convolution principle was given and used to prove a uniqueness theorem with no restrictions imposed on the elasticity or thermal conductivity tensors except the symmetry conditions.

The elastic properties of spherically anisotropic piezoelectric composites were studied by Poon *et al.*[26] Sharma[27] studied the existence of longitudinal and transverse waves in anisotropic thermoelastic media. The two-dimensional (2D) problem of a generalized thermoelastic diffusion material having a thermal source in a transversely isotropic thermoelastic half-space was studied by Abbas *et al.*[28] Abbas and Othman[29] studied a thermal shock problem in an isotropic hollow cylinder having temperature dependent elastic moduli. Lotfy[30] introduced a 2D problem of fibre-reinforced thermoelasticity under the effect of a magnetic field. The effect of rotation on plane waves in a generalized thermo-microstretch elastic solid for a mode-I crack problem was studied by Lotfy and Othman.

In this paper, we attempt to combine the previous results with the fractional order theory of generalized thermoelasticity with diffusion. Uniqueness and reciprocity theorems are proved. The plane wave propagation in a transversely isotropic thermoelastic diffusive medium is studied. Some special cases are also investigated.

2. Basic equations

The Riemann–Liouville fractional integral is introduced as a natural generalization of the convolution type integral[9,10,32,33]

$$\mathrm{I}^{\alpha}f(t)=\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)\,\mathrm{d}\tau,\quad \alpha>0. \tag{1}$$

The Laplace transform rule for this fractional integral is

$$\mathrm{L}[\mathrm{I}^{\alpha}f(t)]=\frac{1}{s^{\alpha}}\mathrm{L}[f(t)]. \tag{2}$$

The Riemann–Liouville fractional derivative of fractional order α is defined as the left-inverse of the fractional integral I^{α} as

$$\begin{aligned}\mathrm{D}_{\mathrm{RL}}^{\alpha}f(t)&=\mathrm{D}^{n}\mathrm{I}^{n-\alpha}f(t)\\&=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\int_0^t(t-\tau)^{n-\alpha-1}f(\tau)\,\mathrm{d}\tau,\\&\qquad n-1<\alpha<n.\end{aligned} \tag{3}$$

For the Laplace transform, the initial values of the fractional integral $\mathrm{I}^{n-\alpha}f(t)$ and its derivatives of order $k=1,2,\ldots,n-1$ are required, as

$$\begin{aligned}\mathrm{L}\left[\mathrm{D}_{\mathrm{RL}}^{\alpha}f(t)\right]&=s^{\alpha}\mathrm{L}\left[f(t)\right]-\sum_{k=0}^{n-1}s^{n-1-k}\mathrm{D}^{k}\mathrm{I}^{n-\alpha}f(0),\\&\qquad n-1<\alpha<n.\end{aligned} \tag{4}$$

We consider a homogeneous anisotropic generalized thermoelastic diffusive medium initially at uniform temperature T_0. Following Kuang,[34] the basic governing equations in anisotropic, homogeneous thermoelastic solids with fractional order derivative and diffusion include the equations of motion

$$\sigma_{ij,j}+\rho F_i=\rho\ddot{u}_i, \tag{5}$$

the stress–strain–temperature–diffusion relation

$$\begin{aligned}\sigma_{ij}&=c_{ijlm}e_{lm}+s_{ij}\left(T+\frac{\tau_1^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}T\right)\\&\quad+\varepsilon_{ij}\left(P+\frac{(\tau^1)^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}P\right),\end{aligned} \tag{6}$$

the displacement–strain relation

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \tag{7}$$

the energy equation

$$-q_{i,i}=\rho T_0\dot{S}, \tag{8}$$

the entropy–strain–temperature–diffusion relation

$$\rho T_0 S = \rho C_{\rm E}\Big(1+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)T - s_{ij}T_0 e_{ij} + aT_0\Big(1+\frac{(\tau^0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)P, \quad (9)$$

the Fourier's law

$$q_i = -K_{ij}T_{,j}, \quad (10)$$

the equation of mass diffusion

$$\alpha^*_{ij}P_{,ji} = \frac{\partial C}{\partial t}, \quad (11)$$

and the chemical–strain–temperature–diffusion relation

$$C = b\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)P - \varepsilon_{ij}e_{ij} + a\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)T, \quad (12)$$

where $\partial^\alpha/(\partial t^\alpha)$ is the Riemann–Liouville fractional derivative of fractional order α; $T=\theta-T_0$ is the temperature change; θ is the absolute temperature; T_0 is the reference uniform temperature of the body chosen such that $|T/T_0|\ll 1$; ρ, $C_{\rm E}$ are the density and the specific heat at constant strain respectively; q_i is the heat conduction vector; τ^0 and τ^1 are diffusion relaxation times with $\tau^1\geq\tau^0\geq 0$; τ_0 and τ_1 are thermal relaxation times with $\tau_1\geq\tau_0\geq 0$; a is a measure of the thermodiffusion effect; b is a measure of the diffusive effect; C is the concentration; α^*_{ij} are the diffusion parameters; s_{ij} and ε_{ij} are tensors of thermal and diffusion moduli, respectively, $K_{ij}=K_{ji}$ is the thermal conductivity tensor, c_{ijlm} ($c_{ijlm}=c_{lmij}=c_{jilm}=c_{ijml}$) is the tensor of elastic constants, $\sigma_{ij}=\sigma_{ji}$ are the components of the stress tensor, u_i are the components of the displacement vector, $e_{ij}=(u_{i,j}+u_{j,i})/2$ are the components of the strain tensor, and P and S are the chemical potential and the entropy per unit mass, respectively. The comma (,) before an index represents the partial space differentiation.

3. Uniqueness theorem

Theorem 1 For a linear anisotropic thermoelastic diffusion material occuping a regular region V with boundary surface A in the three-dimensional space, there is only one solution of functions: $u_i(\boldsymbol{x},t),\theta(\boldsymbol{x},t),P(\boldsymbol{x},t)$ of class $C^{(m)}(m\geq 2)$, and $\sigma_{ij}(\boldsymbol{x},t),e_{ij}(\boldsymbol{x},t)$ of class $C^{(1)}$, for $\boldsymbol{x}\in V+A$ having coordinates $\boldsymbol{x}=(x_1,x_2,x_3)$ at $t\geq 0$ that satisfy Eqs. (5)–(12) subject to the boundary conditions

$$u_i(\boldsymbol{x},t)=U_i(\boldsymbol{x},t),\ T(\boldsymbol{x},t)=\nu(\boldsymbol{x},t),\ P(\boldsymbol{x},t)=\upsilon(\boldsymbol{x},t), \quad (13)$$

and the initial conditions at $t=0$

$$u_i=u_i^0,\ \frac{\partial u_i}{\partial t}=\frac{\partial u_i^0}{\partial t},\ T=T^0,\ \frac{\partial T}{\partial t}=\frac{\partial T^0}{\partial t},\ P=P^0,\ \frac{\partial P}{\partial t}=\frac{\partial P^0}{\partial t}, \quad (14)$$

where u_i^0, $\partial u_i^0/\partial t$, T^0, $\partial T^0/\partial t$, P^0, $\partial P^0/\partial t$ are known functions. We assume that the Laplace transforms of all field variables exist; the material parameters satisfy the inequalities

$$T_0>0,\ \tau_0>0,\ \tau_1>0,\ \tau^0>0,\ \tau^1>0,\ C_{\rm E}>0,\ \rho>0, \quad (15)$$

and c_{ijlm}, K_{ij}, α^*_{ij} are positive definite.

Proof Let $u_i^{(1)},T^{(1)},P^{(1)},\ldots$ and $u_i^{(2)},T^{(2)},P^{(2)},\ldots$ be two solution sets of Eqs. (5)–(10) with homogeneous initial and boundary conditions. Let us take

$$u_i=u_i^{(1)}-u_i^{(2)},\ T=T^{(1)}-T^{(2)},\ P=P^{(1)}-P^{(2)}. \quad (16)$$

Functions u_i, T, and P satisfy the governing equations and homogeneous initial and boundary conditions

$$\sigma_{ij,j}+\rho F_i=\rho\frac{\partial^2 u_i}{\partial t^2}, \quad (17)$$

$$\sigma_{ij}=c_{ijlm}e_{lm}+s_{ij}\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)T + \varepsilon_{ij}\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)P, \quad (18)$$

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad (19)$$

$$-q_{i,i}=\rho T_0\frac{\partial S}{\partial t}, \quad (20)$$

$$q_i=-K_{ij}T_{,j}, \quad (21)$$

$$\rho S=k\Big(1+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)T - s_{ij}e_{ij} + a\Big(1+\frac{(\tau^0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)P, \quad (22)$$

$$\alpha^*_{ij}P_{,ji}=\frac{\partial C}{\partial t}, \quad (23)$$

$$C=b\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)P - \varepsilon_{ij}e_{ij} + a\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\Big)T, \quad (24)$$

$$u_i(\boldsymbol{x},t)=0,\ T(\boldsymbol{x},t)=0,\ P(\boldsymbol{x},t)=0,\ \boldsymbol{x}\in A,\ t>0, \quad (25)$$

$$u_i(\boldsymbol{x},t)=0,\ T(\boldsymbol{x},t)=0,\ P(\boldsymbol{x},t)=0,\ \boldsymbol{x}\in V,\ t=0, \quad (26)$$

$$\frac{\partial u_i}{\partial t}(\boldsymbol{x},t)=0,\ \frac{\partial T}{\partial t}(\boldsymbol{x},t)=0,\ \frac{\partial P}{\partial t}(\boldsymbol{x},t)=0,\ \boldsymbol{x}\in V,\ t=0, \quad (27)$$

where $k=\rho C_{\rm E}/T_0$. Performing the Laplace integral transform defined as

$$\bar f(x,s)=\mathrm{L}(f(x,t))=\int_0^\infty f(x,t)\,\mathrm{e}^{-st}\,\mathrm{d}t \quad (28)$$

on Eqs. (17)–(25) and omitting the bars for simplicity, we obtain

$$\sigma_{ij,j}+\rho F_i=\rho s^2 u_i, \quad (29)$$

$$\sigma_{ij}=c_{ijlm}e_{lm}+s_{ij}\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)T + \varepsilon_{ij}\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\Big)P, \quad (30)$$

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad (31)$$

$$-q_{i,i}=\rho T_0 sS, \quad (32)$$

$$q_i = -K_{ij}T_{,j}, \tag{33}$$

$$\rho S = k\Big(1+\frac{(\tau_0)^\alpha}{\alpha!}s^{\alpha+1}\Big)T - s_{ij}e_{ij} + a\Big(1+\frac{(\tau^0)^\alpha}{\alpha!}s^{\alpha+1}\Big)P, \tag{34}$$

$$\alpha^*_{ij}P_{,ji} = sC, \tag{35}$$

$$C = b\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\Big)P - \varepsilon_{ij}e_{ij} + a\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)T, \tag{36}$$

$$u_i(\boldsymbol{x},s) = 0,\ T(\boldsymbol{x},t) = 0,\ P(\boldsymbol{x},t) = 0,\ \boldsymbol{x}\in A. \tag{37}$$

Consider the integral

$$\int_V \sigma_{ij}e_{ij}\,\mathrm{d}V = \int_V \sigma_{ij}u_{i,j}\,\mathrm{d}V = \int_V (\sigma_{ij}u_i)_{,j}\,\mathrm{d}V - \int_V \sigma_{ij,j}u_i\,\mathrm{d}V. \tag{38}$$

Using the divergence theorem and taking into consideration Eq. (25), we obtain

$$\int_V (\sigma_{ij}u_i)_{,j}\,\mathrm{d}V = \int_A u_i\sigma_{ij}n_j\,\mathrm{d}A = 0, \tag{39}$$

thus equation (38) takes the form

$$\int_V \sigma_{ij}e_{ij}\,\mathrm{d}V + \int_V \sigma_{ij,j}u_i\,\mathrm{d}V = 0. \tag{40}$$

By use of Eqs. (17) and (18), equation (40) can be written in the form

$$\int_V (c_{ijlm}e_{lm}e_{ij}+\rho s^2u_i^2)\,\mathrm{d}V + s_{ij}\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V Te_{ij}\,\mathrm{d}V + \varepsilon_{ij}\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V Pe_{ij}\,\mathrm{d}V = 0. \tag{41}$$

In order to obtain the integral $\int_V Te_{ij}\,\mathrm{d}V$, let us follow Biot[35] and introduce a vector ϕ_i connected with the entropy through the relation

$$q_i = T_0\frac{\partial\phi_i}{\partial t},\ \rho S = -\phi_{i,i}. \tag{42}$$

Substituting Eq. (42) into Eqs. (33) and (34) and assuming that $\phi_i(\boldsymbol{x},0) = \phi_i(\boldsymbol{x},0) = 0$, we obtain respectively

$$-T_{,i} = \frac{T_0}{K_{ij}}s\phi_i, \tag{43}$$

$$-\phi_{i,i} = k\Big(1+\frac{(\tau_0)^\alpha}{\alpha!}s^{\alpha+1}\Big)T - s_{ij}e_{ij} + a\Big(1+\frac{(\tau^0)^\alpha}{\alpha!}s^{\alpha+1}\Big)P. \tag{44}$$

Multiplying both sides of Eq. (43) by ϕ_i and integrating over the region of the body, we obtain

$$\int_V \Big(T_{,i}+\frac{T_0}{K_{ij}}s\phi_i\Big)\phi_i\,\mathrm{d}V = 0. \tag{45}$$

Using the divergence theorem and taking into consideration Eq. (45), we get

$$\frac{T_0}{K_{ij}}s\int_V \phi_i^2\,\mathrm{d}V - \int_V T\phi_{i,i}\,\mathrm{d}V = 0. \tag{46}$$

Using Eq. (44) in Eq. (46) yields

$$s_{ij}\int_V Te_{ij}\,\mathrm{d}V = \frac{T_0}{K_{ij}}s\int_V \phi_i^2\,\mathrm{d}V + k\Big(1+\frac{(\tau_0)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V T^2\,\mathrm{d}V + a\Big(1+\frac{(\tau^0)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V PT\,\mathrm{d}V. \tag{47}$$

In order to obtain the integral $\int_V Pe_{ij}\,\mathrm{d}V$ in Eq. (41), we introduce the vector function ψ_i defined as

$$C = -\psi_{i,i}. \tag{48}$$

Combining Eqs. (35) and (48) and assuming that $\psi_i(\boldsymbol{x},0) = \dot\psi_i(\boldsymbol{x},0) = 0$, we obtain

$$\alpha^*_{ij}P_{,ji} + s\psi_{i,i} = 0. \tag{49}$$

Equation (49) is satisfied if

$$\alpha^*_{ij}P_{,j} + s\psi_i = 0. \tag{50}$$

Multiplying both sides of Eq. (50) with ψ_i and integrating over the region of the body, we obtain

$$\int_V (\alpha^*_{ij}P_{,j} + s\psi_i)\psi_i\,\mathrm{d}V = 0. \tag{51}$$

Using the divergence theorem and taking into consideration Eq. (25), we obtain

$$\int_V s\psi_i^2\,\mathrm{d}V - \int_V \alpha^*_{ij}P\psi_{i,i}\,\mathrm{d}V = 0. \tag{52}$$

Making use of Eqs. (36) and (48) in Eq. (52) yields

$$\varepsilon_{ij}\int_V Pe_{ij}\,\mathrm{d}V = \frac{s}{\alpha^*_{ij}}\int_V \psi_i^2\,\mathrm{d}V + b\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V P^2\,\mathrm{d}V + a\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V PT\,\mathrm{d}V. \tag{53}$$

Equation (41) with the help of Eqs. (47) and (53) leads to

$$\int_V (c_{ijlm}e_{lm}e_{ij}+\rho s^2u_i^2)\,\mathrm{d}V + \Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\Big[\frac{T_0}{K_{ij}}s\int_V \phi_i^2\,\mathrm{d}V + k\Big(1+\frac{(\tau_0)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V T^2\,\mathrm{d}V + a\Big(1+\frac{(\tau^0)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V PT\,\mathrm{d}V\Big] + \Big(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\Big[\frac{s}{\alpha^*_{ij}}\int_V \psi_i^2\,\mathrm{d}V + b\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V P^2\,\mathrm{d}V + a\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V PT\,\mathrm{d}V\Big] = 0. \tag{54}$$

We note that under the condition

$$\frac{n^2}{4}\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\big[2+((\tau^0)^\alpha+(\tau_1)^\alpha)s\big]^2 < bk\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\Big)^2\Big(1+\frac{(\tau_0)^\alpha}{\alpha!}s^{\alpha+1}\Big), \tag{55}$$

the quadratic form

$$g = \Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\Big(1+\frac{(\tau_0)^\alpha}{\alpha!}s^{\alpha+1}\Big)\int_V T^2\,\mathrm{d}V + b\Big(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\Big)^2\int_V P^2\,\mathrm{d}V + a\Big(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\Big)\times\Big[2+((\tau^0)^\alpha+(\tau_1)^\alpha)\frac{s^{\alpha+1}}{\alpha!}\Big]\int_V PT\,\mathrm{d}V \tag{56}$$

is nonnegative.

In the absence of relaxation time, inequality (55) takes the form given by Aouadi.[36] Since the material parameters are positive and the integrand in Eq. (54) is a sum of squares, thus we conclude that

$$u_i = T = P = e_{ij} = \sigma_{ij} = 0. \tag{57}$$

That is, the Laplace transforms of all the difference functions (16) are zeros, and according to Learch's theorem,[37] the inverse Laplace transform of each is unique, consequently

$$u_i^{(1)} = u_i^{(2)},\ T^{(1)} = T^{(2)},\ P^{(1)} = P^{(2)}. \tag{58}$$

This proves the uniqueness of the solution to the complete system of field equations subjected to the initial and boundary conditions.

4. Reciprocity theorem

We consider a homogeneous anisotropic fractional order generalized thermoelastic body occupying region V and bounded by surface A. We assume that the stresses σ_{ij} and the strains e_{ij} are continuous together with their first derivatives, whereas the displacements u_i, temperature T, and chemical potential P are continuous and have continuous derivatives up to the second order, for $\boldsymbol{x} \in V + A$, $t > 0$.

The components of surface traction, the normal component of the heat flux, and the normal component of the equilibrated stress vector at regular points of ∂V are given by

$$h_i = \sigma_{ij} n_j,\ q = K_{ij} T_{,j} n_i,\ p = \alpha_{ij}^* P_{,j} n_j, \tag{59}$$

where n_j is the outward unit normal of ∂V.

To the system of field equations, we adjoin boundary conditions (13) and homogeneous initial conditions (26) and (27).

We derive the dynamic reciprocity relationship for a fractional order generalized thermoelastic body subjected to the action of body forces $F_i(\boldsymbol{x},t)$, surface traction $h_i(\boldsymbol{x},t)$, chemical flux $p(\boldsymbol{x},t)$, and heat flux $q(\boldsymbol{x},t)$, which satisfies Eqs. (29)–(36), the boundary conditions (13), and the homogeneous initial conditions (26), (27).

We now consider two problems where applied body forces, chemical potentials, and surface temperatures are specified differently. Let the variables involved in these two problems be distinguished by superscripts in parentheses. Thus, we have $u_i^{(1)}, e_{ij}^{(1)}, \sigma_{ij}^{(1)}, T^{(1)}, P^{(1)}, \ldots$. for the first problem and $u_i^{(2)}, e_{ij}^{(2)}, \sigma_{ij}^{(2)}, T^{(2)}, P^{(2)}, \ldots$. for the second problem. Each set of variables satisfies the system of equations (29)–(36).

Using the strain–displacement relation, the assumption $\sigma_{ij} = \sigma_{ji}$, and the divergence theorem, with the aid of Eqs. (29) and (59), we obtain

$$\begin{aligned}\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)}\,\mathrm{d}V &= \int_A h_i^{(1)} u_i^{(2)}\,\mathrm{d}A - \rho \int_V s^2 u_i^{(1)} u_i^{(2)}\,\mathrm{d}V \\ &\quad + \rho \int_V F_i^{(1)} u_i^{(2)}\,\mathrm{d}V. \end{aligned}\tag{60}$$

A similar expression is obtained for the integral $\int_V \sigma_{ij}^{(2)} e_{ij}^{(1)}\,\mathrm{d}V$, from which together with Eq. (60), it follows that

$$\begin{aligned}&\int_V \left[\sigma_{ij}^{(1)} e_{ij}^{(2)} - \sigma_{ij}^{(2)} e_{ij}^{(1)}\right]\mathrm{d}V \\ &= \int_A \left[h_i^{(1)} u_i^{(2)} - h_i^{(2)} u_i^{(1)}\right]\mathrm{d}A \\ &\quad + \rho \int_V \left[F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)}\right]\mathrm{d}V. \end{aligned}\tag{61}$$

Now multiplying Eq. (30) by $e_{ij}^{(2)}$ and $e_{ij}^{(1)}$ for the first and the second problems, respectively, subtracting and integrating over region V and using the symmetry properties of c_{ijlm}, we obtain

$$\begin{aligned}&\int_V \left[\sigma_{ij}^{(1)} e_{ij}^{(2)} - \sigma_{ij}^{(2)} e_{ij}^{(1)}\right]\mathrm{d}V \\ &= \int_V s_{ij}\left(1 + \frac{(\tau_1^\alpha)}{\alpha!} s^{\alpha+1}\right)\left[T^{(1)} e_{ij}^{(2)} - T^{(2)} e_{ij}^{(1)}\right]\mathrm{d}V \\ &\quad + \int_V \varepsilon_{ij}\left(1 + \frac{(\tau^1)^\alpha}{\alpha!} s^{\alpha+1}\right)\left[P^{(1)} e_{ij}^{(2)} - P^{(2)} e_{ij}^{(1)}\right]\mathrm{d}V. \end{aligned}\tag{62}$$

Equating Eqs. (61) and (62), we get the first part of the reciprocity theorem

$$\begin{aligned}&\int_A \left[h_i^{(1)} u_i^{(2)} - h_i^{(2)} u_i^{(1)}\right]\mathrm{d}A + \rho \int_V \left[F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)}\right]\mathrm{d}V \\ &= \int_V s_{ij}\left(1 + \frac{(\tau_1^\alpha)}{\alpha!} s^{\alpha+1}\right)\left[T^{(1)} e_{ij}^{(2)} - T^{(2)} e_{ij}^{(1)}\right]\mathrm{d}V \\ &\quad + \int_V \varepsilon_{ij}\left(1 + \frac{(\tau^1)^\alpha}{\alpha!} s^{\alpha+1}\right)\left[P^{(1)} e_{ij}^{(2)} - P^{(2)} e_{ij}^{(1)}\right]\mathrm{d}V, \end{aligned}\tag{63}$$

which contains the mechanical causes of motion F_i, T_i.

From Eqs. (32)–(34), we obtain

$$\begin{aligned}K_{ij} T_{,ji} &= T_0 s\left[k\left(1 + \frac{(\tau_0)^\alpha}{\alpha!} s^{\alpha+1}\right) T - s_{ij} e_{ij}\right. \\ &\quad \left. + a\left(1 + \frac{(\tau^0)^\alpha}{\alpha!} s^{\alpha+1}\right) P\right]. \end{aligned}\tag{64}$$

To derive the second part, multiplying Eq. (64) by $T^{(2)}$ and $T^{(1)}$ for the first and the second problems, respectively, subtracting and integrating over region V, we obtain

$$\begin{aligned}&\int_V \left[K_{ij}(T_{,j}^{(1)} T^{(2)} - T_{,j}^{(2)} T^{(1)})\right]_{,i}\mathrm{d}V \\ &= T_0 s\left(1 + \frac{(\tau_0)^\alpha}{\alpha!} s^{\alpha+1}\right)\int_V a\left(P^{(1)} T^{(2)} - P^{(2)} T^{(1)}\right)\mathrm{d}V \\ &\quad - T_0 s \int_V s_{ij}\left(e_{ij}^{(1)} T^{(2)} - e_{ij}^{(2)} T^{(1)}\right)\mathrm{d}V. \end{aligned}\tag{65}$$

By using the divergence theorem and with the aid of Eqs. (13) and (59), equation (65) becomes

$$\begin{aligned}&\int_A \left(q^{(1)} \nu^{(2)} - q^{(2)} \nu^{(1)}\right)\mathrm{d}A \\ &= T_0 s\left(1 + \frac{(\tau_0)^\alpha}{\alpha!} s^{\alpha+1}\right)\int_V a\left(P^{(1)} T^{(2)} - P^{(2)} T^{(1)}\right)\mathrm{d}V \\ &\quad - T_0 s \int_V s_{ij}\left(e_{ij}^{(1)} T^{(2)} - e_{ij}^{(2)} T^{(1)}\right)\mathrm{d}V. \end{aligned}\tag{66}$$

Equation (66) constitutes the second part of the reciprocity theorem, which contains the thermal causes of motion ν and q.

Using Eqs. (35) and (36), we obtain

$$\begin{aligned}\alpha_{ij}^* P_{,ji} &= s\left[b\left(1 + \frac{(\tau^1)^\alpha}{\alpha!} s^{\alpha+1}\right) P - \varepsilon_{ij} e_{ij}\right. \\ &\quad \left. + a\left(1 + \frac{(\tau_1)^\alpha}{\alpha!} s^{\alpha+1}\right) T\right]. \end{aligned}\tag{67}$$

Now multiplying Eq. (67) by $P^{(2)}$ and $P^{(1)}$ for the first and the second problems, respectively, subtracting and integrating

over region V, we obtain

$$\begin{aligned}&\int_V \alpha_{ij}^*\left(P_{,ji}^{(1)}P^{(2)}-P_{,ji}^{(2)}P^{(1)}\right)\mathrm{d}V\\&=s\left(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\right)\int_V a\left(P^{(2)}T^{(1)}-P^{(1)}T^{(2)}\right)\mathrm{d}V\\&\quad-s\int_V \varepsilon_{ij}\left(e_{ij}^{(1)}P^{(2)}-e_{ij}^{(2)}P^{(1)}\right)\mathrm{d}V.\end{aligned}\tag{68}$$

By using the divergence theorem and with the aid of Eqs. (13) and (59), equation (68) becomes

$$\begin{aligned}&\int_A \left(p^{(1)}\upsilon^{(2)}-p^{(2)}\upsilon^{(1)}\right)\mathrm{d}A\\&=s\left(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\right)\int_V a\left(P^{(2)}T^{(1)}-P^{(1)}T^{(2)}\right)\mathrm{d}V\\&\quad-s\int_V \varepsilon_{ij}\left(e_{ij}^{(1)}P^{(2)}-e_{ij}^{(2)}P^{(1)}\right)\mathrm{d}V.\end{aligned}\tag{69}$$

Equation (69) constitutes the third part of the reciprocity theorem, which contains the chemical potential causes of motion υ and p.

Eliminating the integrals $\int_V \varepsilon_{ij}(e_{ij}^{(1)}P^{(2)}-e_{ij}^{(2)}P^{(1)})\mathrm{d}V$ and $\int_V s_{ij}(e_{ij}^{(1)}T^{(2)}-e_{ij}^{(2)}T^{(1)})\mathrm{d}V$ from Eqs. (63), (66), and (69), we obtain

$$\begin{aligned}&T_0 s\int_A\left[h_i^{(1)}u_i^{(2)}-h_i^{(2)}u_i^{(1)}\right]\mathrm{d}A\\&+\rho s T_0\int_V\left[F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)}\right]\mathrm{d}V\\&-T_0 s\left(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\right)\left(1+\frac{(\tau^0)^\alpha}{\alpha!}s^{\alpha+1}\right)\\&\times\int_V a\left(P^{(1)}T^{(2)}-P^{(2)}T^{(1)}\right)\mathrm{d}V\\&+\int_A\left(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\right)\left(q^{(1)}\nu^{(2)}-q^{(2)}\nu^{(1)}\right)\mathrm{d}A\\&+T_0 s\int_V a\left(1+\frac{(\tau_1)^\alpha}{\alpha!}s^{\alpha+1}\right)\left(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\right)\\&\times\left(T^{(1)}P^{(2)}-T^{(2)}P^{(1)}\right)\mathrm{d}V\\&+T_0\left(1+\frac{(\tau^1)^\alpha}{\alpha!}s^{\alpha+1}\right)\int_A\left(p^{(1)}\upsilon^{(2)}-p^{(2)}\upsilon^{(1)}\right)\mathrm{d}A=0.\end{aligned}\tag{70}$$

This is the general reciprocity theorem in the Laplace transform domain.

To invert the Laplace transform in Eqs. (63), (66), (69), and (70), we shall use the convolution theorem

$$\begin{aligned}\mathrm{L}^{-1}\{F(s)G(s)\}&=\int_0^t f(t-\xi)g(\xi)\mathrm{d}\xi\\&=\int_0^t g(t-\xi)f(\xi)\mathrm{d}\xi,\end{aligned}\tag{71}$$

and the symbolic notations

$$\begin{aligned}M(f)&=1+\frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^\alpha f(x,\xi)}{\partial\xi^\alpha},\\Q(f)&=1+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^\alpha f(x,\xi)}{\partial\xi^\alpha},\\K(f)&=1+\frac{(\tau_1)^\alpha+(\tau^1)^\alpha}{\alpha!}\frac{\partial^\alpha f(x,\xi)}{\partial\xi^\alpha}\\&\quad+(\tau_1)^\alpha(\tau^1)^\alpha\frac{\partial^{2\alpha}f(x,\xi)}{\partial\xi^{2\alpha}},\\H(f)&=1+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^\alpha f(x,\xi)}{\partial\xi^\alpha},\\L(f)&=1+\frac{(\tau_1)^\alpha+(\tau_0)^\alpha}{\alpha!}\frac{\partial^\alpha f(x,\xi)}{\partial\xi^\alpha}\\&\quad+(\tau_1)^\alpha(\tau_0)^\alpha\frac{\partial^{2\alpha}f(x,\xi)}{\partial\xi^{2\alpha}}.\end{aligned}\tag{72}$$

Inverting Eq. (63), we obtain the first part of the reciprocity theorem in the final form

$$\begin{aligned}&\int_A\int_0^t h_i^{(1)}(x,t-\xi)u_i^{(2)}(x,\xi)\mathrm{d}\xi\,\mathrm{d}A\\&+\rho\int_V\int_0^t F_i^{(1)}(x,t-\xi)u_i^{(2)}(x,\xi)\mathrm{d}\xi\,\mathrm{d}V\\&-\int_V\int_0^t s_{ij}T^{(1)}(x,t-\xi)M(e_{ij}^{(2)})(x,\xi)\mathrm{d}\xi\,\mathrm{d}V\\&-\int_V\int_0^t \varepsilon_{ij}P^{(1)}(x,t-\xi)Q(e_{ij}^{(2)})(x,\xi)\mathrm{d}\xi\,\mathrm{d}V=S_{21}^{12}.\end{aligned}\tag{73}$$

Inverting Eq. (66), we obtain the second part of the reciprocity theorem in the final form

$$\begin{aligned}&\int_A\int_0^t q^{(1)}(x,t-\xi)\nu^{(2)}(x,\xi)\mathrm{d}\xi\,\mathrm{d}A\\&+T_0 s\int_V\int_0^t s_{ij}e_{ij}^{(1)}(x,t-\xi)T^{(2)}(x,\xi)\mathrm{d}\xi\,\mathrm{d}V\\&+T_0\int_V\int_0^t aP^{(1)}(x,t-\xi)\frac{\partial H(T^{(2)})}{\partial\xi}(x,\xi)\mathrm{d}\xi\,\mathrm{d}V=S_{21}^{12}.\end{aligned}\tag{74}$$

Inverting Eq. (69), we obtain the third part of the reciprocity theorem in the final form

$$\begin{aligned}&\int_A\int_0^t p^{(1)}(x,t-\xi)\upsilon^{(2)}(x,\xi)\mathrm{d}\xi\,\mathrm{d}A\\&+\int_V\int_0^t \varepsilon_{ij}e_{ij}^{(1)}(x,t-\xi)\frac{\partial P^{(2)}}{\partial\xi}(x,\xi)\mathrm{d}\xi\,\mathrm{d}V\\&-\int_V\int_0^t aP^{(2)}(x,t-\xi)\frac{\partial M(T^{(1)})}{\partial\xi}(x,\xi)\mathrm{d}\xi\,\mathrm{d}V=S_{21}^{12}.\end{aligned}\tag{75}$$

Finally, inverting Eq. (70), we obtain the general reciprocity theorem in the final form

$$\begin{aligned}&T_0\int_A\int_0^t h_i^{(1)}(x,t-\xi)\frac{\partial u_i^{(2)}}{\partial\xi}(x,\xi)\mathrm{d}\xi\,\mathrm{d}A\\&+\rho T_0\int_V\int_0^t F_i^{(1)}(x,t-\xi)\frac{\partial u_i^{(2)}}{\partial\xi}(x,\xi)\mathrm{d}\xi\,\mathrm{d}V\\&-T_0\int_A\int_0^t aP^{(1)}(x,t-\xi)\frac{\partial L(T^{(2)})}{\partial\xi}(x,\xi)\mathrm{d}\xi\,\mathrm{d}A\\&+\int_A\int_0^t q^{(1)}(x,t-\xi)M(\nu^{(2)})(x,\xi)\mathrm{d}\xi\,\mathrm{d}A\\&+T_0\int_A\int_0^t aT^{(1)}(x,t-\xi)\frac{\partial K(P^{(2)})}{\partial\xi}(x,\xi)\mathrm{d}\xi\,\mathrm{d}A\\&+T_0\int_A\int_0^t p^{(1)}(x,t-\xi)Q(\upsilon^{(2)})(x,\xi)\mathrm{d}\xi\,\mathrm{d}A=S_{21}^{12}.\end{aligned}\tag{76}$$

Here S_{21}^{12} indicates the same expression as that on the left-hand side except that the superscripts (1) and (2) are interchanged.

5. Formulation of the problem

From Eqs. (5) and (6), the equation of motion without body force is

$$c_{ijlm}e_{lm,j}+s_{ij}\Big(T+\frac{\tau_1^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}T\Big)_{,j} +\varepsilon_{ij}\Big(P+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}P\Big)_{,j}=\rho\frac{\partial^2 u_i}{\partial t^2}. \quad (77)$$

From Eqs. (8)–(10), the equation of heat conduction is

$$K_{ij}T_{,ji} = kT_0\Big(\frac{\partial}{\partial t}+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}}\Big)T-s_{ij}T_0\frac{\partial}{\partial t}e_{ij} +aT_0\Big(\frac{\partial}{\partial t}+\frac{(\tau^0)^\alpha}{\alpha!}\frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}}\Big)P. \quad (78)$$

From Eqs. (11) and (12), the equation of chemical potential is

$$\alpha_{ij}^*P_{,ji} = b\Big(\frac{\partial}{\partial t}+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}}\Big)P-\varepsilon_{ij}\dot{e}_{ij} +a\Big(\frac{\partial}{\partial t}+\frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}}\Big)T. \quad (79)$$

6. Transversely isotropic media

Applying the transformation[38]

$$x_1'=x_1\cos\phi+x_2\sin\phi,\ x_2'=-x_1\sin\phi+x_2\cos\phi,\ x_3'=x_3, \quad (80)$$

where ϕ is the angle of rotation in the x_1–x_2 plane, into Eqs. (77)–(79), we obtain the basic equations for the homogeneous transversely isotropic thermodiffusive elastic solid

$$c_{11}u_{1,11}+c_{12}u_{2,21}+c_{13}u_{3,31}+c_{66}(u_{1,22}+u_{2,12}) +c_{44}(u_{1,33}+u_{3,13})-s_1\Big(T+\frac{\tau_1^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}T\Big)_{,1} -\varepsilon_1\Big(P+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}P\Big)_{,1}=\rho\frac{\partial^2 u_1}{\partial t^2}, \quad (81)$$

$$c_{11}u_{2,22}+c_{44}u_{2,33}+c_{13}u_{3,31}+c_{66}(u_{1,21}+u_{2,11}) +(c_{13}+c_{44})u_{3,32}-s_2\Big(T+\frac{\tau_1^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}T\Big)_{,2} -\varepsilon_2\Big(P+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}P\Big)_{,2}=\rho\frac{\partial^2 u_2}{\partial t^2}, \quad (82)$$

$$(c_{13}+c_{44})(u_{1,13}+u_{2,23})+c_{44}(u_{3,11}+u_{3,22})+c_{33}u_{3,33} -s_3\Big(T+\frac{\tau_1^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}T\Big)_{,3}-\varepsilon_3\Big(P+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}P\Big)_{,3} =\rho\frac{\partial^2 u_3}{\partial t^2}, \quad (83)$$

$$K_1(T_{,11}+T_{,22})+K_3T_{,33} =kTT_0\Big(\frac{\partial}{\partial t}+\frac{\tau_0^\alpha}{\alpha!}\frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}}\Big)+aPT_0\Big(\frac{\partial}{\partial t}+\frac{(\tau^0)^\alpha}{\alpha!}\frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}}\Big) +T_0\Big(s_1\frac{\partial}{\partial t}(u_{1,1}+u_{2,2})+s_3\frac{\partial u_{3,3}}{\partial t}\Big), \quad (84)$$

$$\alpha_1^*(P_{,11}+P_{,22})+\alpha_3^*P_{,33} =aTT_0\Big(\frac{\partial}{\partial t}+\frac{\tau_1^\alpha}{\alpha!}\frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}}\Big)+bP\Big(\frac{\partial}{\partial t}+\frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}}\Big) +\Big(\varepsilon_1\frac{\partial}{\partial t}(u_{1,1}+u_{2,2})+\varepsilon_3\frac{\partial u_{3,3}}{\partial t}\Big), \quad (85)$$

where

$$s_{ij}=-s_i\delta_{ij},\ \varepsilon_{ij}=-\varepsilon_i\delta_{ij},\ K_{ij}=K_i\delta_{ij},$$
$$\alpha_{ij}^*=\alpha_i^*\delta_{ij},\quad (i \text{ is not summed})$$
$$s_1=(c_{11}+c_{12})\alpha_1+c_{13}\alpha_3,\ s_3=2c_{13}\alpha_1+c_{33}\alpha_3,$$
$$\varepsilon_1=(c_{11}+c_{12})\alpha_{1c}+c_{13}\alpha_{3c},\ \varepsilon_3=2c_{13}\alpha_{1c}+c_{33}\alpha_{3c}.$$

In Eqs. (81)–(85), we use the contracting subscript notations $11\to1$, $22\to2$, $33\to3$, $23\to4$, $13\to5$, $12\to6$ to relate c_{ijkm} to c_{ln} ($i,j,k,m=1,2,3$ and $l,n=1,2,\ldots,6$).

For the two-dimensional problem, we have

$$u=(u_1,0,u_3),\ P(x_1,0,x_3),\ T(x_1,0,x_3). \quad (86)$$

We define the following dimensionless quantities:

$$x_i'=\frac{\omega_1^*x_i}{c_1},\ u_i'=\frac{\omega_1^*u_i}{c_1},\ t'=\omega_1^*t,\ \tau_0'=\omega_1^*\tau_0,\ \tau_1'=\omega_1^*\tau_1,$$
$$\tau^{0\prime}=\omega_1^*\tau^0,\ \tau^{1\prime}=\omega_1^*\tau^1, P'=\frac{P}{\varepsilon_1},\ T'=\frac{s_1T}{c_{11}},$$
$$c_1^2=\frac{c_{11}}{\rho},\ \omega_1^*=\frac{\rho C_{\rm E}c_{11}}{K_1}. \quad (87)$$

Upon introducing the dimensionless quantities defined by Eq. (87) in Eqs. (81)–(85) with the aid of Eq. (86) and after suppressing the primes, we obtain

$$u_{1,11}+\delta_1u_{3,31}+\delta_2u_{1,33}-\tau_{11}T_{,1}-\delta_3\tau_{10}P_{,1}=\frac{\partial^2u_1}{\partial t^2}, \quad (88)$$

$$\delta_1u_{1,13}+\delta_2u_{3,11}+\delta_4u_{3,33}-\bar{s}\tau_{11}T_{,3}-\delta_5\tau_{10}P_{,3}=\frac{\partial^2u_3}{\partial t^2}, \quad (89)$$

$$\delta_8\frac{\partial u_{1,1}}{\partial t}+\delta_9\frac{\partial u_{3,3}}{\partial t}+\delta_6\tau_{12}T-(T_{,11}+\bar{K}T_{,33}) +\delta_7\tau_{13}P=0, \quad (90)$$

$$\delta_{12}\frac{\partial u_{1,1}}{\partial t}+\delta_{13}\frac{\partial u_{3,3}}{\partial t}+\delta_{10}\tau_{11}\frac{\partial T}{\partial t}-(P_{,11}+\bar{\alpha}P_{,33}) +\delta_{11}\tau_{10}\frac{\partial P}{\partial t}=0, \quad (91)$$

where $\delta_1-\delta_{13}$ and $\tau_{10}-\tau_{13}$ are given in Appendix A.

7. Solution of the problem

For plane harmonic waves, we assume the solution of Eqs. (88)–(91) taking the form

$$(u_1,u_2,u_3,T,P) = (U_1,U_2,U_3,T^*,P^*) \times\exp\big[\iota(\xi(x_1l_1+x_3l_3)-\omega t)\big], \quad (92)$$

where ω is the circular frequency, ξ is the complex wave number, U_1, U_2, U_3, T^*, and ϕ^* are undetermined amplitude vectors that are independent of time t and coordinates x_i ($i=1,3$), and l_1 and l_3 are the direction cosines of the wave normal onto the x_1–x_3 plane with the property $l_1^2+l_3^2=1$.

Substituting the values of u_1, u_2, u_3, T, P from Eq. (92) into Eqs. (88)–(91), we obtain

$$[-\xi^2l_1^2-\xi^2l_3^2\delta_2+\omega^2]U_1+[-\xi^2l_1l_3\delta_1]U_3-\iota\xi l_1\tau_{11}T^* -\iota\xi l_1\delta_3\tau_{10}P^*=0, \quad (93)$$

$$[-\xi^2l_1l_3\delta_1]U_1+[-\xi^2l_1^2\delta_2-\xi^2l_3^2\delta_4+\omega^2]U_3-\iota\xi\bar{s}l_3\tau_{11}T^*$$

$$-\iota\xi l_3\tau_{10}\delta_5 P^* = 0, \tag{94}$$

$$[\xi l_1\omega\delta_8]U_1 + [\xi l_3\omega\delta_9]U_3 + [\delta_6\tau_{12} + \xi^2 l_1^2 + \bar{k}\xi^2 l_3^2]T^* + \delta_7\tau_{13}P^* = 0, \tag{95}$$

$$[\xi l_1\omega\delta_{12}]U_1 + [\xi l_3\delta_{13}\omega]U_3 + \delta_{10}\tau_{11}T^* + [\delta_{11}\tau_{10} + \xi^2 l_1^2 + \bar{\alpha}\xi^2 l_3^2]P^* = 0, \tag{96}$$

where τ_{10}–τ_{15}, $\bar{k}$, $\bar{s}$ and $\bar{\alpha}$ are given in Appendix A.

The non-trivial solution of Eqs. (93)–(96) is ensured by the determinantal equation

$$\begin{vmatrix} -\xi^2 l_1^2 - \xi^2 l_3^2\delta_2 + \omega^2 & -\xi^2 l_1 l_3\delta_1 & -\iota\xi l_1\tau_{11} & -\iota\xi l_1\delta_3\tau_{10} \\ -\xi^2 l_1 l_3\delta_1 & -\xi^2 l_1^2\delta_2 - \xi^2 l_3^2\delta_4 + \omega^2 & -\iota\xi\bar{s}l_3\tau_{11} & -\iota\xi l_3\tau_{10}\delta_5 \\ \xi l_1\omega\delta_8 & \xi l_3\omega\delta_9 & \delta_6\tau_{12} + \xi^2 l_1^2 + \bar{k}\xi^2 l_3^2 & \delta_7\tau_{13} \\ \xi l_1\omega\delta_{12} & \xi l_3\delta_{13}\omega & \delta_{10}\tau_{11} & \delta_{11}\tau_{10} + \xi^2 l_1^2 + \bar{\alpha}\xi^2 l_3^2 \end{vmatrix} = 0. \tag{97}$$

Equation (97) yields the following polynomial characteristic equation in ξ:

$$F_1\xi^8 + F_2\xi^6 + F_3\xi^4 + F_4\xi^2 + F_5 = 0, \tag{98}$$

where coefficients F_1, F_2, F_3, F_4, and F_5 are given in Appendix A. Solving Eq. (98), we obtain eight roots of ξ, that is, $\pm\xi_1$, $\pm\xi_2$, $\pm\xi_3$, and $\pm\xi_4$. Corresponding to these roots, there exist four waves in descending order of their velocities, namely, a quasi-longitudinal (qP) wave, a quasi-transverse (qSV) wave, a quasi-mass diffusion (qMD) wave, and a quasi-thermal (qT) wave. Now we derive the expressions of phase velocity, attenuation coefficient, specific loss, and penetration depth for these waves.

The phase velocity is given by

$$V_i = \frac{\omega}{|\mathrm{Re}(\xi_i)|}, \tag{99}$$

where V_i $(i = 1,2,3,4)$ are, respectively, the velocities of qP, qSV, qMD, and qT waves.

The attenuation coefficient is defined as

$$Q_i = \mathrm{Im}(\xi_i), \tag{100}$$

where Q_i $(i = 1,2,3,4)$ are, respectively, the attenuation coefficients of qP, qSV, qMD, and qT waves.

The specific loss is the ratio of energy (ΔW) dissipated in taking a specimen through a stress cycle to the elastic energy (W) stored in the specimen when the strain is a maximum. The specific loss is the most direct method for defining the internal friction of a material. For a sinusoidal plane wave of small amplitude, Kolksy[39] showed that the specific loss $\Delta W/W$ equals 4π times the absolute value of the imaginary part of ξ to the real part of ξ, i.e.,

$$R_i = \left(\frac{\Delta W}{W}\right)_i = 4\pi\left|\frac{\mathrm{Im}(\xi_i)}{\mathrm{Re}(\xi_i)}\right|, \quad i = 1,2,3,4. \tag{101}$$

The penetration depth is defined as

$$S_i = \frac{1}{\mathrm{Im}(\xi_i)}, \quad i = 1,2,3,4. \tag{102}$$

Now we discuss some particular cases. (i) In the absence of diffusion, i.e., when $\varepsilon_1 = \varepsilon_3 = \alpha_3^* = 0$, the characteristic equation (98) reduces to the characteristic equation corresponding to the transversely isotropic fractional order thermoelastic medium

$$G_1\xi^6 + G_2\xi^4 + G_3\xi^2 + G_4 = 0,$$

where G_1, G_2, G_3, and G_4 are given in Appendix A.

(ii) If $\alpha = 1$ in Eqs. (6), (9), and (12), the corresponding results reduce to the case of the Green–Lindsay theory of thermoelasticity with diffusion.

8. Numerical results and discussion

The material chosen for this purpose is magnesium, whose physical parameters are given by

$c_{11} = 5.974\times10^{10}\ \mathrm{kg\cdot m^{-1}\cdot s^{-2}}$,
$c_{12} = 2.624\times10^{10}\ \mathrm{kg\cdot m^{-1}\cdot s^{-2}}$,
$c_{13} = 2.17\times10^{10}\ \mathrm{kg\cdot m^{-1}\cdot s^{-2}}$,
$c_{33} = 6.17\times10^{10}\ \mathrm{kg\cdot m^{-1}\cdot s^{-2}}$,
$c_{44} = 3.278\times10^{10}\ \mathrm{kg\cdot m^{-1}\cdot s^{-2}}$,
$T_0 = 298\ \mathrm{K}$, $C_\mathrm{E} = 0.104\times10^3\ \mathrm{J\cdot kg^{-1}\cdot K^{-1}}$,
$\alpha_1 = 2.33\times10^{-5}\ \mathrm{K^{-1}}$, $\alpha_3 = 2.48\times10^{-5}\ \mathrm{K^{-1}}$,
$\alpha_{1\mathrm{c}} = 2.65\times10^{-4}\ \mathrm{m^3\cdot kg^{-1}}$, $\alpha_{3\mathrm{c}} = 2.83\times10^{-4}\ \mathrm{m^3\cdot kg^{-1}}$,
$a = 2.9\times10^4\ \mathrm{m^2\cdot s^{-2}\cdot K^{-1}}$, $b = 32\times10^5\ \mathrm{kg^{-1}\cdot m^5\cdot s^{-2}}$,
$\alpha_1^* = 0.75\times10^{-8}\ \mathrm{kg\cdot s\cdot m^{-3}}$, $\alpha_3^* = 0.85\times10^{-8}\ \mathrm{kg\cdot s\cdot m^{-3}}$,
$\rho = 1.74\times10^3\ \mathrm{kg\cdot m^{-3}}$, $K_1 = 0.170\times10^3\ \mathrm{W\cdot m^{-1}\cdot K^{-1}}$,
$K_3 = 0.170\times10^3\ \mathrm{W\cdot m^{-1}\cdot K^{-1}}$,
$\tau_0 = 0.02\ \mathrm{s}$, $\tau_1 = 0.01\ \mathrm{s}$, $\tau^0 = 0.03\ \mathrm{s}$, $\tau^1 = 0.04\ \mathrm{s}$.

We can solve Eq. (98) with the help of Matlab software. After solving Eq. (98) and using the formulas given by Eqs. (99)–(102), we can compute the values of phase velocity, attenuation coefficient, specific loss, and penetration depth for intermediate values of frequency ω and different values of fractional order derivative, i.e., $\alpha = 0.5$, 1.0, 1.5. In all the figures, the horizontal lines, square boxes, and vertical lines corresponds to $\alpha = 0.5$, 1.0, and 1.5, respectively.

Figure 1 shows that for fractional orders $\alpha = 1.0$, 1.5, phase velocity V_1 decreases smoothly, whereas for $\alpha = 0.5$, V_1 increases sharply for small values of ω and decreases as ω increases further. It is evident from Fig. 2 that V_2 increases smoothly with the increase in fractional order. Figure 3 shows

that V_3 first increases rapidly and then shows the constant behavior for $\alpha = 0.5$, 1.0, whereas for $\alpha = 1.5$, it decreases smoothly as ω increases further. Figure 4 indicates that V_4 first decreases smoothly and then becomes constant as ω increases further for all fractional orders. In the frequency range $20 < \omega < 80$, the maximum value of V_4 occurs for $\alpha = 0.5$.

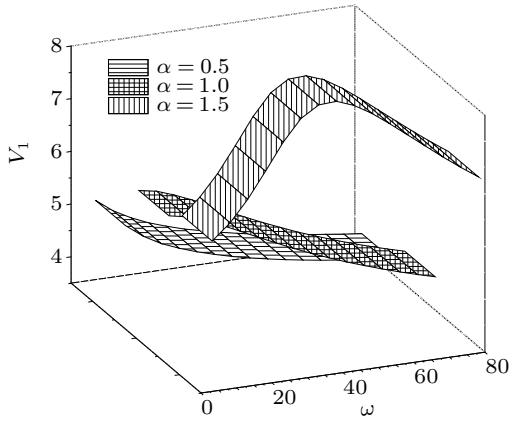


Fig. 1. Variation of phase velocity V_1 with frequency ω.

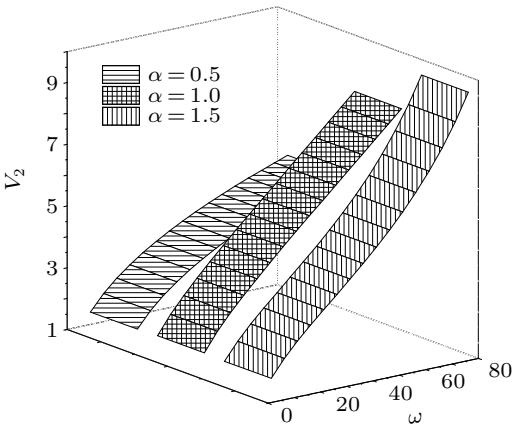


Fig. 2. Variation of phase velocity V_2 with frequency ω.

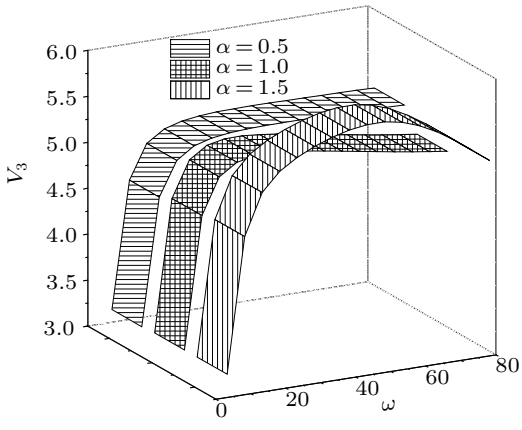


Fig. 3. Variation of phase velocity V_3 with frequency ω.

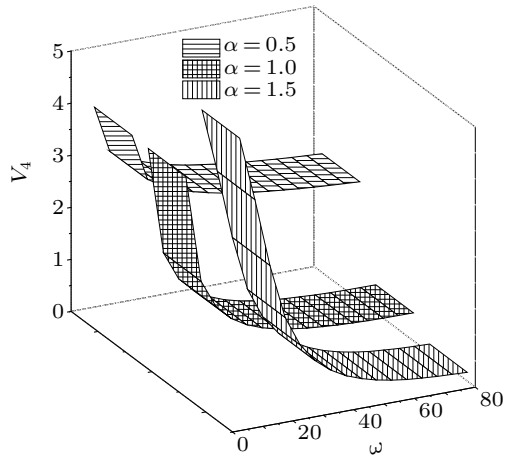


Fig. 4. Variation of phase velocity V_4 with frequency ω.

It is noticed from Fig. 5 that Q_1 increases with small variations for all fractional orders with the maximum value occurring for $\alpha = 1.5$. Figure 6 shows that Q_2 increases smoothly as fractional order α increases. From Fig. 7, it is evident that Q_3 shows the same behavior and variation as Q_1 but with different magnitudes. Figure 8 indicates that Q_4 increases as frequency ω increases for all fractional orders.

Figure 9 shows that for $\alpha = 0.5$, R_1 increases initially and then becomes constant, whereas for $\alpha = 1.0$, 1.5, R_1 increases smoothly with frequency. Figure 10 indicates that R_2 shows the behavior opposite to that of R_1. It is evident from Figure 11 that R_3 increases initially and then becomes constant for $\alpha = 0.5$, 1.0 and increases smoothly for $\alpha = 1.5$. Figure 12 shows that R_4 decreases rapidly for small values of ω and becomes constant as ω increases further with higher values for fractional order $\alpha = 0.5$. Figure 13 shows that penetration depth S_1 sharply decreases initially and then shows the constant behavior with difference in their magnitude values. Figure 14 indicates that S_2 decreases smoothly for all fractional orders and shows constant behavior lastly. Figures 15 and 16 depict that S_3 and S_4 decrease sharply and then show the constant behavior as ω increases further.

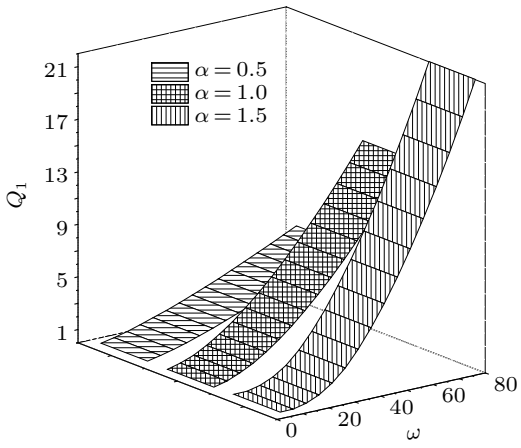


Fig. 5. Variation of attenuation coefficient Q_1 with frequency ω.

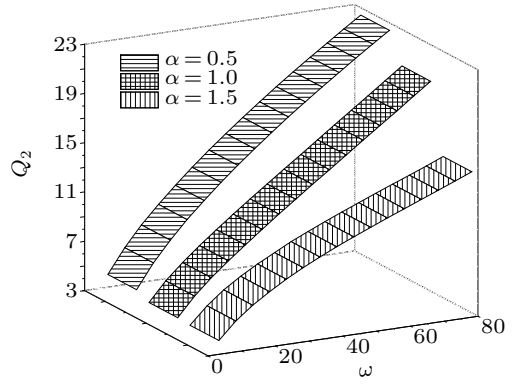


Fig. 6. Variation of attenuation coefficient Q_2 with frequency ω.

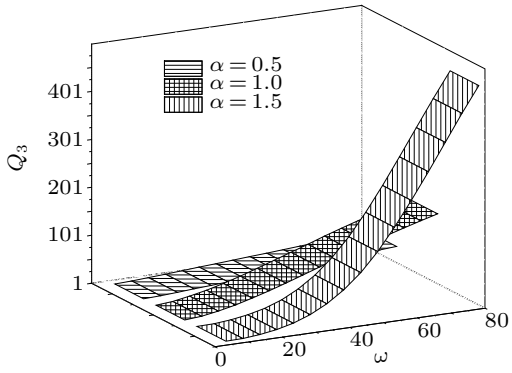


Fig. 7. Variation of attenuation coefficient Q_3 with frequency ω.

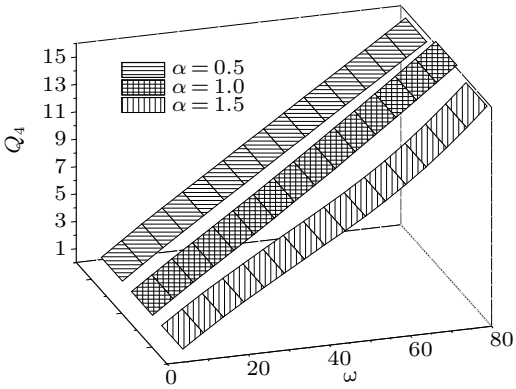


Fig. 8. Variation of attenuation coefficient Q_4 with frequency ω.

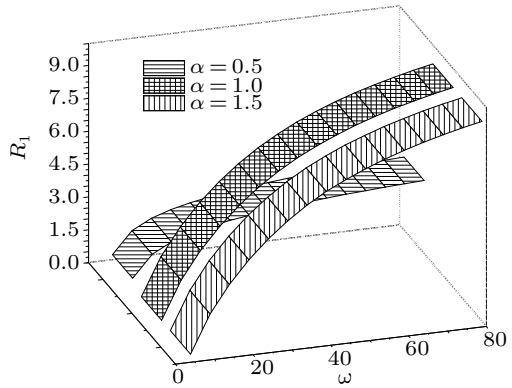


Fig. 9. Variation of specific loss R_1 with frequency ω.

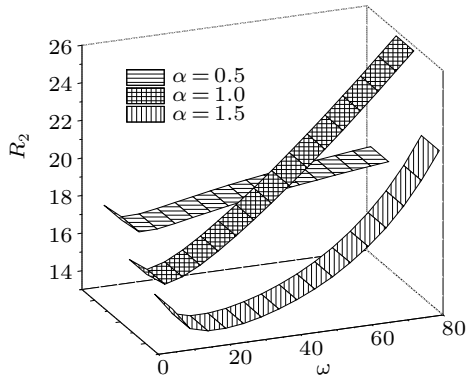


Fig. 10. Variation of specific loss R_2 with frequency ω.

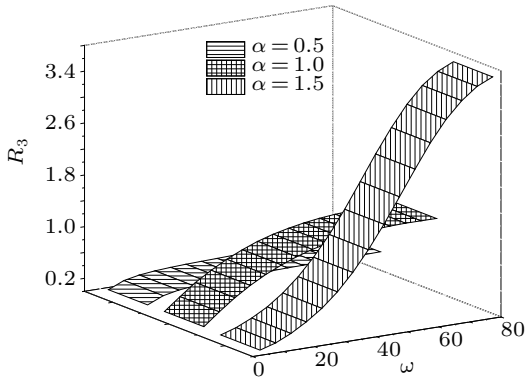


Fig. 11. Variation of specific loss R_3 with frequency ω.

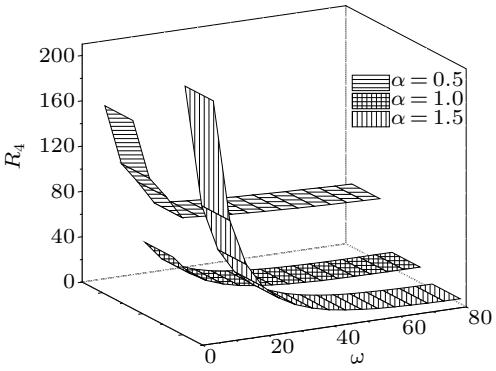


Fig. 12. Variation of specific loss R_4 with frequency ω.

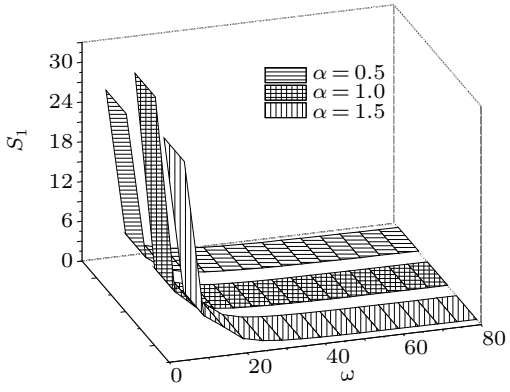


Fig. 13. Variation of penetration depth S_1 with frequency ω.

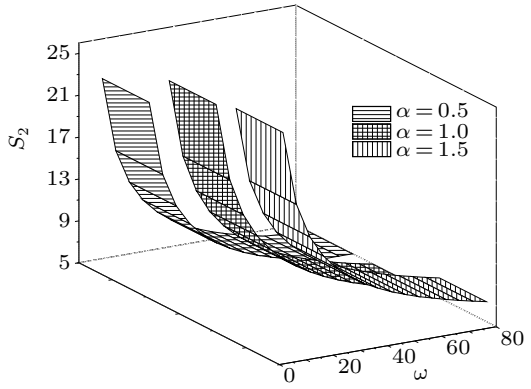


Fig. 14. Variation of penetration depth S_2 with frequency ω.

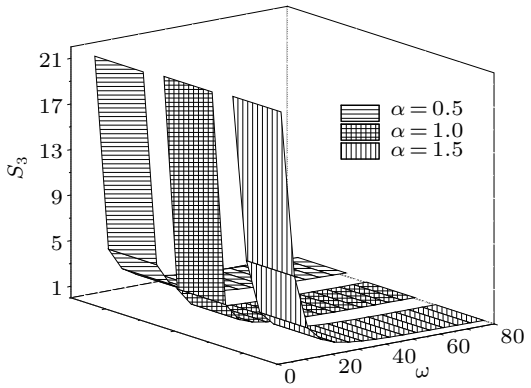


Fig. 15. Variation of penetration depth S_3 with frequency ω.

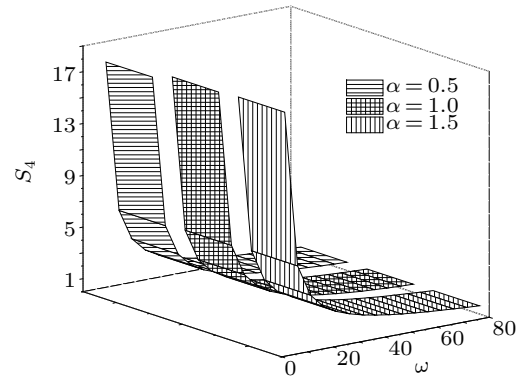


Fig. 16. Variation of penetration depth S_4 with frequency ω.

9. Conclusion

A model of anisotropic, homogeneous thermoelastic diffusion solid with a fractional order derivative based on the theory of Green and Lindsay is given. The uniqueness theorem of the solution of the initial boundary value problem is proved, and the dynamic reciprocity theorem is derived for the given model. The governing equations for the transversely isotropic thermoelastic material with fractional order derivative and diffusion are presented. For the two-dimensional problem, there exist a quasi-longitudinal wave, a quasi-transverse wave, a quasi-mass diffusion wave, and a quasi-thermal wave. The phase velocity, attenuation coefficient, specific loss, and penetration depth are computed numerically and presented graphically. Some particular cases are also discussed. From the figures, it is observed that phase velocities V_1, V_2, V_3, attenuation coefficients Q_1, Q_3, and specific losses R_3, R_4 are higher for fractional order $\alpha = 1.5$.

Appendix A

$$\delta_1 = \frac{c_{13}+c_{44}}{c_{11}},\ \delta_2 = \frac{c_{44}}{c_{11}},\ \delta_3 = \frac{\varepsilon_1^2}{c_{11}},\ \delta_4 = \frac{c_{33}}{c_{11}},$$

$$\delta_5 = \frac{\varepsilon_1\varepsilon_3}{c_{11}},\ \delta_6 = \frac{c_{11}^2 C_{\mathrm{E}}}{K_1 s_1 \omega_1^*},\ \delta_7 = \frac{nT_0\varepsilon_1 c_{11}}{\rho K_1 \omega_1^*},$$

$$\delta_8 = \frac{s_1 c_{11} T_0}{\rho K_1 \omega_1^*},\ \delta_9 = \frac{s_3 c_{11} T_0}{\rho K_1 \omega_1^*},\ \delta_{10} = \frac{aT_0 c_{11}^2}{\rho \alpha_1^* s_1 \omega_1^*},$$

$$\delta_{11} = \frac{bc_1^2\varepsilon_1}{\alpha_1^*\omega_1^*},\ \delta_{12} = \frac{c_1^2\varepsilon_1}{\alpha_1^*\omega_1^*},\ \delta_{13} = \frac{c_1^2\varepsilon_3}{\alpha_1^*\omega_1^*},$$

$$\bar{\alpha} = \frac{\alpha_3^*}{\alpha_1^*},\ \bar{k} = \frac{K_3}{K_1},\ \bar{s} = \frac{s_3}{s_1},\ \tau_{10} = 1 + \frac{(\tau^1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}},$$

$$\tau_{11} = 1 + \frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}},\ \tau_{12} = 1 + \frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}},$$

$$\tau_{13} = 1 + \frac{(\tau^0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}},\ \tau^{10} = 1 + \frac{(\tau^1)^\alpha}{\alpha!}(-\iota\omega)^{\alpha+1},$$

$$\tau^{11} = 1 + \frac{(\tau_1)^\alpha}{\alpha!}(-\iota\omega)^{\alpha+1},\ \tau^{12} = 1 + \frac{(\tau_0)^\alpha}{\alpha!}(-\iota\omega)^{\alpha+1},$$

$$\tau^{13} = 1 + \frac{(\tau^0)^\alpha}{\alpha!}(-\iota\omega)^{\alpha+1},\ \tau^{14} = -\iota\omega + \frac{(\tau^1)^\alpha}{\alpha!}(-\iota\omega)^{\alpha+2},$$

$$\tau^{15} = -\iota\omega + \frac{(\tau^1)^\alpha}{\alpha!}(-\iota\omega)^{\alpha+2},$$

$$F_1 = A_{10}A_{16}(A_0A_4 - A_1^2),$$

$$F_2 = -A_4A_{10}A_{16}\omega^2 - A_0C_1 - A_1C_2 + A_7A_{16}(A_1A_5 + A_2A_4) + A_{10}A_{12}(A_1A_6 + A_3A_4),$$

$$F_3 = C_1\omega^2 - A_0D_1 - A_1D_2 + A_7C_3 - A_{12}C_4,$$

$$F_4 = D_1 - A_0A_9A_{15} + A_3A_7A_{14} - A_{12}(A_3A_9 - A_2A_{11}),$$

$$F_5 = \omega^4 A_9A_{15},$$

$$A_0 = l_1^2 + \delta_2 l_3^2,\ A_1 = -l_1l_3\delta_1,\ A_2 = -\iota l_1\tau^{11},$$

$$A_3 = -\iota l_1\delta_3\tau^{10},\ A_4 = l_1^2\delta_2 + l_3^2\delta_4,\ A_5 = -\iota\bar{s}l_3\tau^{11},$$

$$A_6 = -\iota l_3\tau^{10}\delta_5,\ A_7 = l_1\omega\delta_8,\ A_8 = l_3\omega\delta_9,$$

$$A_9 = \delta_6\tau^{12},\ A_{10} = l_1^2l_3^2\bar{K},\ A_{11} = \delta_7\tau^{13},\ A_{12} = l_1\omega\delta_{12},$$

$$A_{13} = l_3\omega\delta_{13},\ A_{14} = \delta_{10}\tau^{14},\ A_{15} = \delta_{11}\tau^{15},\ A_{16} = l_1^2 + \bar{\alpha}l_3^2,$$

$$C1 = -(A_4(A_9A_{16} + A_{10}A_{15}) - \omega^2A_{10}A_{16} + A_5A_8A_{16} + A_6A_{10}A_{13}),$$

$$C2 = A_1(A_9A_{16} + A_{10}A_{15}) - A_2A_8A_{16} + A_3A_{10}A_{13},$$

$$C_3 = A_1(A_5A_{15} - A_6A_{14}) - A_2(\omega^2A_{16} - A_4A_{15} - A_6A_{13}) - A_3(A_4A_{14} + A_5A_{13}),$$

$$C_4 = A_1(A_5A_{11} - A_6A_9) + A_2(A_4A_{11} + A_6A_8)$$

$$+A_3(\omega^2 A_{10} - A_4 A_9 - A_5 A_8),$$

$$D_1 = -A_4 A_9 A_{15} + \omega^2 (A_9 A_{16} + A_{10} A_{15}) - A_8 (A_5 A_{15} - A_6 A_{14}) + A_{13} (A_5 A_{11} - A_6 A_9),$$

$$D_2 = A_1 (A_9 A_{15} - A_{11} A_{14}) - A_8 (A_2 A_{15} - A_3 A_{14}) + A_{13} (A_2 A_{11} - A_3 A_9).$$

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